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THE ORDER OF THE SECOND GROUP OF UNITS OF THE RING $\mathbf{Z}[i]/ < \beta >$

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Abstract. In this article we introduce a newly defined function $\phi_G^2(\beta)$ that represents the order of the second group of units of the ring $R = \mathbf{Z}[i]/ < \beta >$. We examine some of the properties of this function that are similar to that of the Euler Phi function $\phi(n)$.

Keywords: commutative ring; Euler Phi function; Gaussian primes; group of units; generalized group of units; ring of Gaussian integers.

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1. Introduction

The Euler Phi function, $\phi(n)$, has been generalized through different approaches and was studied in many domains. In 1983, J.T Cross [1], extended the defintion of the Euler Phi function to the domain of Gaussian integers $\mathbf{Z}[i]$, where $\phi_G(\beta)$ represents the order of the group of units of the ring $\mathbf{Z}[i]/ < \beta >$ where β is a non zero Gaussian integer. That is $\phi_G(\beta) = |U(\beta)|$. Cross gave a complete charecterization for the group of units $\Phi_G(\beta)$ as shown in the following theorem.

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Theorem 1.1.

(1) $U(\pi^n) \cong \mathbf{Z}_{q^n - q^{n-1}}$.

(2) $U(p^n) \cong \mathbf{Z}_{p^{n-1}} \times \mathbf{Z}_{p^{n-1}} \times \mathbf{Z}_{p^2-1}$.

(3) $U(1+i) \cong \mathbf{Z}_1$.

(4) $U((1+i)^2) \cong \mathbf{Z}_2$.

(5) $U((1+i)^n) \cong \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_{2^{m-2}} \times \mathbf{Z}_4$ if $n = 2m$.

(6) $U((1+i)^n) \cong \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_4$ if $n = 2m+1$.

In 2006, a generalization for the group of units of any finite commutative ring with identity was introduced by El -Kassar and Chehade [2]. They proved that the group of units of a commutative ring R with identity, $U(R)$, supports a ring structure and this made it possible to define the second group of units of R as $U^2(R) = U(U(R))$. Extending this definition to the kth level, the kth group of units is defined as $U^k(R) = U(U^{k-1}(R))$. This generalization allowed to formulate a new generalization to the Euler Phi function that represents the order of the generalized group of units of the ring R and is denoted by $\phi^k(R)$. In [4], Assaf gave an explicit formula, $\phi^2(R)$, for the order of the second group of units in the domain of integers.

In this paper, we introduce a newly defined function, $\phi_G^2(\beta)$, the order of the second group of units $U^2(\beta) = \Phi_G^2(\beta)$ of the ring $\mathbf{Z}[i]/ < \beta >$.

Below we state two theorems that were discussed in [2].

Theorem 1.2. *If a group $(G,*)$ is isomorphic to the additive group of a ring $(R,+,.)$, then there is an operation $\otimes$ on G such that $(G,*,\otimes)$ is a ring isomorphic to $(R,+,.)$.*

Theorem 1.3. *If $R \cong R_1 \oplus R_2 \oplus ... \oplus R_i$, then the group of units $U(R)$ and $U(R_1) \times U(R_2) \times ... \times U(R_i)$ are isomorphic.*

In [4], Assaf gave the definition of the second group of units in the domain of integers and listed some related properties.

Throughout this paper,

- n is a positive integer.
- $\alpha = 1+i$.

- β and γ always denote any non- zero, non unit Gaussian integers.
- p and p_j always denote prime integers that are congruent to 3 modulo 4.
- π and π_j always denote Gaussian prime integers of the form $a+bi$ where a and b are non zero integers.
- $q = \pi\overline{\pi} = a^2+b^2$.
- q and q_j always denote prime integers that are congruent to 1 modulo 4.

2. Definition of $\phi_G^2(\beta)$

In this section, we define the new function $\phi_G^2(\beta)$ and prove its multilplivcate property.

Definition 2.1. *Define $\phi_G^2(\beta)$ to be the order of the second group of units $U^2(\mathbf{Z}[i]/<\beta>) = \Phi_G^2(\beta)$. Thus we write, $\phi_G^2(\beta) = |\Phi_G^2(\beta)|$.*

Lemma 2.1. *The function ϕ_G^2 is multiplicative.*

Proof. Let $\gcd(\beta,\gamma)\sim 1$, then $\mathbf{Z}[i]/<\beta\gamma>\cong \mathbf{Z}[i]/<\beta>\oplus \mathbf{Z}[i]/<\gamma>$. Applying theorem 1.3 for $k=2$ and $R=\mathbf{Z}[i]/<\beta\gamma>$ we get,

$U^2(\mathbf{Z}[i]/<\beta\gamma>) \cong U^2(\mathbf{Z}[i]/<\beta>)\times U^2(\mathbf{Z}[i]/<\gamma>)$. Then

$|U^2(\mathbf{Z}[i]/<\beta\gamma>)| = |U^2(\mathbf{Z}[i]/<\beta>)|.|U^2(\mathbf{Z}[i]/<\gamma>)|$. Consequently,

$\phi_G^2(\beta\gamma) = \phi_G^2(\beta)\phi_G^2(\gamma)$. □

The fact the ϕ_G^2 is a multiplicative function gives a very important step in finding a general formula of the order of the second group of units.

Lemma 2.2. *If $\beta=\alpha$ or $\beta=\alpha^2$, then $\phi_G^2(\beta)=1$, else $\phi_G^2(\beta)$ is even.*

Proof. If $\beta=\alpha$ or $\beta=\alpha^2$, then $U^2(\beta)$ is trivial, see [3]. Hence $|U^2(\beta)| = \phi_G^2(\beta) = 1$. Now, suppose that $\beta\neq\alpha$ and $\beta\neq\alpha^2$, then applying the fundemental theorem of abelian groups we get, $U(\beta)\cong \mathbf{Z}_{n_1}\times Z_{n_2}\times ...\times Z_{n_k}$ and in ring structure $U(\beta)\cong \mathbf{Z}_{n_1}\oplus\mathbf{Z}_{n_2}\oplus ...\oplus\mathbf{Z}_{n_k}$. Moreover, $U^2(\beta)\cong U(\mathbf{Z}_{n_1})\times U(\mathbf{Z}_{n_2})\times ...\times U(\mathbf{Z}_{n_k})$. Then $\phi_G^2(\beta) = \phi(n_1)\phi(n_2)...\phi(n_k)$ a product of even integers which is again even. □

3. The Explicit formula for $\phi^2_G(\beta)$

In order to determine an explicit formula for ϕ^2_G, we have to examine the cases when $\beta = \alpha^n$, π^n, and p^n.

Lemma 3.1. *Let n be a positive integer, then*

$$\phi^2_G(\alpha^n) = \begin{cases} 1 & if \ n \le 2 \\ 2 & if \ 3 \le \ n \le 4 \\ 2^{n-4} & if\, n \ge 5. \end{cases}$$

Proof. Let $n \ge 5$, then using theorem 1.1 we have

$$\Phi_G(\alpha^n) \cong \begin{cases} \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_{2^{m-2}} \times \mathbf{Z}_4 \ if \ n = 2m. \\ \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_4 \ if \ n = 2m+1. \end{cases}$$

If $n = 2m$, $\Phi^2_G(\alpha^n) \cong U(\mathbf{Z}_{2^{m-1}}) \times U(\mathbf{Z}_{2^{m-2}}) \times U(\mathbf{Z}_4)$.

Hence,

$$|\Phi^2_G(\alpha^n)| = |U(\mathbf{Z}_{2^{m-1}})|.|U(\mathbf{Z}_{2^{m-2}})|.|U(\mathbf{Z}_4)| = \phi(2^{m-1}).\phi(2^{m-2}).\phi(4).$$

Consequently, $\phi^2_G(\alpha^n) = 2^{n-4}$.

If $n = 2m+1$, then $\Phi^2_G(\alpha^n) \cong U(\mathbf{Z}_{2^{m-1}}) \times U(\mathbf{Z}_{2^{m-1}}) \times U(\mathbf{Z}_4)$. So,

$$|\Phi^2_G(\alpha^n)| = |U(\mathbf{Z}_{2^{m-1}})|.|U(\mathbf{Z}_{2^{m-1}})|.|U(\mathbf{Z}_4)| = \phi(2^{m-1}).\phi(2^{m-1}).\phi(4).$$

Hence, $\phi^2_G(\alpha^n) = 2^{n-4}$.

If $n = 1$ or 2, then $\Phi^2_G(\alpha^n)$ is trivial. Therefore, $\phi^2_G(\alpha^n) = 1$.

If $n = 3$, $\Phi_G(\alpha^n) \cong \mathbf{Z}_4$.Then $\phi^2_G(\alpha^n) = |\Phi^2_G(\alpha^n)| = |U(\mathbf{Z}_4)| = 2$.

If $n = 4$, $\Phi_G(\alpha^n) \cong \mathbf{Z}_2 \times \mathbf{Z}_4$. This gives $\phi^2_G(\alpha^n) = |\Phi^2_G(\alpha^n)| = |U(\mathbf{Z}_2)|.|U(\mathbf{Z}_4)| = 2$.

□

Lemma 3.2. $\phi^2_G(\pi^n) = \phi(\phi_G(\pi^n))$.

Proof. Using theoerm 1.1, we have $\Phi_G(\pi^n) \cong \mathbf{Z}_{q^n-q^{n-1}}$. Then, $\Phi_G^2(\pi^n) \cong U(\mathbf{Z}_{q^n-q^{n-1}})$. Therefore, $|U(\mathbf{Z}_{q^n-q^{n-1}})| = \phi(q^n - q^{n-1}) = \phi(q^{n-1}(q-1))$. Consequently, $\phi_G^2(\pi^n) = \phi(\phi_G(\pi^n))$.

□

Lemma 3.3. $\phi_G^2(p^n) = \begin{cases} \phi(\phi_G(p)), & \text{if } n = 1 \\ (p-1)^2 p^{2n-4}\phi(\phi_G(p^n)), & \text{if } n \geq 2. \end{cases}$

Proof. Theorem 1.1 gives $\Phi_G(p) \cong \mathbf{Z}_{p^2-1}$ when $n = 1$. Thus, $\Phi_G^2(p) \cong U(\mathbf{Z}_{p^2-1})$ and $|\Phi_G^2(p)| = |U(\mathbf{Z}_{p^2-1})|$. Consequently, $\phi_G^2(p) = \phi(p^2-1) = \phi(\phi_G(p))$.

For $n \geq 2$, we have $\Phi_G(p^n) \cong \mathbf{Z}_{p^{n-1}} \times \mathbf{Z}_{p^{n-1}} \times \mathbf{Z}_{p^2-1}$. Thus,

$$\Phi_G^2(p) \cong U(\mathbf{Z}_{p^{n-1}}) \times U(\mathbf{Z}_{p^{n-1}}) \times U(\mathbf{Z}_{p^2-1})$$

and $|\Phi_G^2(p)| = |U(\mathbf{Z}_{p^{n-1}})|.|U(\mathbf{Z}_{p^{n-1}})|.|U(\mathbf{Z}_{p^2-1})|$. Consequently,

$$\phi_G^2(p^n) = \phi(p^{n-1})\phi(p^{n-1})\phi(p^2-1) = (p-1)^2 p^{2n-4}\phi(\phi_G(p)).$$

□

Notation 3.1. *For simplicity, we denote by $f_p(n)$ to be a function defined as*

$$f_p(n) = \begin{cases} 1 & \text{if } n = 1 \\ (p-1)^2 p^{2n-4} & \text{if } n \geq 2. \end{cases}$$

Using this notation, we can write $\phi_G^2(p^n) = f_p(n)\phi(\phi_G(p^n))$.

Following lemmas 3.1, 3.2 and 3.3, we can write an explicit form of the order of the second group of units of the ring of Gaussian integers modulo any non zero and non unit Gaussian integer β given in the following theorem.

Theorem 3.1. *Let $\beta = \alpha^n \left(\prod_{t=1}^{r} p_t^{n_t}\right)\left(\prod_{s=1}^{j} \pi_s^{k_s}\right)$ be the decomposition of β into product of distinct prime powers, with n, n_t, k_s are non negative integers for $1 \leq t \leq r$ and $1 \leq s \leq j$. Then*

$$\phi_G^2(\beta) = \begin{cases} \left(\prod_{t=1}^{r} f_{p_t}(n_t)\phi(\phi_G(p_t^{n_t}))\right)\left(\prod_{s=1}^{j} \phi(\phi_G(\pi_s^{n_s}))\right) & \text{if } n \leq 2 \\ 2\left(\prod_{t=1}^{r} f_{p_t}(n_t)\phi(\phi_G(p_t^{n_t}))\right)\left(\prod_{s=1}^{j} \phi(\phi_G(\pi_s^{k_s}))\right) & \text{if } 3 \leq n \leq 4 \\ 2^{n-4}\left(\prod_{t=1}^{r} f_{p_t}(n_t)\phi(\phi_G(p_t^{n_t}))\right)\left(\prod_{s=1}^{j} \phi(\phi_G(\pi_s^{k_s}))\right) & \text{if } n \geq 5. \end{cases}$$

Proof. Using the multiplicative property of ϕ_G^2, we can write

$$\phi_G^2(\beta) = \phi_G^2(\alpha^n)\left(\prod_{t=1}^{r}\phi_G^2\left(p_t^{n_t}\right)\right)\left(\prod_{s=1}^{j}\phi_G^2\left(\pi_s^{k_s}\right)\right).$$

Applying lemmas 3.1, 3.2 and 3.3, we deduce the explicit form stated for $\phi_G^2(\beta)$.

□

Example 3.1. *Let* $\beta = 3.7^2(1+i)^3(1+2i)^2$, *then* $\phi_G^2(\beta) = \phi_G^2(3.5^2(1+i)^3(1+2i)^2) = \phi_G^2((1+i)^3)\phi_G^2(3)\phi_G^2(5^2)\phi_G^2((1+2i)^2)$. *Hence,* $\phi_G^2(\beta) = 2\phi(9-1)(7-1)^2 7^{2(2)-4}\phi(49-1)\phi(5(5-1)) = 36864$.

4. **Properties for $\phi_G^2(\beta)$**

It is well known that $\phi_G(\gamma)|\phi_G(\beta)$ whenever $\gamma|\beta$. We will prove this property for ϕ_G^2.

Lemma 4.1. *Let* $\beta = \alpha^n$ *and let* $\gamma|\beta$, *then* $\phi_G^2(\gamma)|\phi_G^2(\beta)$.

Proof. Since $\gamma|\beta$, then $\gamma = \alpha^k$, where $1 \le k \le n$. Now, consider all possible cases of n. Refering to lemma 3.1, we have the following three cases. If $n \le 2$, then $\phi_G^2(\beta) = \phi_G^2(\gamma) = 1$. Consequently, $\phi_G^2(\gamma)|\phi_G^2(\beta)$. If $n = 3$ or $n = 4$, then $\phi_G^2(\beta) = 2$ and $\phi_G^2(\gamma) = 1$ or $\phi_G^2(\gamma) = 2$. Hence, $\phi_G^2(\gamma)|\phi_G^2(\beta)$. If $n \ge 5$, then $\phi_G^2(\beta) = 2^{n-4}$. Since $1 \le k \le n$, then we have $k < 4$ or $k \ge 5$. If $k < 4$, then $\phi_G^2(\gamma) = 1$ or $\phi_G^2(\gamma) = 2$, thus $\phi_G^2(\gamma)|\phi_G^2(\beta)$. If $k \ge 5$, then $\phi_G^2(\gamma) = 2^{k-4}$. Since $k \le n$, then $2^{k-4}|2^{n-4}$ and we get $\phi_G^2(\gamma)|\phi_G^2(\beta)$. □

Lemma 4.2. *Let* $\beta = \pi^n$ *and let* $\gamma|\beta$, *then* $\phi_G^2(\gamma)|\phi_G^2(\beta)$.

Proof. Since $\gamma|\beta$, then $\gamma = \pi^k$, where $1 \le k \le n$. Lemma 3.2 gives $\phi_G^2(\beta) = \phi(q^{n-1}(q-1))$ and $\phi_G^2(\gamma) = \phi(q^{k-1}(q-1))$. Since $k \le n$, then $q^{k-1}(q-1)|q^{n-1}(q-1)$. And $\phi(m)|\phi(n)$ whenever $m|n$, we conclude that $\phi_G^2(\gamma)|\phi_G^2(\beta)$. □

Lemma 4.3. *Let* $\beta = p^n$ *and let* $\gamma|\beta$, *then* $\phi_G^2(\gamma)|\phi_G^2(\beta)$.

Proof. Since $\gamma|\beta$, then we write $\gamma = p^k$, where $1 \le k \le n$. If $n = 1$, then $k = n = 1$ and then $\phi_G^2(\gamma) = \phi_G^2(\beta)$. If $n > 1$, then $\phi_G^2(\beta) = (p-1)^2 p^{2n-4}\phi(\phi_G(p))$. Since $k \le n$, then we have

$k=1$ or $k>1$. If $k=1$, then $\phi_G^2(\gamma)=\phi(\phi_G(p))$ and in this case $\phi_G^2(\gamma)|\phi_G^2(\beta)$. If $k>1$, then $\phi_G^2(\ \gamma)=(p-1)^2p^{2k-4}\phi(\phi_G(p))$. But $p^{2k-4}|p^{2n-4}$ for $k\le n$, then $\phi_G^2(\gamma)|\phi_G^2(\beta)$. □

Theorem 4.1. *If $\gamma|\beta$, then $\phi_G^2(\gamma)|\phi_G^2(\beta)$.*

Proof. Let $\beta=\alpha^n\left(\prod_{t=1}^{r}p_t^{n_t}\right)\left(\prod_{s=1}^{j}\pi_s^{k_s}\right)$ be the decomposition of β into product of distinct prime powers, with n, n_t and k_s are non negative integers for $1\le t\le r$ and $1\le s\le j$. As $\gamma|\beta$, we write $\gamma=\alpha^{n'}\left(\prod_{t=1}^{r}p_t^{n'_t}\right)\left(\prod_{s=1}^{j}\pi_s^{k'_s}\right)$ with $0\le n'\le n$, $0\le n'_t\le n_t$ and $0\le k'_s\le k_s$. Since ϕ_G^2 is multiplicative, then

$$\phi_G^2(\beta)=\phi_G^2(\alpha^n)\left(\prod_{t=1}^{r}\phi_G^2\left(p_t^{n_t}\right)\right)\left(\prod_{s=1}^{j}\phi_G^2\left(\pi_s^{k_s}\right)\right)$$

and $\phi_G^2(\gamma)=\phi_G^2(\alpha^{n'})\left(\prod_{t=1}^{r}\phi_G^2\left(p_t^{n'_t}\right)\right)\left(\prod_{s=1}^{j}\phi_G^2\left(\pi_s^{k'_s}\right)\right)$. Using lemmas 4.1, 4.2 and 4.3, we get $\phi_G^2(\alpha^{n'})|\phi_G^2(\alpha^n)$, $\phi_G^2(\pi_s^{k'_s})|\ \phi_G^2(\pi_s^{k_s})$ and $\phi_G^2(p_t^{n'_t})|\phi_G^2(p_t^{n_t})$. Hence the result. □

Note that for a non trivial ring R with identity, it is always true that $|U(R)|<|R|$ and consequently $|U^2(R)|<|U(R)|$. In the next proposition, we aim to find a least upper bound for $\phi_G^2(\beta)$.

Proposition 4.1. $\phi_G^2(\beta)\le\dfrac{\phi_G(\beta)}{2}$ *if $\beta\neq\alpha$.*

Proof. We consider the three cases according to the three different types of the Gaussian primes, then we use the multiplicative property to complete the proof for any non zero, non unit Gaussian integer β.

First, we consider the case where $\beta=\alpha^n$.

If $n=2$, then $\phi_G^2(\alpha^2)=1$. But $\phi_G(\ \alpha^2)=2$, Consequently, $\phi_G^2(\alpha^2)=\dfrac{\phi_G(\alpha^2)}{2}$.

If $n=3$, then $\phi_G^2(\alpha^3)=2$ and $\phi_G(\alpha^3)=4$. Again $\phi_G^2(\alpha^3)=\dfrac{\phi_G(\alpha^3)}{2}$.

If $n=4$, then $\phi_G^2(\alpha^4)=2$ and $\phi_G(\alpha^4)=8$. Hence, $\phi_G^2(\alpha^4)<\dfrac{\phi_G(\alpha^4)}{2}$.

If $n\ge5$, then $\phi_G^2(\alpha^n)=2^{n-4}$ and $\phi_G(\alpha^n)=2^{n-1}$. Hence, $\phi_G^2(\ \alpha^n)<\dfrac{\phi_G(\alpha^n)}{2}$. Consequently, $\phi_G^2(\ \alpha^n)\le\dfrac{\phi_G(\alpha^n)}{2}$ for $n\ge2$.

Next, consider the case when $\beta=p^n$.

If $n = 1$, then $\phi_G^2(p) = \phi(\phi_G(p)) = \phi(p^2-1)$. Since p^2-1 is even, then $\phi(p^2-1) \leq \frac{p^2-1}{2}$ and $\phi_G^2(p) = \phi(p^2-1) \leq \frac{p^2-1}{2} = \frac{\phi_G(p)}{2}$.

If $n > 1$, then $\phi_G^2(\ p^n) = (p-1)^2 p^{2n-4} \phi(\phi_G(p)) < p^{2n-2}\phi(p^2-1)$. Then $\phi_G^2(\ p^n) < \frac{p^{2n-2}(p^2-1)}{2} = \frac{\phi_G(p^n)}{2}$. Consequently, $\phi_G^2(\ p^n) \leq \frac{\phi_G(p^n)}{2}$.

Now, consider the case when $\beta = \pi^n$.

Then $\phi_G^2(\pi^n) = \phi(q^{n-1}(q-1))$. Since $q^{n-1}(q-1)$ is even, then $\phi(q^{n-1}(q-1)) \leq \frac{q^{n-1}(q-1)}{2} = \frac{\phi_G(\pi^n)}{2}$. Therefore, $\phi_G^2(\pi^n) \leq \frac{\phi_G(\pi^n)}{2}$.

Finally, let $\beta = \alpha^n \left(\prod_{t=1}^{r} p_t^{n_t}\right)\left(\prod_{s=1}^{j} \pi_s^{k_s}\right)$ be the decomposition of β into product of distinct prime powers, with n, n_t, k_s are non negative integers for $1 \leq t \leq r$ and $1 \leq s \leq j$. Then,

$$\phi_G^2(\beta) = \phi_G^2(\alpha^n)\left(\prod_{t=1}^{r}\phi_G^2\left(p_t^{n_t}\right)\right)\left(\prod_{s=1}^{j}\phi_G^2\left(\pi_s^{k_s}\right)\right).$$

Using the above results, we can write

$$\begin{aligned} & \phi_G^2(\alpha^n)\left(\prod_{t=1}^{r}\phi_G^2\left(p_t^{n_t}\right)\right)\left(\prod_{s=1}^{j}\phi_G^2\left(\pi_s^{k_s}\right)\right) \\ \leq & \frac{\phi_G(\alpha^n)\left(\prod_{t=1}^{r}\phi_G\left(p_t^{n_t}\right)\right)\left(\prod_{s=1}^{j}\phi_G\left(\pi_s^{k_s}\right)\right)}{2^{i+j+1}} \\ < & \frac{\phi_G(\alpha^n)\left(\prod_{t=1}^{r}\phi_G\left(p_t^{n_t}\right)\right)\left(\prod_{s=1}^{j}\phi_G\left(\pi_s^{k_s}\right)\right)}{2}. \end{aligned}$$

Consequently, $\phi_G^2(\ \beta) \leq \frac{\phi_G(\beta)}{2}$.

□

5. Conclusion

Using the decomposition of the generalized second group of units of a quotient ring of Gaussian integers $R = \mathbf{Z}[i]/ < \beta >$, we were able to generalize the Euler Phi function. The new generalization, $\phi_G^2(\beta)$, represents the order of the generalized second group of units of the ring R, $\Phi_G^2(\beta)$. An explicit formula and a least upper bound for $\phi_G^2(\beta)$ were given.

Conflict of Interests

The author declares that there is no conflict of interests.

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