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NUMERICAL SOLUTIONS OF SECOND ORDER MATRIX DIFFERENTIAL EQUATIONS USING BASIS SPLINES

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Abstract. This paper aims to present a general framework of the cubic, quintic and septic B-splines functions to develop a numerical method for obtaining approximation solution numerical solution of the matrix differential equations of second order with boundary conditions. Numerical examples are included to illustrate the practical implementation of the proposed method. The results reveal that the proposed approach is very effective, convenient and quite accurate to such considered problems compared with cubic splines with constant term method.

Keywords: matrix differential equations; cubic b-splines; quintic b-splines; septic b-splines; Kronecker product; Frobenius norm.

2010 AMS Subject Classification: 35A24.

Introduction

Given the matrix boundary value problem

$$\left.\begin{aligned} Y''(x) &= f\left(x, Y(x), Y'(x)\right) \\ Y(a) &= Y_a, Y'(a) = Y_b \end{aligned}\right\}, a \le x \le b,\ [a,\ b] \subset \mathrm{R} \tag{1}$$

where $Y_a, Y_b, Y(x) \in C^{m\times n}$ and matrix function $f:[a,b]\times C^{m\times n}\times C^{m\times n}\to C^{m\times n}$, are frequent in different fields in physics and engineering. Equation (1) is similar to the statement of Newton's law of motion for coupled mechanical system. Models of this kind are frequently appear in molecular dynamics, quantum mechanics and for scattering methods, where one solves scalar or vectorial

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problems with boundary value conditions [1- 6]. We define the Kronecker product of $A \in C^{m\times n}$ and $B \in C^{p\times q}$, denoted by $A \otimes B$ [7]

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \tag{2}$$

The column vector operator on a matrix $A \in C^{m\times n}$ is given by [7]:

$$\text{Vec}(A) = \begin{bmatrix} A_{\bullet 1} \\ \vdots \\ A_{\bullet n} \end{bmatrix}, \quad \text{where } A_{\bullet k} = \begin{bmatrix} A_{1k} \\ \vdots \\ A_{mk} \end{bmatrix} \tag{3}$$

If $Y \in C^{m\times n}$ and $X \in C^{p\times q}$, then the derivative of a matrix with respect to a matrix is defined by [7]:

$$\frac{\partial Y}{\partial X} = \begin{pmatrix} \frac{\partial Y}{\partial x_{11}} & \cdots & \frac{\partial Y}{\partial x_{1q}} \\ \vdots & \ddots & \vdots \\ \frac{\partial Y}{\partial x_{p1}} & \cdots & \frac{\partial Y}{\partial x_{pq}} \end{pmatrix}, \text{ where } \frac{\partial Y}{\partial x_{rs}} = \begin{pmatrix} \frac{\partial Y_{11}}{\partial x_{rs}} & \cdots & \frac{\partial Y_{1n}}{\partial x_{rs}} \\ \vdots & \ddots & \vdots \\ \frac{\partial Y_{m1}}{\partial x_{rs}} & \cdots & \frac{\partial Y_{mn}}{\partial x_{rs}} \end{pmatrix} \tag{4}$$

If $X \in C^{p\times q}$, $Y \in C^{q\times v}$ and $Z \in C^{m\times n}$, then the derivative of a matrix product with respect to another matrix is given by [7]:

$$\frac{\partial XY}{\partial Z} = \frac{\partial X}{\partial Z}\left[I_n \otimes Y\right] + \left[I_m \otimes X\right]\frac{\partial Y}{\partial Z}, \tag{5}$$

where I_m and I_n denote the identity matrices of dimensions m and n, respectively.

If $X \in C^{p\times q}$, $Y \in C^{u\times v}$ and $Z \in C^{m\times n}$, then the chain rule is defined by [7]:

$$\frac{\partial Z}{\partial X} = \left[\frac{\partial\left[Vec(Y)\right]^T}{\partial X} \otimes I_m\right]\left[I_q \otimes \frac{\partial Z}{\partial\left[Vec(Y)\right]}\right] \tag{6}$$

and the derivative of a Kronecker product of matrices with respect to a matrix is given by [7]:

$$\frac{\partial\left(X \otimes Y\right)}{\partial Z} = \frac{\partial X}{\partial Z} \otimes Y + \left[I_m \otimes U_1\right]\left[\frac{\partial Y}{\partial Z} \otimes X\right]\left[I_n \otimes U_2\right] \tag{7}$$

where U_1 and U_2 are permutation matrices.

If $A \in C^{m\times n}$, the frobenius norm of A is given by [8]:

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2} \tag{8}$$

The following relationship between the 2-norm and frobenius norm holds [8]:

$$\|A\|_2 \le \|A\|_F \le \sqrt{n}\,\|A\|_2 \tag{9}$$

Cubic B - splines are used in [9-13], matrix differential equations are discussed in [14-16] and B-splines are presented in [17]. The paper is organized as follows: In section 2, we developed the proposed method. In section 3, some numerical examples were discussed. Finally, in section 4, we gave summary of the suggested method.

2. Analysis of B-splines method

Let $x_0, x_1, ..., x_N$ be $(N+1)$ grid points in the interval $[a, b]$, so that $x_i = a + ih,\ i = 0,\ 1,\ ...,\ n;\ x_0 = a,\ x_N = b,\ h = (b-a)/N$. Then B-splines are presented as follows:

2.1 Cubic B-splines

The cubic B-splines are

$$B_i(x) = \frac{1}{h^3}\begin{cases} (x - x_{i-2})^3 & [x_{i-2}, x_{i-1}], \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3 & [x_{i-1}, x_i], \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3 & [x_i, x_{i+1}], \\ (x_{i+2} - x)^3 & [x_{i+1}, x_{i+2}], \\ 0 & \text{elsewhere.} \end{cases} \tag{10}$$

$$(i = -1, 0, 1, ..., n+1).$$

We consider the B-spline function to the solutions $\overset{pq}{y}(x)$ of the problem (1):

$$\overset{pq}{y}(x) = \sum_{i=-1}^{N+1} \overset{pq}{C_i}(x) \overset{pq}{B_i}(x); \quad 1 \le p \le n,\ 1 \le q \le m \tag{11}$$

where constants $\overset{pq}{C_i}(x)$'s are to be determined. To solve second order matrix boundary value problems, the B_i, B'_i and B''_i at the nodal points are needed. Their coefficients are summarized in Table 1.

Table 1. values of B_i, B_i' and B_i''.

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
B_i	0	1	4	1	0
B_i'	0	$-3/h$	0	$3/h$	0
B_i''	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

By substituting (11) in (1), we find

$$\sum_{i=-1}^{N+1} \overset{pq}{C_i}(x)\overset{pq}{B_i''}(x) = f\left(x,\ \sum_{i=-1}^{N+1} \overset{pq}{C_i}(x)\overset{pq}{B_i}(x),\ \sum_{i=-1}^{N+1} \overset{pq}{C_i}(x)\overset{pq}{B_i'}(x)\right), \tag{12}$$

and boundary conditions can be written as

$$\begin{aligned} &\sum_{i=-1}^{N+1} \overset{pq}{C_i}(x)\overset{pq}{B_i}(x) = \overset{pq}{y}_a; \quad x = a, \\ &\sum_{i=-1}^{N+1} \overset{pq}{C_i}(x)\overset{pq}{B_i}(x) = \overset{pq}{y}_b; \quad x = b. \end{aligned} \tag{13}$$

The spline solution of equation (1) is obtained by solving the following matrix equation. The value of the spline functions at the points $\{x_i\}_{i=0}^{N}$ are determined using Table 1 and substitute into equations (12) and (13). Then a system $q(N+3)\times q(N+3);\ 1 \le q \le m$ of linear equations can be written as follows

$$AE = F. \tag{14}$$

Where,

$$E = \left[\overset{11}{C}_{-1},\ \overset{11}{C}_{0},\ \ldots,\ \overset{11}{C}_{N+1},\ \ldots,\ \overset{nm}{C}_{-1},\ \overset{nm}{C}_{0},\ \ldots,\ \overset{nm}{C}_{N+1}\right]^T$$

$$F = \left[\overset{11}{y}_a,\ \overset{11}{f}(x_0),\ \ldots,\ \overset{11}{f}(x_N),\ \overset{11}{y}_b,\ \ldots,\ \overset{nm}{y}_a,\ \overset{nm}{f}(x_0),\ \ldots,\ \overset{nm}{f}(x_N),\ \overset{nm}{y}_b\right]^T$$

2.2 Quintic B-splines

The quintic B-splines are

$$B_i(x) = \frac{1}{h^5}\begin{cases} (x - x_{i-3})^5 & [x_{i-3}, x_{i-2}], \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 & [x_{i-2}, x_{i-1}], \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5 & [x_{i-1}, x_i], \\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5 + 15(x_{i+1} - x)^5 & [x_i, x_{i+1}], \\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5 & [x_{i+1}, x_{i+2}], \\ (x_{i+3} - x)^5 & [x_{i+2}, x_{i+3}], \\ 0 & \text{elsewhere}, \end{cases} \tag{15}$$

$$(i = -1, 0, 1, \ldots, n+1).$$

Let
$$\overset{pq}{y}(x) = \sum_{i=-2}^{N+2} \overset{pq}{C_i}(x)\overset{pq}{B_i}(x); \quad 1 \le p \le n,\ 1 \le q \le m \tag{16}$$

be the B-spline function to the solutions $\overset{pq}{y}(x)$ of the problem (1), where constants $\overset{pq}{C_i}(x)$'s are to be determined. To solve second order matrix boundary value problems, the B_i, B'_i and B''_i at the nodal points are needed. Their coefficients are summarized in Table 2.

Table 2. values of B_i, B_i' and B_i''.

	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
B_i	0	1	26	66	26	1	0
B_i'	0	$-5/h$	$-50/h$	0	$50/h$	$5/h$	0
B_i''	0	$20/h^2$	$40/h^2$	$-120/h^2$	$40/h^2$	$20/h^2$	0

By substituting (16) in (1), we find

$$\sum_{i=-2}^{N+2}\overset{pq}{C_i}(x)\overset{pq}{B_i''}(x)=f\left(x,\ \sum_{i=-2}^{N+2}\overset{pq}{C_i}(x)\overset{pq}{B_i}(x),\ \sum_{i=-2}^{N+2}\overset{pq}{C_i}(x)\overset{pq}{B_i'}(x)\right), \tag{17}$$

, boundary conditions can be written as

$$\begin{aligned}&\sum_{i=-2}^{N+2}\overset{pq}{C_i}(x)\overset{pq}{B_i}(x)=\overset{pq}{y}_a;\quad x=a,\\&\sum_{i=-2}^{N+2}\overset{pq}{C_i}(x)\overset{pq}{B_i}(x)=\overset{pq}{y}_b;\quad x=b.\end{aligned} \tag{18}$$

and we need an extra conditions:

$$\begin{aligned}&\sum_{i=-2}^{N+2}\overset{pq}{C_i}(x)\overset{pq}{B_i'}(x)=\overset{pq}{y'}_a;\quad x=a,\\&\sum_{i=-2}^{N+2}\overset{pq}{C_i}(x)\overset{pq}{B_i'}(x)=\overset{pq}{y'}_b;\quad x=b.\end{aligned} \tag{19}$$

The spline solution of equation (1) is obtained by solving the following matrix equation. The value of the spline functions at the points $\{x_i\}_{i=0}^{N}$ are determined using Table 2 and substitute into equations (17 - 19). Then a system $q(N+5)\times q(N+5);\ 1\le q\le m$ of linear equations can be written as follows

$$AE=F. \tag{20}$$

Where,

$$E=\left[\overset{11}{C}_{-2},\ \overset{11}{C}_{-1},\ \ldots,\ \overset{11}{C}_{N+2},\ \ldots,\ \overset{nm}{C}_{-2},\ \overset{nm}{C}_{-1},\ \ldots,\ \overset{nm}{C}_{N+2}\right]^T$$

$$F=\left[\overset{11}{y}_a,\ \overset{11}{y'}_a,\ \overset{11}{f}(x_0),\ \ldots,\overset{11}{f}(x_N),\ \overset{11}{y}_b,\ \overset{11}{y'}_b,\ \ldots,\ \overset{nm}{y}_a,\ \overset{nm}{y'}_a,\ \overset{nm}{f}(x_0),\ \ldots,\ \overset{nm}{f}(x_N),\ \overset{nm}{y}_b,\ \overset{nm}{y'}_b\right]^T$$

2.3 Septic B-splines

The septic B-splines are

$$B_i(x)=\frac{1}{h^7}\begin{cases}(x-x_{i-4})^7 & [x_{i-4},x_{i-3}],\\ (x-x_{i-4})^7-8(x-x_{i-3})^7 & [x_{i-3},x_{i-2}],\\ (x-x_{i-4})^7-8(x-x_{i-3})^7+28(x-x_{i-2})^7 & [x_{i-2},x_{i-1}],\\ (x-x_{i-4})^7-8(x-x_{i-3})^7+28(x-x_{i-2})^7-56(x-x_{i-1})^7 & [x_{i-1},x_i],\\ (x_{i+4}-x)^7-8(x_{i+3}-x)^7+28(x_{i+2}-x)^7-56(x_{i+1}-x)^7 & [x_i,x_{i+1}],\\ (x_{i+4}-x)^7-8(x_{i+3}-x)^7+28(x_{i+2}-x)^7 & [x_{i+1},x_{i+2}],\\ (x_{i+4}-x)^7-8(x_{i+3}-x)^7 & [x_{i+2},x_{i+3}],\\ (x_{i+4}-x)^7 & [x_{i+3},x_{i+4}],\\ 0 & \text{elsewhere,}\end{cases} \tag{21}$$

$$(i=-1,0,1,\ldots,n+1).$$

The B-spline function to the solutions $\overset{pq}{y}(x)$ of the problem (1) is given as

$$\overset{pq}{y}(x)=\sum_{i=-3}^{N+3}\overset{pq}{C_i}(x)\overset{pq}{B_i}(x);\quad 1\le p\le n,\ 1\le q\le m \tag{22}$$

where constants $\overset{pq}{C_i}(x)$'s are to be determined. In order to solve second order matrix boundary value problems, the B_i, B'_i and B''_i at the nodal points are needed. Their coefficients are summarized in Table 3.

Table 3. values of B_i, B_i' and B_i''.

	x_{i-4}	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
B_i	0	1	120	1191	2416	1191	120	1	0
B_i'	0	$-7/h$	$-392/h$	$-1715/h$	0	$1715/h$	$392/h$	$7/h$	0
B_i''	0	$42/h^2$	$1008/h^2$	$630/h^2$	$-3360/h^2$	$630/h^2$	$1008/h^2$	$42/h^2$	0
B_i'''	0	$-210/h^3$	$-1680/h^3$	$3990/h^3$	0	$-3990/h^3$	$1680/h^3$	$210/h^3$	0

By substituting (22) in (1), we find

$$\sum_{i=-3}^{N+3} \overset{pq}{C_i}(x)\overset{pq}{B_i''}(x) = f\left(x,\ \sum_{i=-3}^{N+3} \overset{pq}{C_i}(x)\overset{pq}{B_i}(x),\ \sum_{i=-3}^{N+3} \overset{pq}{C_i}(x)\overset{pq}{B_i'}(x) \right), \tag{23}$$

, boundary conditions can be written as

$$\begin{aligned} \sum_{i=-3}^{N+3} \overset{pq}{C_i}(x)\overset{pq}{B_i}(x) &= \overset{pq}{y}_a;\quad x=a, \\ \sum_{i=-3}^{N+3} \overset{pq}{C_i}(x)\overset{pq}{B_i}(x) &= \overset{pq}{y}_b;\quad x=b. \end{aligned} \tag{24}$$

and we need an extra conditions:

$$\begin{aligned} \sum_{i=-3}^{N+3} \overset{pq}{C_i}(x)\overset{pq}{B_i'}(x) &= \overset{pq}{y'}_a;\quad x=a, \\ \sum_{i=-3}^{N+3} \overset{pq}{C_i}(x)\overset{pq}{B_i'}(x) &= \overset{pq}{y'}_b;\quad x=b. \end{aligned} \tag{25}$$

$$\begin{aligned} \sum_{i=-3}^{N+3} \overset{pq}{C_i}(x)\overset{pq}{B_i'''}(x) &= \overset{pq}{y'''}_a;\quad x=a, \\ \sum_{i=-3}^{N+3} \overset{pq}{C_i}(x)\overset{pq}{B_i'''}(x) &= \overset{pq}{y'''}_b;\quad x=b. \end{aligned} \tag{26}$$

The spline solution of equation (1) is obtained by solving the following matrix equation. The value of the spline functions at the points $\{x_i\}_{i=0}^{N}$ are determined using Table 3 and substitute into equations (23 - 26). Then a system $q(N+7)\times q(N+7);\ 1\le q\le m$ of linear equations can be written as follows

$$AE = F. \tag{27}$$

Where,

$$E = \left[\overset{11}{C}_{-3},\ \overset{11}{C}_{-2},\ \ldots,\ \overset{11}{C}_{N+3},\ \ldots,\ \overset{nm}{C}_{-3},\ \overset{nm}{C}_{-2},\ \ldots,\ \overset{nm}{C}_{N+3} \right]^T$$

$$F = \left[\overset{11}{y}_a,\ \overset{11}{y'}_a,\ \overset{11}{y'''}_a,\ \overset{11}{f}(x_0),\ \ldots, \overset{11}{f}(x_N),\ \overset{11}{y}_b,\ \overset{11}{y'}_b,\ \overset{11}{y'''}_b,\ \ldots,\ \overset{nm}{y}_a,\ \overset{nm}{y'}_a,\ \overset{nm}{y'''}_a,\ \overset{nm}{f}(x_0),\ \ldots,\ \overset{nm}{f}(x_N),\ \overset{nm}{y}_b,\ \overset{nm}{y'}_b,\ \overset{nm}{y'''}_b \right]^T$$

3. Numerical examples

In this section, we present some examples of matrix differential equations of second order. We take $h = 0.1$ on the interval $[0,\ 1]$ and the results are generated with Mathematica using Find Root function to solve the emerging algebraic equations. At each point, we evaluated the difference between approximate solution and exact solution, and then take the Frobenius norm of this difference.

Example 1. A non-linear differential vector system [16]

Let

$$Y''(x)=\begin{pmatrix} 1-\cos(x)+\sin(y'_{21}(x))+\cos(y'_{21}(x)) \\ \dfrac{1}{4+(y_{11}(x))^2}-\dfrac{1}{5-(\sin(x))^2} \end{pmatrix},\ 0\le x\le 1. \tag{28}$$

This example has an exact solution $Y(x)=\begin{pmatrix}\cos(x)\\ \pi x\end{pmatrix}$. Thus, we can compare our numerical estimates with this solution to obtain the exact errors of the approximation which summarized in Table 4.

Table 4. Approximation for Example 1.

x	Cubic B-splines errors	Quintic B-splines errors	Septic B-splines errors	Cubic splines errors [16]
0	0	0	1.96262×10^{-17}	0
0.1	3.41618×10^{-5}	3.85149×10^{-9}	1.20717×10^{-12}	4.16114×10^{-6}
0.2	6.00216×10^{-5}	8.70339×10^{-9}	2.54358×10^{-12}	1.66032×10^{-5}
0.3	7.77071×10^{-5}	1.15496×10^{-8}	3.10449×10^{-12}	3.72028×10^{-5}
0.4	8.74294×10^{-5}	1.32375×10^{-8}	3.55659×10^{-12}	6.57658×10^{-5}
0.5	8.94801×10^{-5}	1.35948×10^{-8}	3.60134×10^{-12}	1.02012×10^{-4}
0.6	8.42278×10^{-5}	1.27541×10^{-8}	3.42381×10^{-12}	1.45563×10^{-4}
0.7	7.21139×10^{-5}	1.07256×10^{-8}	2.88345×10^{-12}	1.96167×10^{-4}
0.8	5.36475×10^{-5}	7.77746×10^{-9}	2.26503×10^{-12}	2.52912×10^{-4}
0.9	2.94003×10^{-5}	3.33570×10^{-9}	1.04252×10^{-12}	3.15643×10^{-4}
1.0	0	4.44089×10^{-16}	4.57757×10^{-16}	3.83638×10^{-4}

Example 2. Incomplete second - order differential system [16]

The problem

$$Y''(x)+AY(x)=0,\ 0\le x\le 1. \tag{29}$$

Where $A=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ and corresponding exact solution $Y(x)=\begin{pmatrix}\sin(x) & 0\\ x\cos(x) & \sin(x)\end{pmatrix}$. Thus, we can find the exact errors of the approximation which summarized in Table 5.

Table 5. Approximation for Example 2.

x	Cubic B-splines errors	Quintic B-splines errors	Septic B-splines errors	Cubic splines errors [16]
0	1.20185×10^{-17}	5.48450×10^{-15}	4.01297×10^{-15}	0
0.1	7.86131×10^{-5}	1.21289×10^{-8}	4.88643×10^{-12}	1.0072×10^{-6}
0.2	1.51856×10^{-4}	2.96468×10^{-8}	1.11081×10^{-11}	6.3032×10^{-6}
0.3	2.14489×10^{-4}	4.30700×10^{-8}	1.49671×10^{-11}	2.0059×10^{-5}
0.4	2.61535×10^{-4}	5.34986×10^{-8}	1.85289×10^{-11}	4.6213×10^{-5}
0.5	2.88399×10^{-4}	5.92703×10^{-8}	2.02749×10^{-11}	8.8359×10^{-5}
0.6	2.90988×10^{-4}	5.96807×10^{-8}	2.07159×10^{-11}	1.4964×10^{-4}
0.7	2.65820×10^{-4}	5.35877×10^{-8}	1.86006×10^{-11}	2.3267×10^{-4}
0.8	2.10116×10^{-4}	4.14174×10^{-8}	1.56682×10^{-11}	3.3941×10^{-4}
0.9	1.21887×10^{-4}	1.86540×10^{-8}	7.56441×10^{-12}	4.7114×10^{-4}
1.0	2.11526×10^{-15}	1.57009×10^{-16}	1.11022×10^{-16}	6.2838×10^{-4}

Example 3. Second - order polynomial matrix equation [16]

We consider the following problem

$$Y''(x)+A_0Y'(x)+A_1Y(x)=0,\ 0\le x\le 1. \tag{30}$$

where $A_0=\begin{pmatrix}-1 & 1\\ 0 & -2\end{pmatrix}$, $A_1=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ and the exact solution $Y(x)=\begin{pmatrix}e^x & -1+e^x-xe^x\\ 0 & e^x\end{pmatrix}$. Thus, we summarized the exact errors at each point in Table 6.

Table 6. Approximation for Example 3.

x	Cubic B-splines errors	Quintic B-splines errors	Septic B-splines errors	Cubic splines errors [16]
0	2.22045×10^{-16}	3.45840×10^{-15}	4.73069×10^{-15}	0
0.1	1.61317×10^{-4}	2.77917×10^{-8}	2.50356×10^{-12}	1.53895×10^{-5}
0.2	3.11539×10^{-4}	6.85535×10^{-8}	2.67982×10^{-11}	6.67523×10^{-5}
0.3	4.44650×10^{-4}	1.00132×10^{-7}	3.61804×10^{-11}	1.63924×10^{-4}
0.4	5.53169×10^{-4}	1.26114×10^{-7}	4.54015×10^{-11}	3.18789×10^{-4}
0.5	6.27861×10^{-4}	1.42833×10^{-7}	5.06466×10^{-11}	5.45654×10^{-4}
0.6	6.57397×10^{-4}	1.48247×10^{-7}	5.33134×10^{-11}	8.61682×10^{-4}
0.7	6.27962×10^{-4}	1.38157×10^{-7}	4.95015×10^{-11}	1.28740×10^{-3}
0.8	5.22786×10^{-4}	1.12056×10^{-7}	4.38239×10^{-11}	1.84731×10^{-3}
0.9	3.21599×10^{-4}	5.34319×10^{-8}	2.26900×10^{-11}	2.57055×10^{-3}
1.0	5.36709×10^{-15}	1.35064×10^{-15}	1.47288×10^{-15}	3.49171×10^{-3}

4. Conclusion

In this paper, we presented a numerical treatment for the second-order matrix differential equations using B-spline functions of different types. The computational results are found to be in good agreement with the exact solutions by finding frobenius norm and are compared with Ref. [16] as shown in Tables 4, 5 and 6.

Conflict of Interests

The authors declare that there is no conflict of interests.

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