σ-CONVERGENT DIFFERENCE SEQUENCE SPACES OF SECOND ORDER DEFINED BY ORLICZ FUNCTION

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Abstract. In this paper, we introduce the sequence space $V_\sigma(M, p, r, \triangle^2)$, where $M$ is an Orlicz function, $p = (p_m)$ is any sequence of strictly positive real numbers and $r \geq 0$ and study some of the properties and inclusion relations that arise on the said space.

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1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively.

We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \ or \ \mathbb{C}\},$$

the space of all real or complex sequences.

Let $\ell_\infty$, $c$ and $c_0$ denote the Banach spaces of bounded, convergent and null sequences respectively.

The following subspaces of $\omega$ were first introduced and discussed by Maddox [12-13].

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$$\ell(p) = \{ x \in \omega : \sum_k |x_k|^{p_k} < \infty \},$$

$$\ell_\infty(p) = \{ x \in \omega : \sup_k |x_k|^{p_k} < \infty \},$$

$$c(p) = \{ x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \text{ for some } l \in C \},$$

$$c_0(p) = \{ x \in \omega : \lim_k |x_k|^{p_k} = 0 \},$$

where \( p = (p_k) \) is a sequence of strictly positive real numbers.

The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value. (see [13])

Let \( X \) be a linear space. A function \( g : X \rightarrow R \) is called paranorm, if for all \( x, y, z \in X \),

(P1) \( g(x) = 0 \) if \( x = \theta \),

(P2) \( g(-x) = g(x) \),

(P3) \( g(x+y) \leq g(x) + g(y) \),

(P4) If \( (\lambda_n) \) is a sequence of scalars with \( \lambda_n \rightarrow \lambda \) \( (n \rightarrow \infty) \) and \( x_n, a \in X \) with \( x_n \rightarrow a \) \( (n \rightarrow \infty) \), in the sense that \( g(x_n - a) \rightarrow 0 \) \( (n \rightarrow \infty) \), in the sense that \( g(\lambda_n x_n - \lambda a) \rightarrow 0 \) \( (n \rightarrow \infty) \).

An Orlicz function is a function \( M : [0, \infty) \rightarrow [0, \infty) \), which is continuous, non-decreasing and convex with \( M(0) = 0, M(x) > 0 \) for \( x > 0 \) and \( M(x) \rightarrow \infty \) as \( x \rightarrow \infty \).

Lindenstrauss and Tzafriri [10] used the idea of Orlicz functions to construct the sequence space

$$\ell_M = \{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \}$$

The space \( \ell_M \) is a Banach space with the norm

$$||x|| = \inf \{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \}$$

The space \( \ell_M \) is closely related to the space \( \ell_p \) which is an Orlicz sequence space with \( M(x) = x^p \) for \( 1 \leq p < \infty \).

An Orlicz function \( M \) is said to satisfy \( \triangle_2 \) condition for all values of \( x \) if there exists a constant \( K > 0 \) such that \( M(Lx) \leq KLM(x) \) for all values of \( L > 1 \).

A sequence space \( E \) is said to be solid or normal if \( (x_k) \in E \) implies \( (\alpha_k x_k) \in E \) for all sequence
of scalars $(\alpha_k)$ with $|\alpha_k| < 1$ for all $k \in N$.

For Orlicz function and related results see ([2],[8],[17]).

Let $\sigma$ be an injection on the set of positive integers $\mathbb{N}$ into itself having no finite orbits and $T$ be the operator defined on $\ell_{\infty}$ by $T(x_k) = (x_{\sigma(k)})$.

A positive linear functional $\Phi$, with $\|\Phi\| = 1$, is called a $\sigma$-mean or an invariant mean if $\Phi(x) = \Phi(Tx)$ for all $x \in \ell_{\infty}$.

A sequence $x$ is said to be $\sigma$-convergent, denoted by $x \in V_{\sigma}$, if $\Phi(x)$ takes the same value, called $\sigma - \lim x$, for all $\sigma$-means $\Phi$. We have

$$V_{\sigma} = \{x = (x_k) : \sum_{m=1}^{\infty} t_{m,n}(x) = L \text{ uniformly in } n, L = \sigma - \lim x\},$$

where for $m \geq 0, n > 0$,

$$t_{m,n}(x) = \frac{x_k + x_{\sigma(k)} + \ldots + x_{\sigma^m(k)}}{m+1}, \text{ and } t_{-1,n} = 0.$$

where $\sigma^m(k)$ denotes the $m^{th}$ iterate of $\sigma$ at $n$. In particular, if $\sigma$ is the translation, a $\sigma$-mean is often called a Banach limit and $V_{\sigma}$ reduces to $f$, the set of almost convergent sequences.

Subsequently the spaces of invariant mean and Orlicz function have been studied by various authors. See ([1],[11],[15],[16],[19]).

The idea of Difference sequence sets

$$X_\triangle = \{x = (x_k) \in \omega : \triangle x = (x_k - x_{k+1}) \in X\},$$

where $X = \ell_{\infty}, c$ or $c_0$ was introduced by Kizmaz [9].

Kizmaz [9] defined the sequence spaces,

$$\ell_{\infty}(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in \ell_{\infty}\},$$

$$c(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c\},$$
$c_0(\triangle) = \{ x = (x_k) \in \omega : (\triangle x_k) \in c_0 \},$

where $\triangle x = (x_k - x_{k+1})$. These are Banach spaces with the norm

$$||x||_\triangle = |x_1| + ||\triangle x||_\infty.$$ 

After then Mikael [14] defined the sequence spaces,

$$\ell_\infty(\triangle^2) = \{ x = (x_k) \in \omega : (\triangle^2 x_k) \in \ell_\infty \},$$

$$c(\triangle^2) = \{ x = (x_k) \in \omega : (\triangle^2 x_k) \in c \},$$

$$c_0(\triangle^2) = \{ x = (x_k) \in \omega : (\triangle^2 x_k) \in c_0 \},$$

and showed that these are Banach spaces with norm

$$||x||_\triangle = |x_1| + |x_2| + ||\triangle^2 x||_\infty.$$ 

For difference sequences see([3-5],[8],[9]).

Recently Ebadullah[6] introduced and studied the sequence space

$$V_\sigma(M, p, r) = \{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|t_{m,n}(x)|}{\rho})]^{p_m} < \infty \text{ uniformly in } n, \rho > 0 \}.$$ 

Where $M$ is an Orlicz function, $p = (p_m)$ is any sequence of strictly positive real numbers and $r \geq 0$.

After then Ebadullah[7] introduced the sequence space

$$V_\sigma(M, p, r, \triangle) = \{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|t_{m,n}(\triangle x)|}{\rho})]^{p_m} < \infty \text{ uniformly in } n, \rho > 0 \}.$$ 

and discussed the following sequence spaces:

For $M(x) = x$ we get

$$V_\sigma(p, r, \triangle) = \{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} |t_{m,n}(\triangle x)|^{p_m} < \infty \text{ uniformly in } n \}$$
For $p_m = 1$, for all $m$, we get

$$V_\sigma(M, r, \triangle) = \{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} [M\left(\frac{|t_{m,n}(\triangle x)|}{\rho}\right)]^{p_m} < \infty \text{ uniformly in } n, \rho > 0\}$$

For $r = 0$ we get

$$V_\sigma(M, p, \triangle) = \{ x = (x_k) : \sum_{m=1}^{\infty} [M\left(\frac{|t_{m,n}(\triangle x)|}{\rho}\right)]^{p_m} < \infty \text{ uniformly in } n, \rho > 0\}$$

For $M(x) = x$ and $r=0$ we get

$$V_\sigma(p, \triangle) = \{ x = (x_k) : \sum_{m=1}^{\infty} |t_{m,n}(\triangle x)|^{p_m} < \infty \text{ uniformly in } n, \rho > 0\}$$

For $p_k = 1$, for all $m$ and $r=0$, we get

$$V_\sigma(M, \triangle) = \{ x = (x_k) : \sum_{m=1}^{\infty} [M\left(\frac{|t_{m,n}(\triangle x)|}{\rho}\right)] < \infty \text{ uniformly in } n, \rho > 0\}$$

For $M(x) = x$, $p_m = 1$, for all $m$ and $r=0$, we get

$$V_\sigma(\triangle) = \{ x = (x_k) : \sum_{m=1}^{\infty} |t_{m,n}(\triangle x)| < \infty \text{ uniformly in } n\}.$$

### 2. Main results

In this article we introduce the sequence space

$$V_\sigma(M, p, r, \triangle^2) = \{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} [M\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}\right)]^{p_m} < \infty \text{ uniformly in } n, \rho > 0\}.$$

Where $M$ is an Orlicz function, $p = (p_m)$ is any sequence of strictly positive real numbers and $r \geq 0$.

Now we define the sequence spaces as follows:
For $M(x) = x$ we get

$$V_\sigma(p, r, \triangle^2) = \{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r}|t_{m,n}(\triangle^2 x)|^{p_m} < \infty \text{ uniformly in } n\}$$

For $p_m = 1$, for all $m$, we get

$$V_\sigma(M, r, \triangle^2) = \{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r}[M\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}\right)]^{p_m} < \infty \text{ uniformly in } n, \rho > 0\}$$

For $r = 0$ we get

$$V_\sigma(M, p, \triangle^2) = \{x = (x_k) : \sum_{m=1}^{\infty} [M\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}\right)]^{p_m} < \infty \text{ uniformly in } n, \rho > 0\}$$

For $M(x) = x$ and $r=0$ we get

$$V_\sigma(p, \triangle^2) = \{x = (x_k) : \sum_{m=1}^{\infty} |t_{m,n}(\triangle^2 x)|^{p_m} < \infty \text{ uniformly in } n, \rho > 0\}$$

For $p_k = 1$, for all $m$ and $r=0$, we get

$$V_\sigma(M, \triangle^2) = \{x = (x_k) : \sum_{m=1}^{\infty} [M\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}\right)] < \infty \text{ uniformly in } n, \rho > 0\}$$

For $M(x) = x$, $p_m = 1$, for all $m$ and $r=0$, we get

$$V_\sigma(\triangle^2 x) = \{x = (x_k) : \sum_{m=1}^{\infty} |t_{m,n}(\triangle^2 x)| < \infty \text{ uniformly in } n\}.$$

**Theorem 2.1.** The sequence space $V_\sigma(M, p, r, \triangle^2)$ is a linear space over the field $\mathbb{C}$ of complex numbers.

**Proof.** Let $x, y \in V_\sigma(M, p, r, \triangle^2)$ and $\alpha, \beta \in \mathbb{C}$ then there exists positive numbers $\rho_1$ and $\rho_2$ such that

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho_1}\right)]^{p_m} < \infty,$$
and

\[ \sum_{m=1}^{\infty} \frac{1}{m^r} [M\left(\frac{|t_{m,n}(\triangle^2x)|}{\rho_2}\right)]^{p_m} < \infty \]

uniformly in \( n \).

Define \( \rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2) \).

Since \( M \) is non-decreasing and convex we have

\[ \sum_{m=1}^{\infty} \frac{1}{m^r} [M\left(\frac{|\alpha t_{m,n}(\triangle^2x)| + \beta t_{m,n}(\triangle^2y)|}{\rho_3}\right)]^{p_m} \]

\[ \leq \sum_{m=1}^{\infty} \frac{1}{m^r} [M\left(\frac{|\alpha t_{m,n}(\triangle^2x)|}{\rho_3}\right) + M\left(\frac{\beta t_{m,n}(\triangle^2y)|}{\rho_3}\right)]^{p_m} \]

\[ \leq \sum_{m=1}^{\infty} \frac{1}{m^r} \frac{1}{2} [M\left(\frac{t_{m,n}(\triangle^2x)}{\rho_1}\right) + M\left(\frac{t_{m,n}(\triangle^2y)}{\rho_2}\right)] < \infty \]

uniformly in \( n \).

This proves that \( V_\sigma(M, p, r, \triangle^2) \) is a linear space over the field \( \mathbb{C} \) of complex numbers.

**Theorem 2.2.** For any Orlicz function \( M \) and a bounded sequence \( p = (p_m) \) of strictly positive real numbers, \( V_\sigma(M, p, r, \triangle^2) \) is a paranormed space with

\[ g(x) = \inf \left\{ \rho \frac{p_n}{r} : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} [M\left(\frac{|t_{m,n}(\triangle^2x)|}{\rho}\right)]^{p_m} \right)^{\frac{1}{p}} \leq 1, \text{ uniformly in } n \right\} \]

where \( H = \max(1, \sup \rho_m) \).

**Proof.** It is clear that \( g(\triangle^2x) = g(-\triangle^2x) \).

Since \( M(0) = 0 \), we get

\( \inf \{ \rho \frac{p_n}{r} \} = 0 \), for \( x = 0 \)

Now for \( \alpha=\beta=1 \), we get

\( g(\triangle^2x + \triangle^2y) \leq g(\triangle^2x) + g(\triangle^2y) \).
For the continuity of scalar multiplication let \( l \neq 0 \) be any complex number. Then by the definition we have

\[
g(\triangle^2 x) = \inf_{n \geq 1} \{ \rho_{\triangledown}^n : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} [M \left( \frac{|t_{m,n}(\triangle^2 x)|}{\rho} \right)]^{p_m} \right)^{\frac{1}{p}} \leq 1, \text{ uniformly in } n \}
\]

\[
g(\triangle^2 x) = \inf_{n \geq 1} \{ (|l|s)^{\triangledown} : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} [M \left( \frac{|t_{m,n}(\triangle^2 x)|}{(|l|s)} \right)]^{p_m} \right)^{\frac{1}{p}} \leq 1, \text{ uniformly in } n \}
\]

where \( s = \frac{\rho}{|l|} \).

Since \( |l|^{p_m} \leq \max(1, |l|^H) \), we have

\[
g(\triangle^2 x) \leq \max(1, |l|^H) \inf_{n \geq 1} \{ (|l|s)^{\triangledown} : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} [M \left( \frac{|t_{m,n}(\triangle^2 x)|}{(|l|s)} \right)]^{p_m} \right)^{\frac{1}{p}} \leq 1, \text{ uniformly in } n \}
\]

\[
g(\triangle^2 l x) \leq \max(1, |l|^H)g(\triangle^2 x)
\]

Therefore \( g(\triangle^2 x) \) converges to zero when \( g(\triangle^2 x) \) converges to zero in \( V_\sigma(M, p, r, \triangle^2) \).

Now let \( x \) be fixed element in \( V_\sigma(M, p, r, \triangle^2) \). There exists \( \rho > 0 \) such that

\[
g(\triangle^2 x) = \inf_{n \geq 1} \{ \rho_{\triangledown}^n : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} [M \left( \frac{|t_{m,n}(\triangle^2 x)|}{\rho} \right)]^{p_m} \right)^{\frac{1}{p}} \leq 1, \text{ uniformly in } n \}
\]

Now

\[
g(\triangle^2 x) = \inf_{n \geq 1} \{ \rho_{\triangledown}^n : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} [M \left( \frac{|t_{m,n}(\triangle^2 x)|}{\rho} \right)]^{p_m} \right)^{\frac{1}{p}} \leq 1, \text{ uniformly in } n \} \rightarrow 0 \text{ as } l \rightarrow 0.
\]

This completes the proof.
Theorem 2.3. Suppose that $0 < p_m < t_m < \infty$ for each $m \in N$ and $r > 0$. Then

(a) $V_\sigma(M, p, \triangle^2) \subseteq V_\sigma(M, t, \triangle^2)$.
(b) $V_\sigma(M, \triangle^2) \subseteq V_\sigma(M, r, \triangle^2)$

Proof. (a) Suppose that $x \in V_\sigma(M, p, \triangle^2)$. This implies that $[M\left(\frac{|t_{i,n}(\triangle^2 x)|}{\rho}\right)]^{p_m} \leq 1$ for sufficiently large value of $i$, say $i \geq m_0$ for some fixed $m_0 \in N$.

Since $M$ is non-decreasing, we have

$$\sum_{m=m_0}^{\infty} [M\left(\frac{|t_{i,n}(\triangle^2 x)|}{\rho}\right)]^{p_m} \leq \sum_{m=m_0}^{\infty} [M\left(\frac{|t_{i,n}(\triangle^2 x)|}{\rho}\right)]^{p_m} < \infty.$$ 

Hence $x \in V_\sigma(M, t, \triangle^2)$.

(b) The proof is trivial.

Corollary 2.4. $0 < p_m \leq 1$ for each $m$, then $V_\sigma(M, p, \triangle^2) \subseteq V_\sigma(M, \triangle^2)$

If $p_m \geq 1$ for all $m$, then $V_\sigma(M, \triangle^2) \subseteq V_\sigma(M, p, \triangle^2)$.

Theorem 2.5. The sequence space $V_\sigma(M, p, r, \triangle^2)$ is solid.

Proof. Let $x \in V_\sigma(M, p, r, \triangle^2)$. This implies that

$$\sum_{m=1}^{\infty} \frac{1}{m^r} [M\left(\frac{|t_{i,n}(\triangle^2 x)|}{\rho}\right)]^{p_m} < \infty.$$ 

Let $\alpha_m$ be a sequence of scalars such that $|\alpha_m| \leq 1$ for all $m \in N$. Then the result follows from the following inequality.

$$\sum_{m=1}^{\infty} \frac{1}{m^r} [M\left(\frac{|\alpha_m t_{i,n}(\triangle^2 x)|}{\rho}\right)]^{p_m} \leq \sum_{m=1}^{\infty} \frac{1}{m^r} [M\left(\frac{|t_{i,n}(\triangle^2 x)|}{\rho}\right)]^{p_m} < \infty.$$
Hence $\alpha x \in V_\sigma(M, p, r, \triangle^2)$ for all sequence of scalars $(\alpha_m)$ with $|\alpha_m| \leq 1$ for all $m \in N$ whenever $x \in V_\sigma(M, p, r, \triangle^2)$.

**Corollary 2.6.** The sequence space $V_\sigma(M, p, r, \triangle^2)$ is monotone.

**Theorem 2.7.** Let $M_1, M_2$ be Orlicz function satisfying $\triangle^2$ condition and $r, r_1, r_2 \geq 0$. Then we have

(a) If $r > 1$ then $V_\sigma(M_1, p, r, \triangle^2) \subseteq V_\sigma(M0M_1, p, r, \triangle^2),$

(b) $V_\sigma(M_1, p, r, \triangle^2) \cap V_\sigma(M_2, p, r, \triangle^2) \subseteq V_\sigma(M_1 + M_2, p, r, \triangle^2),$

(c) If $r_1 \leq r_2$ then $V_\sigma(M, p, r_1, \triangle^2) \subseteq V_\sigma(M, p, r_2, \triangle^2).$

**Proof.** (a) Since $M$ is continuous at 0 from right, for $\varepsilon > 0$ there exists $0 < \delta < 1$ such that $0 \leq c \leq \delta$ implies $M(c) < \varepsilon$.

If we define

$I_1 = \{m \in N : M_1\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}\right) \leq \delta \text{ for some } \rho > 0\},$

$I_2 = \{m \in N : M_1\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}\right) > \delta \text{ for some } \rho > 0\},$

we get

$$M(M_1\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}\right)) \leq \left\{\frac{2M(1)}{\delta}\right\}M_1\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}\right).$$

Hence for $x \in V_\sigma(M_1, p, r, \triangle^2)$ and $r > 1$

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M0M_1\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}\right)]^p_m = \sum_{m \in I_1} \frac{1}{m^r}[M0M_1\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}\right)]^p_m + \sum_{m \in I_2} \frac{1}{m^r}[M0M_1\left(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}\right)]^p_m.$$
\[ \sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M_0 M_1 \left( \frac{t_m n (\Delta^2 x)}{\rho} \right) \right]^{p_m} \leq \max \left( e^h, e^H \right) \sum_{m=1}^{\infty} \frac{1}{m^r} + \max \left( \left\{ \frac{2M_1}{\delta} \right\}^h, \left\{ \frac{2M_1}{\delta} \right\}^H \right) \]

where \(0 < h = \inf p_m \leq p_m \leq H = \sup p_m < \infty\)

Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES


