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RIEMANN BOUNDARY VALUE PROBLEM WITH DEGENERATE COEFFICIENTS AND BASES FROM THE DOUBLE SYSTEM OF EXPONENTS IN A GENERALIZED LEBESGUE SPACES

GULIYEVA F. A.

Institute of Mathematics and Mechanics of NAS of Azerbaijan, Azerbaijan

Abstract: In this paper a double system of exponents with degenerate coefficients is considered. Basicity of this system is studied in a generalized Lebesgue space $L_{p(\cdot)}$. Method of boundary value problems of the theory of analytic functions is applied. In addition, a specific Riemann boundary value problem with degenerate coefficients is obtained. At first, this problem is studied in the Hardy classes with variable summability. The obtained results are applied to the study of basicity of considered double system of exponents in $L_{p(\cdot)}$, when the coefficients are assumed a degeneration at the ends of the segment $[-\pi,\pi]$.

Keywords: system of exponents; degeneration; basicity; variable exponent.

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1. Introduction

Consider the system of exponents

$$\left\{A(t)\rho^{+}(t)e^{\mathrm{int}}\,;B(t)\rho^{-}(t)e^{-ikt}\right\}_{n\geq 0;k\geq 1}, \tag{1}$$

with the degenerate coefficients

$$\rho^{\pm}(t)=|t-\pi|^{\beta_{\pi}^{\pm}}|t+\pi|^{\beta_{-\pi}^{\pm}}\prod_{k=0}^{r^{\pm}}|t-t_k^{\pm}|^{\beta_k^{\pm}},$$

*Corresponding author

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where $T^{\pm} \equiv \{t_k^{\pm}\}_1^{r^{\pm}} \subset (-\pi,\pi)$, $t_k^{\pm} \neq 0, \forall k, t_0^{\pm} = 0$; $\{\beta_\pi^{\pm}; \beta_k^{\pm}, k = \overline{0,r^{\pm}}\} \subset R$, and $A(t) \equiv |A(t)| e^{i\alpha(t)}$, $B(t) \equiv |B(t)| e^{i\beta(t)}$ are complex valued functions on $[-\pi,\pi]$. Interest in the study of basis properties of the system of the form (1):

$$E \equiv \{\exp i(n + \alpha\, sign\, n)t\}_{n \in Z}$$

in $L_p \equiv L_p(-\pi,\pi)$, $1 < p < +\infty$, being a special case of (1), dates back to the well-known work of Paley R., Wiener N. [1] and N.Levinson [2]. The final results in this direction are obtained in [3-7]. The weighted case of Lebesgue space is considered in [8-10;25]. It should be noted that the system of exponents with degenerate coefficient have been first considered in [11]. Then the results of [11] were generalized with respect to the degenerate coefficient in [12]. The most general case with different coefficients of degeneration is considered in [13;14]. When the degeneration are missing, the final results on basicity of the system of the form (1) in L_p in the general case with respect to the coefficients $A(t)$ and $B(t)$ have been obtained in [15;16;18-20].

In the present paper a double system of exponents with degenerate coefficients is considered. Basicity of this system is studied in a generalized Lebesgue space $L_{p(\cdot)}$. Method of boundary value problems of the theory of analytic functions is applied. In addition, a specific Riemann boundary value problem with degenerate coefficients is obtained. At first, this problem is studied in the Hardy classes with variable summability. The obtained results are applied to the study of basicity of considered double system of exponents in $L_{p(\cdot)}$, when the coefficients are assumed a degeneration at the ends of the segment $[-\pi,\pi]$. It should be noted that the boundary value problems and basicity problems associated with variable summability had previously been considered in [21-25]. The basicity of the system E in Lebesgue space with variable summability was previously studied in [21-23; 31; 32].

2. Needful information and main assumptions

We will use the usual notations. N – will be a set of all positive integers; Z – will be a set of all integers; $Z_+ = \{0\} \cup N$; R – will be the set of all real numbers; C – will stand for the field of complex numbers; $\overline{(\cdot)}$ – is the complex conjugate; δ_{nk} – is the Kronecker symbol; $\chi_A(\cdot)$ – is the characteristic function of the set A. $\omega = \{z \in C : |z| < 1\}$ be the unit circle, $\gamma \equiv \partial\omega$ be the unit circumference.

Let $p : [-\pi, \pi] \to [1, +\infty)$ be some Lebesgue measurable function. By $\mathcal{L}_0$ denote the class of all functions measurable on $[-\pi, \pi]$ (with respect to Lebesgue measure). Denote

$$I_p(f) \stackrel{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)} dt.$$

Let

$$\mathcal{L} \equiv \{f \in \mathcal{L}_0 : I_p(f) < +\infty\}.$$

With respect to the usual linear operations of addition and multiplication by a number, $\mathcal{L}$ is a linear space as $p^+ = \sup\limits_{[-\pi,\pi]} \underline{vrai}\, p(t) < +\infty$. With respect to the norm

$$\|f\|_{p(\cdot)} \stackrel{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

$\mathcal{L}$ is a Banach space, and we denote it by $L_{p(\cdot)}$. Let

$$WL \stackrel{def}{\equiv} \Big\{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \frac{1}{2} \Rightarrow$$

$$\Rightarrow |p(t_1) - p(t_2)| \le \frac{C}{-\ln|t_1 - t_2|}\Big\}.$$

Throughout this paper $q(\cdot)$ will denote the conjugate of a function $p(\cdot)$: $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Denote $p^- = \inf\limits_{[-\pi,\pi]} \underline{vrai}\, p(t)$. The following generalized Hölder inequality is true

$$\int_{-\pi}^{\pi} |f(t) g(t)| dt \le c(p^-; p^+) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $c(p^-; p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

It is easy to prove

Statement 1. *Let* $p \in WL$, $p(t) > 0, \forall t \in [-\pi, \pi]$; $\{\alpha_i\}_1^m \subset R$. *The weight function*

$$\rho(t) = \prod_{i=1}^{m} |t - \tau_i|^{\alpha_i},$$

belongs to the space $L_{p(\cdot)}$*, if the following inequalities*

$$\alpha_i > -\frac{1}{p(\tau_i)}, \quad \forall i = \overline{1,m};$$

are satisfied, where $-\pi = \tau_1 < \tau_2 < ... < \tau_m = \pi$.

The following fact plays an important role in obtaining the main results.

Property B **[27].** *If* $p(t): 1 < p^- \le p^+ < +\infty$*, then the class* $C_0^\infty(-\pi,\pi)$ *(class of finite and indefinitely differentiable functions) is everywhere dense in* $L_{p(\cdot)}$.

By S we denote the singular integral

$$Sf = \frac{1}{2\pi i}\int_\gamma \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \gamma .$$

Let $\rho: [-\pi,\pi] \to (0,+\infty)$. Define weight class $L_{p(\cdot),\rho(\cdot)}$:

$$L_{p(\cdot),\rho(\cdot)} : L_{p(\cdot),\rho(\cdot)} \overset{def}{\equiv} \left\{ f : \ \rho f \in L_{p(\cdot)} \right\},$$

furnished with a norm $\|f\|_{p(\cdot),\rho(\cdot)} \overset{def}{\equiv} \|\rho f\|_{p(\cdot)}$. The validity of the following statement is established in [30].

Statement 2 **[30].** *Let* $p \in WL$, $1 < p^-$ *and the weight* $\rho(\cdot)$ *be defined by*

$$\rho(t) = \prod_{k=1}^{m} |t - \tau_k|^{\alpha_k} ,$$

where $\{\tau_k\}_1^m \subset [-\pi,\pi]$ *are different points,* $\{\alpha_k\}_1^m \subset R$.

Then, singular operator S is acting boundedly from $L_{p(\cdot),\rho(\cdot)}$ *to* $L_{p(\cdot),\rho(\cdot)}$ *if and only if*

$$-\frac{1}{p(\tau_k)} < \alpha_k < \frac{1}{q(\tau_k)}, \quad k = \overline{1,m}.$$

More details on these and other facts one can see works [26-30].

Define the weighted Hardy classes $H^\pm_{p(\cdot),\rho}$. By $H^+_{p_0}$ we denote the usual Hardy class of functions which are analytic inside ω, where $p_0 \in [1,+\infty)$ is some number. Assume

$$H^\pm_{p(\cdot),\rho} \equiv \left\{ f \in H_1^+ : f^+ \in L_{p(\cdot),\rho}(\partial\omega) \right\},$$

where f^+ are non-tangential boundary values on $\partial\omega$ of $f(\cdot)$.

The weighted Hardy class ${}_mH^-_{p(\cdot),\rho}$ of functions which are analytic in $C\setminus\overline{\omega}$ $(\overline{\omega}=\omega\bigcup\partial\omega)$, with their orders $k\le m$ at infinity is defined similarly to the classical one. Let $f(z)$ be the analytic function in $C\setminus\overline{\omega}$ $(\overline{\omega}=\omega\bigcup\partial\omega)$, of finite order $k\le m$ at infinity, i.e.

$$f(z)=f_1(z)+f_2(z),$$

where $f_1(z)$ is a polynomial of degree $k\le m$, $f_2(z)$ is a regular part of Laurent series expansion of $f(z)$ *in the neighborhood of an infinitely remote point.* If the function $\varphi(z)\equiv\overline{f_2\left(\frac{1}{\overline{z}}\right)}$ belongs to the class $H^+_{p(\cdot),\rho}$, then we will say that the function $f(z)$ belongs to the class ${}_mH^-_{p(\cdot),\rho}$.

As the norms in these spaces we accept

$$\|f\|_{H^+_{p(\cdot),\rho}}\equiv\left\|f^+\rho\right\|_{L_{p(\cdot)}},\ \forall f\in H^+_{p(\cdot),\rho},$$

$$\|f\|_{{}_mH^-_{p(\cdot),\rho}}\equiv\left\|f^-\rho\right\|_{L_{p(\cdot)}},\ \forall f\in{}_mH^-_{p(\cdot),\rho},$$

where by $f^\pm$ is denoted non-tangential boundary values on $\partial\omega$ of f.

Restrictions of functions from $H^+_{p(\cdot),\rho}$ to the unit circumference we denote by $L^+_{p(\cdot),\rho}$, i.e.

$$L^+_{p(\cdot),\rho}\equiv\left\{f:\exists g\in H^+_{p(\cdot),\rho},\ f\equiv g/_{\partial\omega}\right\}.$$

Let us note that if $f\in H^+_{p(\cdot),\rho}$, then $\|f\|_{H^+_{p(\cdot),\rho}}=\left\|f^+\right\|_{p(\cdot),\rho}$, where $f^+=f/_{\partial\omega}$.

A similar fact also holds with respect to space ${}_mH^-_{p(\cdot)}$. Assume

$${}_mL^-_{p(\cdot)}\equiv\left\{f:\exists g\in{}_mH^-_{p(\cdot)},\ f=g/_{\partial\omega}\right\}.$$

In obtaining the main result we also need the following

Statement 3 [31]. *Let $p(\cdot)\in WL$, $p^->1$, and the weigh $\rho(\cdot)$ satisfy all the conditions of Statement 2. Then the system of exponents $\left\{e^{\mathrm{int}}\right\}_{n\ge0}$ ($\left\{e^{-\mathrm{int}}\right\}_{n\ge m}$) forms a basis for the weighted space $H^+_{p(\cdot),\rho}$ (${}_mH^-_{p(\cdot),\rho}$).*

Assume the following main assumptions:

1) $\arg A(t)$, $\arg B(t)$ are piecewise Holder functions on $[-\pi,\pi]$: $S\equiv\{s_i\}_1^r$:

$-\pi < s_1 < ... < s_r < \pi -$ be the points of discontinuity of the function $\theta(t) \equiv \arg A(t) - \arg B(t)$ on $(-\pi, \pi)$ and $h_k = \theta(s_k + 0) - \theta(s_k - 0)$, $k = \overline{1, r}$, $h_\pi = \theta(-\pi + 0) - \theta(\pi - 0)$, $h_0 = \theta(+0) - \theta(-0)$ are corresponding jumps at these points.

2) $A(t), B(t)$ are measurable on $(-\pi, \pi)$ and the following inequality is fulfilled

$$\sup_{(-\pi,\pi)} vrai \left\{ |A(t)|^{\pm 1}, |B(t)|^{\pm 1} \right\} < +\infty ;$$

3) Let $T^{\pm} \cap S = \varnothing; T^{+} \cap T^{-} = \varnothing$ and denote

$$\Omega^{\pm} \equiv \left\{ \beta_0^{\pm}; \beta_\pi^{\pm}; \beta_{-\pi}^{\pm}; \beta_k^{\pm}, k = \overline{1, r^{\pm}} \right\}, \ \Omega = \Omega^{+} \cup \Omega^{-} .$$

Assume

$$p_\beta = \begin{cases} p(t_k^{\pm}), \ \beta = \beta_k^{\pm}, k = \overline{0, r^{\pm}} , \\ p(\pi), \ \beta = \beta_\pi^{\pm} \wedge \beta = \beta_{-\pi}^{\pm} . \end{cases}$$

Throughout this paper, q_β will denote the conjugate of p_β: $p_\beta^{-1} + q_\beta^{-1} = 1$.

3. Riemann boundary value problem with degenerate coefficients in generalized Hardy classes

Consider the following non-homogeneous conjugate problem in classes $H^{+}_{p(\cdot),\omega^{+}} \times_m H^{-}_{p(\cdot),\omega^{-}}$:

$$\begin{cases} F^{+}(\tau) + G(\tau) F^{-}(\tau) = \tilde{f}(\arg \tau), |\tau| = 1, \\ F^{-}(\infty) = 0, \end{cases} \tag{2}$$

where $\tilde{f}(t) \equiv \dfrac{f(t)}{\rho^{+}(t)}$, $\forall f \in L_{p(\cdot)}(-\pi, \pi)$, the coefficient $G(\tau)$ is defined by

$$G(e^{it}) \equiv \frac{\rho^{-}(t) A(t)}{\rho^{+}(t) B(t)} .$$

By solution of the problem (2) in $H^{+}_{p(\cdot),\omega^{+}} \times_m H^{-}_{p(\cdot),\omega^{-}}$ we mean the following: to find a pair of functions $(F^{+}(z); F^{-}(z)) \in H^{+}_{p(\cdot);\omega^{+}} \times_m H^{-}_{p(\cdot);\omega^{-}}$, whose non-tangential boundary values on the unit circumference γ a.e. satisfy the relation (2), where $\omega^{\pm} = |\rho^{\pm}(\cdot)|^{p(\cdot)}$.

By $Z(z)$ we denote the canonical solution ([17]) of the corresponding homogeneous problem

$$F^{+}(\tau) + G(\tau) F^{-}(\tau) = 0, \ \tau \in \gamma . \tag{3}$$

$Z(z)$ has an expression $Z(z)=\prod_{k=1}^{3} Z_k(z)$, where $Z_k(z)$ is defined by the expression

$$Z_k(z) \equiv \begin{cases} X_k^+(z) \ , \ |z|<1 \ , \\ \left[X_k^-(z)\right]^{-1}, \ |z|>1, k=\overline{1,3}; \end{cases}$$

$$X_1^{\pm}(z)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\frac{\rho^-(t)}{\rho^+(t)}\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_2^{\pm}(z)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left|\frac{A(t)}{B(t)}\right|\frac{e^{it}+z}{e^{it}-z}dt\right\} \ ,$$

$$X_3^{\pm}(z)=\exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi}\theta(t)\frac{e^{it}+z}{e^{it}-z}dt\right\}.$$

Note that the sign "+" ("-") corresponds to the case $|z|<1$ ($|z|>1$).

As it is established in [17] we have the relation

$$\left\|Z_2^-\left(e^{it}\right)\right\|_{\infty}^{\pm 1}<+\infty \ ,$$

where $\|\cdot\|_{\infty}$ is an ordinary norm in $L_{\infty}(-\pi,\pi)$. Regarding the boundary values $Z_1^-\left(e^{it}\right)$ the following expression is valid

$$\left|Z_1^-\left(e^{it}\right)\right|=\left[\frac{\rho^+(t)}{\rho^-(t)}\right]^{\frac{1}{2}}.$$

It directly follows from Sokhotskii-Plemelj formula [17].

Let $\theta(t)\equiv\theta_0(t)+\theta_1(t)$ be a Jordan decomposition of function $\theta(t)$ on a continuous part of $\theta_0(t)$ and on the jump function $\theta_1(t)$. Assume $h_\pi=\theta(-\pi+0)-\theta(\pi-0)$. It is clear that $h_\pi=h^{(1)}+h^{(0)}$, where

$$h^{(1)}=\theta_1(-\pi+0)-\theta_1(\pi-0) \ , \ h^{(0)}=\theta_0(-\pi)-\theta_0(\pi) \ .$$

Denote

$$u_0(t)\equiv\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h^{(0)}}{2\pi}}\exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta_0(s)ctg\frac{t-s}{2}ds\right\}.$$

According to the results of the monograph [17], the function $u_0^{\pm 1}(t)$ is integrable with any degree of $p \in (0,+\infty)$ on $(-\pi,\pi)$. For the solution of the homogeneous problem we obtain the relation

$$\frac{F^+(\tau)}{Z^+(\tau)} = -\frac{F^-(\tau)}{Z^-(\tau)}, \text{ a.e. } \tau \in \gamma .$$

To obtain the general solution of the homogeneous problem (3), let us show that the boundary values of a piecewise analytic function $\Phi(z)$:

$$\Phi(z) \equiv \begin{cases} \dfrac{F^+(z)}{Z^+(z)}, |z| < 1, \\ -\dfrac{F^-(z)}{Z^-(z)}, |z| > 1, \end{cases}$$

belong to $L_1(-\pi,\pi)$. Since, by definition of the solution, the expression $F^-\left(e^{it}\right)\rho^-(t)$ belongs to $L_{p(\cdot)}(-\pi,\pi)$, then it is sufficient to show that the expression

$$Y_0(t) \equiv \left|Z^-\left(e^{it}\right)\rho^-(t)\right|^{-1} ,$$

belongs to the space $L_{q(\cdot)}(-\pi,\pi)$. We have $Z^-\left(e^{it}\right) = \prod_{k=1}^{3} Z_k^-\left(e^{it}\right)$. Using Sokhotskii-Plemelj formulas it is easy to obtain the following expression for the boundary values $Z_3^-\left(e^{it}\right)$:

$$\left|Z_3^-\left(e^{it}\right)\right| = u_0(t)u(t)\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_\pi}{2\pi}} ,$$

where

$$u(t) \equiv \left\{\sin\left|\frac{t}{2}\right|^{-\frac{h_0}{2\pi}}\right\}\prod_{k=1}^{r}\left\{\sin\left|\frac{t-s_k}{2}\right|\right\}^{-\frac{h_k}{2\pi}} .$$

Taking into account above expression for $Y_0(t)$, we obtain

$$Y_0(t) \sim u_0^{-1}(t)u^{-1}(t)\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{\frac{h_\pi}{2\pi}}\left[\rho^+(t)\rho^-(t)\right]^{-\frac{1}{2}} ,$$

(the symbol $\sim$ denotes an asymptotic equivalence) in other words

$$Y_0(t) \sim u_0^{-1}(t)\left\{\sin\left|\frac{t}{2}\right|^{\frac{h_0}{2\pi}}\right\}\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{\frac{h_\pi}{2\pi}}\prod_{k=1}^{r}\left\{\sin\left|\frac{t-s_k}{2}\right|^{\frac{h_k}{2\pi}}\right\}\left[\rho^+(t)\rho^-(t)\right]^{-\frac{1}{2}} .$$

Thus

$$Y_0(t) \sim \left\{ \sin\left|\frac{t}{2}\right| \right\}^{\frac{h_0}{2\pi}} \left[\rho^+(t)\rho^-(t)\right]^{-\frac{1}{2}}, \text{ as } t \to 0,$$

and as a result

$$Y_0(t) \sim |t|^{\frac{h_0}{2\pi} - \frac{\beta_0^+ + \beta_0^-}{2}}, \text{ as } t \to 0.$$

Similarly, we have

$$Y_0(t) \sim \left\{ \sin\left|\frac{t-\pi}{2}\right| \right\}^{\frac{h_\pi}{2\pi}} \left[\rho^+(t)\rho^-(t)\right]^{-\frac{1}{2}}, \text{ as } t \to \pm\pi,$$

and as a result

$$Y_0(t) \sim \left\{ |t-\pi|^{\frac{h_\pi}{2\pi} - \frac{\beta_\pi^+ + \beta_\pi^-}{2}} \right\}, \text{ as } t \to +\pi,$$

$$Y_0(t) \sim \left\{ |t+\pi|^{\frac{h_\pi}{2\pi} - \frac{\beta_{-\pi}^+ + \beta_{-\pi}^-}{2}} \right\}, \text{ as } t \to -\pi.$$

Paying attention to Statement 1, we obtain from these representations that $Y_0(t)$ belongs to the space $L_{q(\cdot)}(-\pi,\pi)$, when the inequalities

$$\frac{h_k}{2\pi} > -\frac{1}{q(s_k)}, k = \overline{1,r}; \beta_k^\pm < \frac{2}{q(t_k^\pm)}, k = \overline{1,r^\pm};$$

$$\beta_0^\pm < \frac{2}{q(0)}; \beta_\pi^\pm < \frac{2}{q(\pi)}; \beta_{-\pi}^\pm < \frac{2}{q(\pi)};$$

$$\frac{h_0}{2\pi} - \frac{\beta_0^+ + \beta_0^-}{2} > -\frac{1}{q(0)}; \frac{h_\pi}{2\pi} - \frac{\beta_\pi^+ + \beta_\pi^-}{2} > -\frac{1}{q(\pi)};$$

$$\frac{h_\pi}{2\pi} - \frac{\beta_{-\pi}^+ + \beta_{-\pi}^-}{2} > -\frac{1}{q(\pi)},$$

are fulfilled. When these inequalities are fulfilled, then the function $\Phi(z)$ belongs to the Hardy class $H_1^\pm$. Then, from the uniqueness theorem [17] we obtain that $\Phi(z) \equiv P_m(z)-$ is a polynomial of order $\le m$, i.e. $F(z) \equiv Z(z)P_m(z)$.

Let us show that $F(z) \equiv (F^+(z); F^-(z))$ belongs to the class $H^+_{p(\cdot);\omega^+} \times_m H^-_{p(\cdot);\omega^-}$. It is sufficient to prove that $[Y_0(t)]^{-1}$ belongs to space $L_{p(\cdot)}(-\pi,\pi)$. It is clear that the following relations hold

$$\left|Y_0(t)\right|^{-1} \sim |t|^{\frac{\beta_0^+ + \beta_0^-}{2} - \frac{h_0}{2\pi}}, \text{ as } t \to 0;$$

$$\left|Y_0(t)\right|^{-1} \sim |t-\pi|^{\frac{\beta_\pi^+ + \beta_\pi^-}{2} - \frac{h_\pi}{2\pi}}, \text{ as } t \to +\pi;$$

$$\left|Y_0(t)\right|^{-1} \sim |t+\pi|^{\frac{\beta_{-\pi}^+ + \beta_{-\pi}^-}{2} - \frac{h_\pi}{2\pi}}, \text{ as } t \to -\pi.$$

From these representations follows that the function $[Y_0(t)]^{-1}$ belongs to the space $L_{p(\cdot)}(-\pi,\pi)$, if the inequalities

$$\beta_k^\pm > -\frac{2}{p(t_k^\pm)}, k=\overline{1,r^\pm}; \frac{h_k}{2\pi} < \frac{1}{p(s_k)}, k=\overline{1,r};$$

$$\frac{h_0}{2\pi} - \frac{\beta_0^+ + \beta_0^-}{2} < -\frac{1}{p(0)}; \frac{h_\pi}{2\pi} - \frac{\beta_\pi^+ + \beta_\pi^-}{2} < \frac{1}{p(\pi)}.$$

$$\beta_0^\pm > -\frac{2}{p(0)} \ ; \ \beta_\pi^\pm > -\frac{2}{p(\pi)} \ ; \ \beta_{-\pi}^\pm > -\frac{2}{p(\pi)} \ ;$$

$$\frac{h_\pi}{2\pi} - \frac{\beta_{-\pi}^+ + \beta_{-\pi}^-}{2} < \frac{1}{p(\pi)},$$

are fulfilled. As a result, we obtain that when these inequalities are fulfilled, then $F(z)$ belongs to the class $H^+_{p(\cdot);\omega^+} \times_m H^-_{p(\cdot);\omega^-}$, and thus, it is the general solution of the problem (3).

Consequently, the following theorem is true.

Theorem 1. *Let $p(\cdot) \in WL, p^- > 1$ and the functions $A(t)$ and $B(t)$ satisfy the conditions 1)-3). If the following inequalities are fulfilled*

$$-\frac{1}{q(s_k)} < \frac{h_k}{2\pi} < \frac{1}{p(s_k)}, k=\overline{1,r}; -\frac{2}{p_\beta} < \beta < \frac{2}{q_\beta}, \forall \beta \in \Omega;$$

$$-\frac{1}{q(\pi)} < \frac{h_\pi}{2\pi} - \frac{\beta_{-\pi}^+ + \beta_{-\pi}^-}{2} < \frac{1}{p(\pi)} \ ; \tag{4}$$

$$-\frac{1}{q(0)} < \frac{h_0}{2\pi} - \frac{\beta_0^+ + \beta_0^-}{2} < \frac{1}{p(0)};$$

$$-\frac{1}{q(\pi)} < \frac{h_\pi}{2\pi} - \frac{\beta_\pi^+ + \beta_\pi^-}{2} < \frac{1}{p(\pi)},$$

then the general solution of the homogeneous conjugate problem (3) in classes $H^{+}_{p(\cdot);\omega^{+}} \times_m H^{-}_{p(\cdot);\omega^{-}}$ *has the form* $F(z) \equiv Z(z) P_m(z)$, *where* $P_m(z)$ *is an arbitrary polynomial of order* $\leq m$.

From this theorem, we get the following

Corollary 1. *Let all the conditions of Theorem 1 be fulfilled. Then the homogeneous problem (3) has only the trivial solution in classes* $H^{+}_{p(\cdot);\omega^{+}} \times_m H^{-}_{p(\cdot);\omega^{-}}$ *under the condition* $F(\infty) = 0$.

Now, let us consider the non-homogeneous conjugate problem (2) in classes $H^{+}_{p(\cdot);\omega^{+}} \times_m H^{-}_{p(\cdot);\omega^{-}}$. It is obvious that if it is solvable, then under condition $F(\infty) = 0$ the solution is unique. Let us consider the function $F_1(z)$:

$$F_1(z) \equiv \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \frac{\tilde{f}(\sigma)}{Z^{+}(e^{i\sigma})} \frac{d\sigma}{1 - ze^{-i\sigma}},$$

where $Z(z)$ is the canonical solution of the homogeneous problem (3). Let

$$Z_0(\tau) \equiv \rho^{+}(\arg \tau) Z^{+}(\tau), \tau \in \Gamma.$$

Thus

$$F_1^{+}(e^{it}) \rho^{+}(t) = f(t) + \frac{Z_0(e^{it})}{2\pi} \int_{-\pi}^{\pi} \frac{f(\sigma)}{Z_0(e^{i\sigma})} \frac{d\sigma}{1 - ze^{i(t-\sigma)}}.$$

It is clear that, the following holds

$$\rho^{+}(\arg \tau) Z^{+}(\tau) \sim \rho^{-}(\arg \tau) Z^{-}(\tau).$$

Taking into account this relation from inequality (4) we obtain that the weight $|Z_0(\tau)|$ satisfies all the conditions of Statement 2 [30]. Therefore, the expression

$$[K f](t) = \frac{Z_0(e^{it})}{2\pi} \int_{-\pi}^{\pi} \frac{f(\sigma)}{Z_0(e^{i\sigma})} \frac{d\sigma}{1 - ze^{i(t-\sigma)}},$$

belongs to the space $L_{p(\cdot)}(-\pi, \pi)$, more precisely the operator K continuously acts $L_{p(\cdot)}(-\pi, \pi)$. As a result, we obtain the validity of the following

Theorem 2. *Let all the conditions of Theorem 1 be fulfilled. Then under the condition* $F(\infty) = 0$ *the non-homogeneous conjugate problem (2) is solvable for* $\forall f \in L_{p(\cdot)}(-\pi, \pi)$, *and it has a* *unique solution in classes* $H^{+}_{p(\cdot);\omega^{+}} \times_m H^{-}_{p(\cdot);\omega^{-}}$, *and the solution* $F(z)$ *is expressed by Cauchy type integral*

$$F(z)=\frac{Z(z)}{2\pi}\int_{-\pi}^{\pi}\frac{f(\sigma)}{\rho^{+}(\sigma)Z\left(e^{i\sigma}\right)}\frac{d\sigma}{1-ze^{i(t-\sigma)}},$$

where $Z(z)$ *is an appropriate canonical solution of the homogeneous problem.*

4. Basicity of the system of exponents in $L_{p(\cdot)}$

Now, we turn to the study of the basicity of the system (1) in $L_{p(\cdot)}(-\pi,\pi)$. Take $\forall f \in L_{p(\cdot)}(-\pi,\pi)$ and consider the non-homogeneous conjugate problem in classes $H^{+}_{p(\cdot);\omega^{+}} \times_m H^{-}_{p(\cdot);\omega^{-}}$:

$$\begin{cases} F^{+}(\tau)+G(\tau)F^{-}(\tau)=\dfrac{f(\arg\tau)}{\rho^{+}(\arg\tau)}, \tau\in\Gamma, \\ F^{-}(\infty)=0. \end{cases} \tag{5}$$

As has already been shown, if all conditions of Theorem 2 are fulfilled, the problem (5) has a unique solution for $\forall f \in L_{p(\cdot)}(-\pi,\pi)$. Consequently, $F^{+}(z)\in H_1^{+}$ и $F^{-}(z)\in_{-1} H_1^{-}$. On the other hand, it is clear that $F^{\pm}\left(e^{it}\right)\in L_{p(\cdot),\omega^{\pm}}$, and, as a result, $F^{+}\left(e^{it}\right)\in L^{+}_{p(\cdot),\omega^{+}}$, $F^{-}\left(e^{it}\right)\in_{-1} L^{-}_{p(\cdot),\omega^{-}}$. Require the fulfilment of the following inequalities

$$-\frac{1}{p_\beta}<\beta<\frac{1}{q_\beta}, \quad \forall\beta\in\Omega . \tag{6}$$

If the inequalities (6) hold, then all conditions of the Statement 3 are fulfilled, i.e. the system $\left\{e^{\mathrm{int}}\right\}_{n\ge 0}$ ($\left\{e^{-\mathrm{int}}\right\}_{n\ge 1}$) forms a basis for $L^{+}_{p(\cdot);\omega^{+}}$ (${}_{-1}L^{-}_{p(\cdot);\omega^{-}}$). Consequently, the function $F^{+}\left(e^{it}\right)$ ($F^{-}\left(e^{it}\right)$) can be expanded in a biorthogonal series with respect to the system $\left\{e^{\mathrm{int}}\right\}_{n\ge 0}$ ($\left\{e^{-\mathrm{int}}\right\}_{n\ge 1}$) in space $L^{+}_{p(\cdot);\omega^{+}}$ (${}_{-1}L^{-}_{p(\cdot);\omega^{-}}$).

Expanding the function $F^{+}\left(e^{it}\right)$ ($F^{-}\left(e^{it}\right)$) in relation (5) with respect to the systems $\left\{e^{\mathrm{int}}\right\}_{n\ge 0}$ ($\left\{e^{-\mathrm{int}}\right\}_{n\ge 1}$), respectively we obtain that the function f is expanded in series with respect to the system (1) in $L_{p(\cdot)}$. The fact that this series is unique follows from the (only) trivial solvability of the corresponding homogeneous problem (3), under the condition $F(\infty)=0$. Combining the inequalities (4), (6) we obtain the following results regarding the basicity of the system (1) in $L_{p(\cdot)}$.

Theorem 3. *Let* $p(\cdot)\in WL, p^{-}>1$ *and the functions* $A(t)$ *and* $B(t)$ *satisfy the conditions 1)-3). If the following inequalities are fulfilled*

$$-\frac{1}{p_\beta}<\beta<\frac{1}{q_\beta}, \quad \forall\beta\in\Omega;$$

$$-\frac{1}{q(s_k)}<\frac{h_k}{2\pi}<\frac{1}{p(s_k)} k=\overline{1,r}; \quad -\frac{1}{q(0)}<\frac{h_0}{2\pi}-\frac{\beta_0^{+}+\beta_0^{-}}{2}<\frac{1}{p(0)};$$

$$-\frac{1}{q(\pi)}<\frac{h_\pi}{2\pi}-\frac{\beta_\pi^{+}+\beta_\pi^{-}}{2}<\frac{1}{p(\pi)}; \; -\frac{1}{q(\pi)}<\frac{h_\pi}{2\pi}-\frac{\beta_{-\pi}^{+}+\beta_{-\pi}^{-}}{2}<\frac{1}{p(\pi)},$$

then the system of exponents (1) forms a basis for $L_{p(\cdot)}$.

From these results, it is not difficult to deduce the following

Statement 4. *Let* $p(\cdot)\in WL, p^{-}>1$ *and* $\rho\in L_1$ *be a weighted function of the power form and* $\{\rho\vartheta_n\}_{n\in N}\subset L_{p(\cdot)}$, *be some system. Then this system forms a basis for* $L_{p(\cdot)}$ *if and only if the system* $\{\vartheta_n\}_{n\in N}$ *forms a basis for the weighted space* $L_{p(\cdot),\omega}$, *where* $\omega=\rho^{p(\cdot)}(\cdot)$.

Remark 1. *Taking* $\rho^{+}=\rho^{-}=\rho$, *from the previous results, we can obtain similar results with respect to the basicity of the double system of exponents* $\{A(t)e^{\mathrm{int}};B(t)e^{-ikt}\}_{n\geq 0;k\geq 1}$ *with degenerate coefficients* $A(\cdot)$ *and* $B(\cdot)$ *in the weighted space* $L_{p(\cdot),\omega}$ *with a norm* $\|\cdot\|_{p(\cdot),\omega}$, *where* $\omega=\rho^{p(\cdot)}(\cdot)$. *In this case in contrast to the work [10], it is assumed various orders of degeneracy at points* $t=\pm\pi$.

5. Examples

Let us consider some particular cases of the obtained results.

5.1. As the functions $A(t)$ and $B(t)$ take

$$A(t)\equiv e^{i\alpha(t)}; \quad B(t)\equiv e^{-i\alpha(t)},$$

where $\alpha(t)$ is a piecewise Hölder function on $[-\pi;\pi]$ with the discontinuity points $\{s_k\}_1^r: -\pi<s_1<...<s_r<\pi, s_k\neq 0, k=\overline{1,r}$. We have $\theta(t)=\arg B(t)-\arg A(t)=-2\alpha(t)$. Thus

$$h_k=-2(\alpha(s_k+0)-\alpha(s_k-0)), k=\overline{1,r}; \quad h_0=-2(\alpha(+0)-\alpha(-0));$$

$$h_{\pi} = -2(\alpha(-\pi) - \alpha(\pi)).$$

Regarding the degenerate coefficients accept the following relations

$$\beta_k^{\pm} = 0,\ k = \overline{1, r^{\pm}};\ \beta_0^{+} = \beta_0^{-} = \beta_0,\ \beta_{\pi}^{+} = \beta_{\pi}^{-} = \beta_{\pi}, \beta_{-\pi}^{+} = \beta_{-\pi}^{-} = \beta_{-\pi}.$$

Let

$$\rho(t) = |t|^{\beta_0} |t - \pi|^{\beta_{\pi}} |t + \pi|^{\beta_{-\pi}}.$$

Applying Theorem 3 to the system

$$\left\{\rho(t) e^{i(nt + \alpha(t)) sign\, n}\right\}_{n \in Z}, \tag{7}$$

we obtain the following

Statement 5. *Let* $p(\cdot) \in WL, p^{-} > 1$ *and the following inequalities be satisfied*

$$-\frac{1}{p_{\beta}} < \beta < \frac{1}{q_{\beta}},\ \forall \beta \in \{\beta_0; \beta_{\pi}; \beta_{-\pi}\};$$

$$-\frac{1}{p(s_k)} < \frac{\alpha(s_k + 0) - \alpha(s_k - 0)}{\pi} < \frac{1}{q(s_k)},\ k = \overline{1, r};$$

$$-\frac{1}{p(0)} < \frac{\alpha(+0) - \alpha(-0)}{\pi} + \beta_0 < \frac{1}{q(0)};$$

$$-\frac{1}{p(\pi)} < \frac{\alpha(-\pi) - \alpha(\pi)}{\pi} + \beta_{\pi} < \frac{1}{q(\pi)};$$

$$-\frac{1}{p(\pi)} < \frac{\alpha(-\pi) - \alpha(\pi)}{\pi} + \beta_{-\pi} < \frac{1}{q(\pi)}.$$

Then, the system (7) forms a basis for $L_p, 1 < p < +\infty$.

5.2. Consider the case $\alpha(t) = \alpha t + \beta\, signt,\ t \in [-\pi; \pi]$. In this case, we have

$$\alpha(+0) - \alpha(-0) = 2\beta;\ \alpha(-\pi) - \alpha(\pi) = -2(\alpha\pi + \beta).$$

So, the following statement is true.

Statement 6. *Let* $p(\cdot) \in WL, p^{-} > 1$ *and with respect to the parameters* $\alpha, \beta \in R$ *the following inequalities are true*

$$-\frac{1}{p_{\gamma}} < \gamma < \frac{1}{q_{\gamma}}, \forall \gamma \in \{\beta_0; \beta_{\pi}; \beta_{-\pi}\},$$

$$-\frac{1}{2p(0)} < \beta + \frac{\beta_0}{2} < \frac{1}{2q(0)};\ -\frac{1}{2q(\pi)} < \alpha + \frac{\beta}{\pi} - \frac{\beta_{\pi}}{2} < \frac{1}{2p(\pi)};$$

$$-\frac{1}{2q(\pi)} < \alpha + \frac{\beta}{\pi} - \frac{\beta_{-\pi}}{2} < \frac{1}{2p(\pi)}, \qquad (8)$$

where

$$p_\gamma = \begin{cases} p(0), & \gamma = \beta_0, \\ p(\pi), & \gamma = \beta_{\pm\pi}. \end{cases}$$

Then the system of exponents

$$\left\{\rho(t) e^{i(n+\alpha\, sign\, n)t + \beta\, sign\, t\ sign\, n}\right\}_{n\in Z},$$

forms a basis for $L_{p(\cdot)}$.

5.3. For $\beta = 0$ from the Example 5.2 we obtain

Corollary 2. *Let* $p(\cdot) \in WL, p^- > 1$ *and (8) and the inequality*

$$-\frac{1}{2q(\pi)} < \alpha - \frac{\beta_\pi}{2} < \frac{1}{2p(\pi)}; -\frac{1}{2q(\pi)} < \alpha - \frac{\beta_{-\pi}}{2} < \frac{1}{2p(\pi)},$$

are fulfilled. Then the system of exponents

$$\left\{\rho(t) e^{i(n+\alpha\, sign\, n)t}\right\}_{n\in Z},$$

forms a basis for $L_{p(\cdot)}$.

5.4. Let us take the weighted function ω of the form

$$\omega(t) = |t|^{\gamma_0} |t - t_1|^{\gamma_1} |t - \pi|^{\gamma_2} |t + \pi|^{\gamma_3},$$

where $t_1 \in (-\pi, \pi)$, $t_1 \neq 0$. Paying attention to the Statement 3 from Theorem 3 we obtain

Corollary 3. *Let* $p(\cdot) \in WL, p^- > 1$ *and the following inequalities be satisfied*

$$-1 < \gamma_k < p_{\gamma_k} - 1;\ k = \overline{0,3};$$
$$-\frac{1}{2q_{\gamma_k}} < \alpha - \frac{\gamma_k}{2p_{\gamma_k}} < \frac{1}{2p_{\gamma_k}},\ \ k = 2,3,$$

where

$$p_{\gamma_k} = \begin{cases} p(0), & k = 0, \\ p(t_1), & k = 1, \\ p(\pi), & k = 2,3, \end{cases}$$

and $\dfrac{1}{p_{\gamma_k}} + \dfrac{1}{q_{\gamma_k}} = 1$. *Then the system of exponents*

$$\left\{e^{i(n+\alpha\, sign\, n)t}\right\}_{n\in Z},$$

forms a basis for $L_{p(\cdot),\omega}$.

Note that the cases $\gamma_2 = \gamma_3$ and $p(x) \equiv p = const$ coincides with the results of $[9;10]$.

5.5. Let us provide another interesting result, which follows directly from Statement 4.

Corollary 4. *Let* $p(\cdot) \in WL, p^- > 1$ *and the following inequalities be satisfied*

$$-1 < \gamma_k < p_k - 1, k = \overline{0,3}.$$

Then the system of exponents

$$\left\{e^{i(nt+\alpha\, sign\, n|t|)}\right\}_{n\in Z},$$

forms a basis for $L_{p(\cdot),\rho}$, *for* $\forall \alpha \in R$.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Paley R., Wiener N. Fourier Transforms in the Complex Domain. Amer. Math. Soc. Coll. Publ. 19, New York, 1934.

[2] Levinson N. Gap and density theorems, Amer. Math. Soc. Coll. Publ. 26, New York, 1940.

[3] Кадец М.И. О точном значении константы Палея-Винера. ДАН СССР, 1964.

[4] Young R.M. An Introduction to Nonharmonic Fourier series. Springer, 1980.

[5] Sedletskii A.M., Biorthogonal expansions of functions in series of exponents on intervals of the real axis, Russian Math. Surveys, 37(5)(1982), 57–108

[6] Moiseev E.I. On basicity of systems of sines and cosines. DAN SSSR, 275(4)(1984), 794-798.

[7] He X., Volkmer H. Riesz bases of solutions of Sturm-Liouville equations, J. Fourier Anal. Appl., 7(3)(2001), 297-307

[8] Hunt R.A., Young W.S. A weighted norm inequality for Fourier series. Bull. Amer. Math. Soc., 80(2)(1974), 274-277.

[9] Moiseev E.I. On Basicity of Systems of Sines and Cosines in Weighted Space. Diff. uravneniya, 34(1)(1998), 40-44.

[10] Pukhov S.S., Sedletskii A.M., Bases of exponentials, sines, and cosines in weighted spaces on a finite interval, Dokl. Akad. Nauk, 425(4)(2009), 452-455.

[11] Babenko K.I., On conjugate functions, Doklady-Akad.-Nauk-SSSR, 62(2)(1948), 157-160.

[12] Gaposhkin V.F. On a generalization of M. Riesz theorem on conjugated functions. Mat. sbornik, 46(3)(1958), 111-115.

[13] Veliev S.G. Bases from subsets of eigenfunctions of two discontinuous differential operators. Mathematical Physics, Analysis, Geometry, 12(2)2005, 148-157.

[14] Bilalov B.T., Veliev S.G. Bases of eigenfunctions of two discontinuous differential operators. Differential Equations, 42(10) (2006): 1503-1506.

[15] Bilalov B.T. Basicity of Some Systems of Exponents, Cosines and Sines. Diff. Uravneniya. 26(1)(1990), 10-16.

[16] Bilalov B.T. Basis Properties of Some Systems of Exponents, Cosines and Sines. Sibirskiy Matem. Jurnal, 45(2)(2004), 264-273

[17] Danilyuk I. I. Irregular boundary value problems in the plane. Nauka, Moscow, 1975.

[18] Bilalov B.T. Basicity of some systems of functions. Diff. uravneniya, 25(1989), 163-164 (*in Russian*).

[19] Bilalov B.T. Necessary and sufficient condition for completeness of some system of functions. *Differ. Uravneniya*, 27(1)(1991), 158-161.

[20] Bilalov B.T. Completeness and minimality of some trigonometric systems. Diff. uravneniya, 28(1)(1992), 170-173 (*in Russian*).

[21] Bilalov B.T., Guseynov Z.G. Basicity of a system of exponents with a piece-wise linear phase in variable spaces, Mediterr. J. Math. 9(3)(2012), 487–498.

[22] Bilalov B.T., Guseynov Z.G. Basicity criterion for perturbed systems of exponents in Lebesgue spaces with variable summability, Dokl. RAN, 436(5)(2011), 586-589 (*in Russian*).

[23] Bilalov B.T., Guseynov Z.G. Bases from exponents in Lebesque spaces of functions with variable summability exponent. Trans. of NAS of Azerbaijan, 28(1)(2008), 43-48.

[24] Najafov T.I., Nasibova N.P. On the Noetherness of the Riemann Problem in Generalized Weighted Hardy Classes, Azerbaijan Journal of Mathematics, 5(2) (2015), 109-125.

[25] Bilalov B.T. On completeness of exponent system with complex coefficients in weight spaces. Trans. of NAS of Azerbaijan, 25(7)(2005), 9 – 14.

[26] Kovacik O., Rakosnik J. On spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. Czechoclovak Math. I., 41(116)(1991), 592–618.

[27] Xianling F., Dun Z. On the spaces $L^{p(x)}(\Omega)$ and $L^{m,p(\cdot)}(\Omega)$. Journal of Math. Anal. and Appl., 263(2001), 424–446.

[28] Sharapudinov I.I. Topology of space $L^{p(x)}([0,1])$. Mat. Zametki, 26(4)(1979), 613 – 632.

[29] Cruz-Uribe D.V., Fiorenza A. Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Springer, 2013.

[30] Kokilashvili V., Samko S. Singular integrals in weighted Lebesgue spaces with variable exponent, Georgian Math. J., 10(1) (2003) 145-156

[31] Muradov T.R. On bases from perturbed system of exponents in Lebesgue spaces with variable summability exponent. J. Inequal. Appl., 2014(2014), Article ID 495.

[32] Sharapudinov I.I. Some problems of approximation theory in spaces $L^{p(x)}(E)$ // Analysis Math., 33(2)(2007), 135–153.