A CHARACTERIZATION OF ONE-SIDED BEST SIMULTANEOUS $L_1$–APPROXIMATION

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Abstract. In this paper, we give a simplified proof of the characterization theorem of a one-sided $L_1$–approximation in [2] and the continuity of the function $F \rightarrow C_{M(F)}(F)$ in [3].

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1. Introduction

Let $X$ be a compact Hausdorff space, $C(X)$ be the space of real-valued continuous functions on $X$. For each $f \in C(X)$, define $\|f\|_1 = \int_X |f|d\mu$, where $\mu$ is an admissible measure defined on $X$, that is, $\mu(O) > 0$ for every non-empty open set $O \subset X$. Let $C_1(X)$ be the space $C(X)$ equipped with the norm $\| \cdot \|_1$.

Let $F$ be a compact subset of $C_1(X)$ and $M$ be a finite-dimensional linear subspace of $C_1(X)$. Define

$$M(F) = \{ g \in M \mid g \leq f \text{ for all } f \in F \} = \bigcap_{f \in F} \{ g \in M \mid g \leq f \},$$

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where \( g \leq f \) if and only if \( g(x) \leq f(x) \) almost everywhere in \( X \). Denote by \( d(F, M(F)) := \inf_{g \in M(F)} \max_{f \in F} \| f - g \|_1 \) and \( C_{M(F)} = \{ g \in M(F) \mid \max_{f \in F} \| f - g \|_1 = d(F, M(F)) \} \). The elements of \( C_{M(F)}(F) \) are called one-sided simultaneous best \( L_1 \)-approximants of \( F \) in \( M \). The set \( M(F) \) is a closed convex subset of the finite dimensional subspace \( M \) of \( X \). In order to ensure that the set \( M(F) \) is non-empty, it is enough to assume that \( M \) contains a strictly positive function. In fact, \( M(F) \neq \emptyset \) if and only if \( M \) contains a positive function. Every bounded set has a one-sided best simultaneous approximation in the closed convex set \( M(F) \) of \( M \) and the set function \( F \rightarrow C_{M(F)}(F) \) is continuous on \( B[C_1(X)] \) with a condition that the sets \( M(\cdot) \) are equal, where \( B[C_1(X)] \) is the space of non-empty bounded subsets in the space \( C_1(X) \) and \( C[C_1(X)] \) the family of non-empty compact subsets in the space \( C_1(X) \).

**Definition.** The Hausdorff metric on \( B[C_1(X)] \) is defined by

\[
H(A, B) = \max\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \}.
\]

The motivation is the one-sided best approximation of an element, studied by [1] and the parametric approximation in [4].

**2. Main results**

**Theorem 1.** Let \( M \) and \( M(F) \) be as defined above and \( g^* \in M(F) \). Then \( g^* \in C_{M(F)}(F) \) if and only if \( \sup_{g \in M(F)} \int_X gd\mu = \int_X g^*d\mu \).

**Proof.** Assume that \( g^* \in C_{M(F)}(F) \). Then, for each \( g \in M(F) \), \( \max_{f \in F} \| f - g^* \|_1 \leq \max_{f \in F} \int_X |f - g^*|d\mu \leq \max_{f \in F} \int_X |f - g|d\mu \).

\[
\Leftrightarrow \max_{f \in F} \int_X (f - g^*)d\mu \leq \max_{f \in F} \int_X (f - g)d\mu
\]

\[
\Leftrightarrow \max_{f \in F} \int_X f d\mu - \int_X g^*d\mu \leq \max_{f \in F} \int_X f d\mu - \int_X gd\mu
\]

\[
\Leftrightarrow \int_X gd\mu \leq \int_X g^*d\mu. \text{ Thus } \sup_{g \in M(F)} \int_X gd\mu = \int_X g^*d\mu.
\]

Conversely, let \( g^* \in M(F) \) and \( \sup_{g \in M(F)} \int_X gd\mu = \int_X g^*d\mu \). Then, for all \( g \in M(F) \),

\[
\max_{f \in F} \| f - g \|_1 = \max_{f \in F} \int_X |f - g|d\mu = \max_{f \in F} \int_X (f - g)d\mu
\]
Conversely, let 

\[ g \in M \]

be a finite-dimensional subspace of \( C_1(X) \). Suppose \( \int_X gd\mu \neq 0 \) for some \( g \in M \) and that there is a \( g_0 \in M \) such that \( g_0 < f \) on \( X \) for all \( f \in F \). Let \( g^* \in M(F) \). Then \( g^* \in C_{M(F)}(F) \) if and only if, for all \( g \in M \) with \( g \leq 0 \) on \( \bigcup_{f \in F} Z(f - g^*) \), we have \( \int_X gd\mu \leq 0 \).

**Proof.** Let \( g^* \in C_{M(F)}(F) \). Then, there is an \( h \in M \) such that \( \int_X hd\mu > 0 \). If \( \bigcup_{f \in F} Z(f - g^*) = \emptyset \), then, there is an \( \epsilon > 0 \) such that \( g^* + \epsilon h \leq f \) for each \( f \in F \). Hence, \( g^* + \epsilon h \in M(F) \) and

\[
\int_X (g^* + \epsilon h)d\mu = \int_X g^*d\mu + \epsilon \int_X hd\mu > \int_X g^*d\mu.
\]

This contradicts the fact that \( g^* \in C_{M(F)}(F) \). Therefore \( \bigcup_{f \in F} Z(f - g^*) \neq \emptyset \).

Assume that there is a \( g \in M \) with \( g \leq 0 \) on \( \bigcup_{f \in F} Z(f - g^*) \) and \( \int_X gd\mu > 0 \). Now by hypothesis, there is a \( g_0 \in M \) such that \( g_0 < f \) on \( X \) for each \( f \in F \). Then, \( \tilde{g} = g^* - g_0 > 0 \) on \( \bigcup_{f \in F} Z(f - g^*) \). Hence, there is a \( \delta > 0 \) such that \( g - \delta \tilde{g} < 0 \) on \( \bigcup_{f \in F} Z(f - g^*) \) and \( \int_X (g - \delta \tilde{g})d\mu = \int_X gd\mu - \delta \int_X \tilde{g}d\mu > 0 \). For each \( f \in C_1(X) \), let

\[ J(f) := \{ x \in X \mid f(x) < 0 \} \]

Then \( \bigcup_{f \in F} (f - g^*) \subset J(g - \delta \tilde{g}) \). The set \( X \setminus J(g - \delta \tilde{g}) \) is compact. Hence there is a constant \( m > 0 \) such that \( m \leq f - g^* \) on \( X \setminus J(g - \delta \tilde{g}) \) and there is a constant \( M \) such that \( g - \delta \tilde{g} \leq M \) on \( X \). Let \( \epsilon = \frac{m}{M} \) and \( \hat{h} = g - \delta \tilde{g} \).

Then \( g^* + \hat{h} \in M(F) \). Taking integral, we obtain \( \int_X g^*d\mu < \int_X \hat{h}d\mu \), which contradicts \( g^* \in C_{m(F)}(F) \).

Conversely, let \( g \in M(F) \). Now for all \( x \in \bigcup_{f \in F} Z(f - g^*) \), there is an \( f \in F \) such that \( f(x) = g^*(x) \). Hence \( g(x) \leq g^*(x) \). Therefore \( g - g^* \leq 0 \) on \( \bigcup_{f \in F} Z(f - g^*) \). By assumption
\[ \int_X (g - g^*) \, d\mu \leq 0 \] which implies that \( \int_X g \, d\mu \leq \int_X g^* \, d\mu \). Thus, \( \sup_{g \in M(F)} \int_X g \, d\mu = \int_X g^* \, d\mu \) and hence \( g^* \in C_{M(F)}(F) \). \( \square \)

**Lemma 3.** [2] Let \( M \) be an \( n \)-dimensional subspace of \( C(X) \) and assume that \( \int_X g \, d\mu \neq 0 \) for some \( g \in M \). Let \( K \) be a closed subset of \( X \) with the property that, if \( g \in M \) satisfies \( g(x) \leq 0 \) for all \( x \in K \), then \( \int_K g \, d\mu < 0 \). Then there exist points \( x_1, x_2, \ldots, x_k \in K \), \( 1 \leq k \leq n \) and positive numbers \( \lambda_1, \lambda_2, \ldots, \lambda_k \) such that \( \int_X g \, d\mu = \sum_{i=1}^k \lambda_i g(x_i) \) for each \( g \in M \).

**Theorem 4.** Let \( M \) be an \( n \)-dimensional linear subspace of \( C^1(X) \) such that \( \int_X g \, d\mu \neq 0 \) for some \( g \in M \). Assume that there is a \( g_0 \in M \) such that \( g_0 < f \) for each \( f \in F \). Then \( g^* \in C_{M(F)}(F) \) if and only if, there are \( 1 \leq k \leq n \) distinct points \( x_1, x_2, \ldots, x_k \in \bigcup_{f \in F} Z(f - g^*) \) and \( k \) positive numbers \( \lambda_1, \lambda_2, \ldots, \lambda_k \) such that \( \int_X g \, d\mu = \sum_{i=1}^k \lambda_i g(x_i) \) for each \( g \in M \).

**Proof.** Assume \( g^* \in C_{M(F)}(F) \). Then, by the above theorem, for each \( g \in M \) with \( g \leq 0 \) on \( \bigcup_{f \in F} Z(f - g^*) \), we have \( \int_X g \, d\mu \leq 0 \). Let \( K = \bigcup_{f \in F} Z(f - g^*) \), then by the lemma, there are points \( x_1, x_2, \ldots, x_k \in K \) and positive numbers \( \lambda_1, \lambda_2, \ldots, \lambda_k \) such that \( \int_X g \, d\mu = \sum_{i=1}^k \lambda_i g(x_i) \) for each \( g \in M \).

Conversely, assume there are \( k \) distinct points \( x_1, x_2, \ldots, x_k \in \bigcup_{f \in F} Z(f - g^*) \) and \( k \) positive numbers \( \lambda_1, \lambda_2, \ldots, \lambda_k \) such that \( \int_X g \, d\mu = \sum_{i=1}^k \lambda_i g(x_i) \) for each \( g \in M \). Then for each \( g \in M(F) \),

\[
\int_X g \, d\mu = \sum_{i=1}^k \lambda_i g(x_i) \\
\leq \sum_{i=1}^k \lambda_i f(x_i) \quad \text{for all } f \in F = \sum_{i=1}^k \lambda_i g^*(x_i) = \int_X g^* \, d\mu. \quad \text{That is} \\
\int_X g \, d\mu = \int_X g^* \, d\mu \text{ for each } g \in M(F). \quad \text{Hence } \sup_{g \in M(F)} \int_X g \, d\mu = \int_X g^* \, d\mu. \quad \text{Therefore} \\
g^* \in C_{M(F)}(F). \quad \square \]
Theorem 5. Let $M$ be a finite-dimensional subspace of $C_1(X)$. For any $F, G \in B[C_1(X)]$ with $M(F) = M(G)$ and any $\epsilon > 0$, there exists $\delta > 0$ such that $H(F, G) < \delta$ implies $H(C_{M(F)}(F), C_{M(G)}(G)) < 2\epsilon$.

Proof. For any $\epsilon > 0$, let $0 < \delta < \frac{\epsilon}{2} < \min(d(F, M(F)), d(G, M(G)))$ where $F, G \in B[C_1(X)]$ with $M(F) = M(G)$. Assume $H(F, G) < \delta$. Then, for any $x \in C_1(X), |d(x, F) - d(x, G)| \leq \delta$. In fact, for any $u \in F$, there exists $v \in G$ such that $\|u - v\| < \delta$. Then $\|u - v\| - \|x - v\| \leq \|u - v\| < \delta$. Then $\|u - x\| \leq \|x - v\| + \delta$.

Thus $d(x, F) \leq d(x, G) + \delta$. Similarly, $d(x, G) \leq d(x, F) + \delta$. For any $x \in C_{M(G)}(G), d(x, F) \leq d(x, G) + \delta$. Thus $d(M(F)), F) \leq d(M(G), G) + \delta$. Hence $|d(M(F)), F) - d(M(G), G)| \leq \delta$.

For any $z \in C_{M(F)}(F)$,

$d(z, G) \leq d(z, F) + \delta = d(M(F), F) + \delta$

$\leq d(M(G), G) + 2\delta$

$\leq d(M(G), G)(1 + \epsilon)$.

There exists $w \in C_{M(G)}(G)$ with $\|z - w\| \leq 2\epsilon$,

so $\sup_{z \in C_{M(F)}(F)} \inf_{w \in C_{M(G)}(G)} \|z - w\| \leq 2\epsilon$. Hence $H(C_{M(F)}(F)), C_{M(G)}(G)) \leq 2\epsilon$. □

References


