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SI-RINGS AND THEIR EXTENSIONS AS 2-PRIMAL RINGS

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Abstract. Let R be a ring, σ an automorphism of R such that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$, where $N(R)$ is the set of nilpotent elements of R and δ a σ-derivation of R such that $\delta(P) \subseteq P$, for all minimal prime ideal P of R. We recall that a ring R is called an *SI*-ring if for $a, b \in R$, $ab = 0$ implies $aRb = 0$. In this paper we show that if R is a commutative Noetherian *SI*-ring, which is also an algebra over $\mathbb{Q}$ and σ and δ be as above, then $R[x;\sigma,\delta]$ is 2-primal.

Keywords: 2-primal rings; minimal prime; automorphism; derivation; *SI*-rings; weak σ-rigid rings; Ore extensions.

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1. Introduction

All rings are associative with identity $1 \neq 0$. Let R be a ring, σ be an endomorphism of R and δ be a σ-derivation of R. Then $\delta : R \to R$ is an additive map such that $\delta(ab) = \delta(a)\sigma(b) +$

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$a\delta(b)$, for all a, $b \in R$. For example let σ be an automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be a defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$

Then δ is a σ-derivation of R if and only if ϕ is a homomorphism. In case σ is the identity map, δ is called just a derivation of R. For example for any endomorphism τ of a ring R and for any $a \in R$, $\rho : R \to R$ defined as $\rho(r) = ra - a\tau(r)$ is a τ-derivation of R.

Now let R be a ring, σ an automorphism of R and δ a σ-derivation of R. Then $R[x;\sigma,\delta] = \{f = \Sigma_{i=0}^{n} x^i a_i,\ a_i \in R,\ n \in \mathbb{N}\}$ subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We denote the Ore extension $R[x;\sigma,\delta]$ by $O(R)$. In case σ is the identity map, we denote the ring of differential operators $R[x;\delta]$ by $D(R)$. If δ is the zero map, we denote the skew polynomial ring $R[x;\sigma]$ by $S(R)$.

The field of rational numbers, the field of complex numbers and the set of positive integers are denoted by $\mathbb{Q}$, $\mathbb{C}$ and $\mathbb{N}$ respectively unless otherwise stated. $Spec(R)$ denotes the set of prime ideals of R. $MinSpec(R)$ denotes the set of minimal prime ideals of R. For a ring R the prime radical is denoted by $P(R)$ and the set of nilpotent elements is denoted by $N(R)$. In this paper we will discuss SI-property over Ore extensions.

Definition 1.1 (Birkenmeier-Heatherly-Lee [4]). A ring R is 2-primal if and only if the set of nilpotent elements and prime radical of R are same if and only if the prime radical is a completely semi prime ideal.

An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$, where $a \in R$.

Example 1.2 1.A reduced ring is 2-primal. For a commutative ring $P(R)$ and $N(R)$ coincide, so it is also 2-primal.

2. Let $R = F[x]$ be the polynomial ring over the field F. Then R is 2-primal with $P(R) = \{0\}$.

3. Let $R = M_2(Q)$, the set of 2×2 matrices over Q. Then $R[x]$ is a prime ring with non-zero nilpotent elements and, so can not be 2-primal.

They also introduced the concept of 2-primal ideal. Shin in [12], showed that every proper ideal of a ring R is 2-primal if and only if every prime ideal of R is completely prime in Proposition (1.13) of [12]. He also proved that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime in Proposition (1.11) of [12]. Birkenmeier-Heatherly-Lee provided various examples relating to this equivalent condition in [4]. The 2-primal property of $O(R)$, where R is a local ring, σ is an automorphism of R and δ is a σ-derivation of R is also discussed by Greg Marks in [9]. The study of 2-primal condition was continued by Hirano [5] and Sun [11], etc.

This article concerns the study of weak σ-rigid *SI*- rings and their extensions in terms of 2-primal rings.

2. Weak σ-rigid rings and *SI*-rings

Definition 2.1(Krempa [6]). Let R be a ring and σ an endomorphism of R. Then σ is said to be a rigid endomorphism if $a\sigma(a) = 0$ implies that $a = 0$, for $a \in R$. The ring R is said to be a σ-rigid ring if there exists a σ-rigid endomorphism R.

For example let $R = \mathbb{C}$, and σ: $\mathbb{C} \to \mathbb{C}$ be the map defined by $\sigma(a+ib) = a-ib$, a, $b \in R$. Then it can be seen that σ is a rigid endomorphism of R.

Definition 2.2 (Kwak [7]). Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 2.3(Example (2) of [7]). Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field.

Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \to R$ defined by $\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.

We note that the above ring is not σ-rigid. For let $0 \neq a \in F$. Then
$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma\Big(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\Big) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Ouyang in [10] introduced weak σ-rigid rings, where σ is an endomorphism of ring R. These rings are related to 2-primal rings.

Definition 2.4 (Ouyang [10]). Let R be a ring and σ an endomorphism of R such that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$. Then R is called a weak σ-rigid ring.

Example 2.5 (Example (2.1) of Ouyang [10]). Let σ be an endomorphism of a ring R such that

R is a σ-rigid ring. Let $A = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$ be a subring of $T_3(R)$, the ring of upper triangular matrices over R. Now σ can be extended to an endomorphism $\overline{\sigma}$ of A by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$. Then it can be seen that A is a weak $\overline{\sigma}$-rigid ring.

Definition 2.6 (Shin [12]). Let R be a ring. Then R is called an *SI*-ring if for $a, b \in R$, $ab = 0$ implies $aRb = 0$.

Example 2.7 1. Let $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in \mathbb{Z} \right\}$.

The only matrices A and B satisfying $AB = 0$ are of the type

$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$; $a, b \in \mathbb{Z}$.

i.e., $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$. Now for all $K = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in R$,

$$AB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{implies } AKB = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \right)$$

$$= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & db \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies R is an *SI*-ring.

2. Reduced rings (i.e., rings without nonzero nilpotent elements) are obviously *SI*-rings, right (left) duo rings are *SI*-rings by ([12], Lemma 1.2). Shin showed that *SI*-rings are 2-primal in ([12], Theorem 1.5), and so reduced rings are 2-primal.

3. We take the rings in Example (5.3) of [12]. Let $F = \mathbb{Z}_2(y)$ be the field of rational functions

over $\mathbb{Z}_2$ with y an indeterminate. Consider the ring $R = \{f(x) \in F[x] \mid xy + yx = 1\}$. Then clearly R is a domain, so it is reduced and hence an *SI*-ring.

Now let R be a commutative Noetherian *SI*-ring, σ be an automorphism of R such that R is a weak σ-rigid ring and δ be a σ-derivation of R such that $\delta(P) \subseteq P$, for all $P \in MinSpec(R)$. Then $O(R)$ is 2-primal if and only if $O(P(R)) = P(O(R))$. This is proved in Proposition 3.15.

3. Main results

Definition 3.1 Symmetric and almost symmetric rings: In Lambek [8], a ring R is called *symmetric* provided $abc = 0$ implies $acb = 0$ for any $a, b, c \in R$.

A ring R is called *almost symmetric* if it satisfies:

$(S1)$ For each element $a \in R$, a^r is an ideal of R, where $a^r = \{b \in R : ab = 0\}$; and

$(S2)$ For any a, b, $c \in R$, if $a(bc)^n = 0$ for a positive integer n, then $ab^m c^m = 0$ for some positive integer m.

Remark 3.2 We define $S1$ condition for Ore extension $O(R)$ as:

For each element $f \in O(R)$, f^r is an ideal of $O(R)$, where $f^r = \{g \in O(R) : fg = 0\}$.

Proposition 3.3 For any ring R, the following are equivalent:

- **(a):** R is an *SI*-ring.
- **(b):** Every minimal prime ideal is completely prime.

Proof. Since R is an *SI*-ring. So, by Proposition (1.5) of Shin [12] R is 2-primal implies that $P(R)$ coincides with the set of all nilpotent elements of R. Therefore, by Proposition (1.11) of Shin [12] every minimal prime ideal is completely prime.

Proposition 3.4 Let R be a ring. Then R is an *SI*-ring implies that $P(R)$ is completely semiprime.

Proof. Since R is an *SI*-ring. So, by Proposition (1.5) of Shin [12] R is 2-primal implies that $P(R)$ is completely semiprime.

Theorem 3.5 Let R be a Noetherian (even commutative) *SI*-ring. Let σ be an automorphism of R such that R is a weak σ-rigid ring. Then R is a $\sigma(*)$-ring.

Proof. Let R be an *SI*-ring. Then R is 2-primal, so $N(R) = P(R)$. Since R is a weak σ-rigid ring, so $a\sigma(a) \in N(R)$ implies that $a \in N(R)$. Therefore, $a\sigma(a) \in P(R)$ implies that $a \in P(R)$. Hence R is a $\sigma(*)$-ring.

Theorem 3.6 Let R be a commutative Noetherian ring. Then R is an *SI*-ring implies that $N(R)$ is completely semiprime.

Proof. The proof is obvious by Proposition () and Theorem (1.5) of Shin [12].

Corollary 3.7 Let R be a commutative Noetherian ring and σ be an automorphism of R. Then R is a weak σ-rigid *SI*-ring if and only if for each minimal prime P of R, $\sigma(P) = P$ and P is a completely prime ideal of R.

Proof. Combining Theorem 3.5 and Theorem (5) of [3].

Theorem 3.8 Let R be an *SI*-ring. Then:

For any minimal prime ideal P of R with $\delta(P) \subseteq P$, $O(P)$ is completely prime ideal of $O(R)$.

Proof. Since R is an *SI*-ring, so every minimal prime ideal of R is completely prime by Proposition 3.3. This implies P is completely prime ideal of R with $\delta(P) \subseteq P$. Which implies that $O(P)$ is a completely prime ideal of $O(R)$ by Proposition (2.2) of [1].

Proposition 3.9 Let R be a ring and f^r be an ideal of $O(R)$ for all $f \in O(R)$. Then $O(R)$ is an *SI*-ring implies $O(R)/P(O(R))$ is also an *SI*-ring.

Proof. Let $O(R)$ be an *SI*-ring. We have to show that $O(R)/P(O(R))$ is an *SI*-ring. Let $f + P(O(R)), g + P(O(R)) \in O(R)/P(O(R))$ be such that $(f + P(O(R)))(g + P(O(R))) = P(O(R))$. This implies that $fg + P(O(R)) = P(O(R))$, i.e., $fg \in P(O(R))$. Now, $O(R)$ is *SI*-ring. Therefore for $j, k \in O(R)$ such that $jk = 0$ implies $j(O(R))k = 0$, i.e., $jlk = 0$, for all $l \in O(R)$......(1) Now for all $h + P(O(R)) \in O(R)/P(O(R))$; $(f + P(O(R)))(h + P(O(R)))(g + P(O(R))) = fhg + P(O(R))$. Since for all $f \in O(R)$, f^r is given to be an ideal of $O(R)$, where $f^r = \{g \in O(R) : fg = 0\}$. This implies by (1) that $fhg = 0$ so that $(f + P(O(R)))(h + P(O(R)))(g + P(O(R))) = P(O(R))$; for all $h + P(O(R)) \in O(R)/P(O(R))$. Hence, $O(R)/P(O(R))$ is an *SI*-ring.

Theorem 3.10 Let R be a ring and f^r be an ideal of $O(R)$ for all $f \in O(R)$. Then $O(R)$ is an *SI*-ring implies $O(R)/P(O(R))$ is a 2-primal ring.

Proof. It is enough to show that $O(R)/P(O(R))$ is an *SI*-ring. Rest is obvious by Theorem (1.5) of Shin [12].

Corollary 3.11 Let R be a ring and f^r be an ideal of $O(R)$ for all $f \in O(R)$. Then $O(R)/P(O(R))$ is a 2-primal ring.

Proof. For this it is enough to show that $O(R)$ is an *SI*-ring (by Lemma (1.2) of [12]). Rest is obvious by above Theorem 3.10.

Theorem 3.12 Let R be commutative Noetherian SI-ring. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ be a σ-derivation of R such that $\delta(P) \subseteq P$, for all $P \in MinSpec(R)$. Then $O(P)$ is completely prime ideal of $O(R)$.

Proof. Since R is weak σ-rigid SI-ring, so we have $\sigma(P) = P$ and P is completely prime ideal of R by Corollary 3.7. So, P is completely prime ideal of R and $\delta(P) \subseteq P$. Therefore, $O(P)$ is a completely prime ideal of $O(R)$ by Theorem 3.8.

Theorem 3.13 Let R be a commutative Noetherian SI-ring. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ be a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. Then $\delta(P) \nsubseteq P$ implies $\sigma(P) \neq P$.

Proof. Suppose $\delta(P) \nsubseteq P$, i.e., let $a \in P$ be such that $\delta(a) \notin P$. To show $\sigma(P) \neq P$. Suppose $\sigma(P) = P$. Then by Corollary 3.7 P is completely prime ideal of R. Therefore for any $a \in P$ there exits $b \notin P$ such that $ab \in P(R)$ by Corollary (1.10) of Shin [12]. Now $\delta(P(R)) \subseteq P(R)$, and therefore $\delta(ab) \in P(R)$ i.e., $\delta(a)\sigma(b) + a\delta(b) \in P(R) \subseteq P$. Now $a\delta(a) \in P$ implies $\delta(a)\sigma(b) \in P$ implies either $\delta(a) \in P$ or $\sigma(b) \in P$.

Case I: If $\delta(a) \in P$, a contradiction.

Case II: If $\sigma(b) \in P$, but $b \notin P$ implies $\sigma(b) \notin \sigma(P) = P$, a contradiction.

Therefore, $\sigma(P) \neq P$.

Proposition 3.14 Let R be a commutative Noetherian SI-ring. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ be a σ-derivation of R such that $\delta(P) \subseteq P$, for all $P \in MinSpec(R)$. Then $O(R)$ is 2-primal if and only if $O(P(R)) = P(O(R))$.

Proof. Let $O(R)$ be 2-primal. Now R is weak σ-rigid SI-ring implies $\sigma(P) = P$ by Corollary 3.7. Then by Theorem 3.12 $P(O(R)) \subseteq O(P(R))$. Let $f(x) = \Sigma_{j=0}^{n} x^j a_j \in O(P(R))$. Now R is a 2-primal subring of $O(R)$ by Theorem (1.5) of Shin [12], which implies that a_j is nilpotent and thus $a_j \in N(O(R)) = P(O(R))$, and so we have $x^j a_j \in P(O(R))$ for each j, $0 \leq j \leq n$, which implies that $f(x) \in P(O(R))$. Hence $O(P(R)) = P(O(R))$.

Conversely suppose $O(P(R)) = P(O(R))$. We will show that $O(R)$ is 2-primal. Let $g(x) = \Sigma_{i=0}^{n} x^i b_i \in O(R)$, $b_n \neq 0$, be such that $(g(x))^2 \in P(O(R)) = O(P(R))$. We will show that $g(x) \in P(O(R))$. Now leading coefficient $\sigma^{2n-1}(b_n)b_n \in P(R) \subseteq P$, for all $P \in MinSpec(R)$.

Now since R is weak σ-rigid *SI*-ring we have $\sigma(P) = P$ and P is completely prime by Corollary (). Therefore we have $b_n \in P$, for all $P \in MinSpec(R)$, i.e., $b_n \in P(R)$. Now since $\delta(P) \subseteq P$ for all $P \in MinSpec(R)$, we get $(\Sigma_{i=o}^{n-1} x^i b_i)^2 \in P(O(R)) = O(P(R))$ and as above we get $b_{n-1} \in P(R)$. With the same process in a finite number of steps we get $b_i \in P(R)$ for all i, $0 \leq i \leq n$. Thus we have $g(x) \in O(P(R))$, i.e., $g(x) \in P(O(R))$. Therefore $P(O(R))$ is a completely semiprime ideal of $O(R)$. Hence $O(R)$ is 2-primal.

Corollary 3.15 Let R be a ring and f^r be an ideal of $O(R)$ for all $f \in O(R)$. Then $O(P(R)) = P(O(R))$.

Proof. Since $O(R)$ is an *SI*-ring by Lemma (1.2) of [12], so it is 2-primal by Theorem (1.5) of [12]. Rest is obvious by Proposition 3.14.

Theorem 3.16 Let R be a commutative Noetherian *SI*-ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ be a σ-derivation of R such that $\delta(P) \subseteq P$, for all $P \in MinSpec(R)$. Then $O(R)$ is 2-primal.

Proof. Let $P_1 \in MinSpec(R)$. Now R is weak σ-rigid *SI*-ring, so Corollary () implies that $\sigma(P_1) = P_1$. Therefore Theorem (2.3) of [2] implies that $O(P_1) \in MinSpec(O(R))$. Similarly for any $P \in MinSpec(O(R))$ such that $\sigma(P \cap R) = P \cap R$, Theorem (2.3) of [2] implies that $P \cap R \in MinSpec(R)$. Therefore, $O(P(R)) = P(O(R))$, and now the result is obvious by using above Proposition 3.14.

Some results for $S(R) = R[x;\sigma]$

Proposition 3.14 and Theorem 3.16 also holds for the ring $S(R)$.

Theorem 3.17 Let R be a Noetherian ring and σ an automorphism of R. Let $S(R) = R[x;\sigma]$ be as usual. Then:

(1) If $P \in MinSpec(S(R))$, then $P = (P \cap R)S(R)$ and there exists $U \in MinSpec(R)$ such that $P \cap R = U^0$.
(2) If $U \in MinSpec(R)$, then $U^0 S(R) \in MinSpec(S(R))$.

Proof. See Theorem (2) of [3].

Proposition 3.18 Let R be a ring and f^r be an ideal of $S(R)$ for all $f \in S(R)$. Then $S(R)$ is an *SI*-ring implies $S(R)/P(S(R))$ is also an *SI*-ring.

Proof. Let $S(R)$ be an SI-ring. We have to show that $S(R)/P(S(R))$ is an SI-ring. Let $f+P(S(R)), g+P(S(R)) \in S(R)/P(S(R))$ be such that $(f+P(S(R)))(g+P(S(R))) = P(S(R))$. This implies that $fg+P(S(R)) = P(S(R))$, i.e., $fg \in P(S(R))$. Now, $S(R)$ is an SI-ring. Therefore for j, $k \in S(R)$ such that $jk = 0$ implies $j(S(R))k = 0$, i.e.,$jlk = 0$, for all $l \in S(R)$....(2) Now for all $h+P(S(R)) \in S(R)/P(S(R))$; $(f+P(S(R)))(h+P(S(R)))(g+P(S(R))) = fhg+P(S(R))$. Since for all $f \in S(R)$, f^r is given to be an ideal of $S(R)$, where $f^r = \{g \in S(R) : fg = 0\}$. This implies by (2) that $fhg = 0$ so that $(f+P(S(R)))(h+P(S(R)))(g+P(S(R))) = P(S(R))$; for all $h+P(S(R)) \in S(R)/P(S(R))$. Hence, $S(R)/P(S(R))$ is an SI-ring.

Theorem 3.19 Let R be a ring and f^r be an ideal of $S(R)$ for all $f \in S(R)$. Then $S(R)$ is an SI-ring implies $S(R)/P(S(R))$ is a 2-primal ring.

Proof. It is enough to show that $S(R)/P(S(R))$ is an SI-ring. Rest is obvious by Theorem (1.5) of Shin [12].

Proposition 3.20 Let R be a commutative Noetherian SI-ring. Let σ be an automorphism of R such that R is a weak σ-rigid ring. Let $P \in MinSpec(R)$ then $P(S(R)) = P[x;\sigma]$ is a completely prime ideal of $S(R) = R[x;\sigma]$.

Proof. Let $P \in MinSpec(R)$. So $\sigma(P) = P$ by Corollary 3.7. Also R is $\sigma(*)$-ring by Theorem 3.5, so by Proposition (4) of [3] $P(S(R))$ is a completely prime ideal of $S(R)$.

Proposition 3.21 Let R be a commutative Noetherian SI-ring. Let σ be an automorphism of R such that R is a weak σ-rigid ring. Then $S(R)$ is 2-primal if and only if $S(P(R)) = P(S(R))$.

Proof. Let $S(R)$ be 2-primal. Then by Proposition (), $P(S(R)) \subseteq S(P(R))$. Let $f(x) = \Sigma_{j=0}^{n} x^j a_j \in S(P(R))$. Now R is a 2-primal subring of $S(R)$ by Theorem (1.5) of Shin [12], which implies that a_j is nilpotent and thus $a_j \in N(S(R)) = P(S(R))$, and so we have $x^j a_j \in P(S(R))$ for each j, $0 \leq j \leq n$, which implies that $f(x) \in P(S(R))$. Hence $S(P(R)) = P(S(R))$.

Conversely suppose $S(P(R)) = P(S(R))$. We will show that $S(R)$ is 2-primal. Let $g(x) = \Sigma_{i=0}^{n} x^i b_i \in S(R)$, $b_n \neq 0$, be such that $(g(x))^2 \in P(S(R)) = S(P(R))$. We will show that $g(x) \in P(S(R))$. Now leading coefficient $\sigma^{2n-1}(b_n)b_n \in P(R) \subseteq P$, for all $P \in MinSpec(R)$. Now since R is weak σ-rigid SI-ring we have $\sigma(P) = P$ and P is completely prime by Corollary 3.7. Therefore we have $b_n \in P$, for all $P \in MinSpec(R)$, i.e., $b_n \in P(R)$. Now since $\sigma(P) = P$ for all $P \in MinSpec(R)$, we get $(\Sigma_{i=o}^{n-1} x^i b_i)^2 \in P(S(R)) = S(P(R))$ and as above we get $b_{n-1} \in P(R)$.

With the same process in a finite number of steps we get $b_i \in P(R)$ for all i, $0 \leq i \leq n$. Thus we have $g(x) \in S(P(R))$, i.e., $g(x) \in P(S(R))$. Therefore $P(S(R))$ is a completely semiprime ideal of $S(R)$. Hence $S(R)$ is 2-primal.

Theorem 3.22 Let R be a commutative Noetherian SI-ring. Let σ be an automorphism of R such that R is a weak σ-rigid ring. Then $S(R)$ is 2-primal.

Proof. We use Theorem 3.17 to get that $S(P(R)) = P(S(R))$, and now the result is obvious by using Proposition 3.21.

Conflict of Interests

The authors declare that there is no conflict of interests.

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