1. Introduction

Many modern electro-mechanical materials and items tend to have a long life under normal-use operating conditions. Hence it is difficult to test their failure times since standard testing procedures are far too lengthy and expensive to be useful. However, accelerated life tests (ALTs)
offer an alternative that manufacturing industries prefer due to the ability to obtain enough failure data in a short period of time.

In an ALT the test items are subjected to higher than usual levels of stress to induce early failures. The stress can be applied in different ways including the constant stress and step stress techniques. A step stress ALT is often preferred to a constant stress ALT because it reduces overall test time and enables quicker failures see [5-8]. We consider here m-step stress ALTs where \( n \) identical units are placed on a life-test with an initial stress level \( x_1 \) which is changed to \( x_2 \) at a fixed time \( \tau_1 \) and the successive failure times are recorded. Then, at the fixed time \( \tau_2 \), the stress is increased to \( x_3 \). Thus the resulting failure times are observed in a naturally ordered manner.

Cumulative exposure models are often useful in the analysis of step-stress experiments. These models relate the life distribution of the test units at one stress level to the distributions at preceding stress levels by assuming that the residual lives of the experimental units depend only on the cumulative exposure that the units have experienced, with no memory of how the stress was accumulated. Moreover, the surviving units will fail according to the cumulative distribution at the same stress level that is currently being tested at, but starting at the previous accumulated stress level. For more discussion see [1].

We develop a model for 3-step stress ALTs based on the lifetime distribution following a generalized exponential distribution see [2]. We then show how the observed ordered failure times can be used to do maximum likelihood estimation and Bayesian estimation of the parameters of the distribution of failure times under normal operating conditions see [9,10]. Finally, we study the performance of these methods in a simulation study under Type-II censoring.

2. Model description

We assume that the lifetime \( T \) follows a two-parameter generalized exponential distribution, denoted \( GE(\alpha, \lambda) \) where \( \lambda \) is a scale parameter and \( \alpha \) is a shape parameter. The two-parameter \( GE(\alpha, \lambda) \) distribution can be used quite effectively in analyzing lifetime data, particularly in place of two-parameter gamma or Weibull distributions see [3,4].
The GE($\alpha, \lambda$) probability density function (pdf) and cumulative distribution function (cdf) are given, respectively, by

$$f(t; \alpha, \lambda) = \alpha \lambda (1 - e^{-\lambda t})^{\alpha-1} e^{-\lambda t}, \quad \alpha, \lambda > 0.$$  \hspace{1cm} (2.1)

and

$$F(t; \alpha, \lambda) = (1 - e^{-\lambda t})^{\alpha}.$$  \hspace{1cm} (2.2)

The survival (sf) and hazard rate functions (hrf) are

$$\bar{F}(t; \alpha, \lambda) = 1 - (1 - e^{-\lambda t})^{\alpha},$$  \hspace{1cm} (2.3)

and

$$h(t; \alpha, \lambda) = \frac{\alpha \lambda (1 - e^{-\lambda t})^{\alpha-1} e^{-\lambda t}}{1 - (1 - e^{-\lambda t})^{\alpha}}.$$  \hspace{1cm} (2.4)

respectively. For any $\lambda$ the GE distribution has an increasing hrf if $\alpha > 1$, while the hrf is decreasing if $\alpha < 1$. Of course, if $\alpha = 1$, then the hrf is constant.

2.1. Basic assumptions

We assume that the lifetime distribution functions at stress levels $x_1, x_2$ and $x_3$ are $F_1, F_2$ and $F_3$, respectively, and that they belong to the same family of distributions. The experiment starts with $n$ identical units, and each unit is subjected to an initial stress $x_1$ with lifetimes following the CDF $F_1(t)$. The time at which a unit failed will be collected and the surviving units will continue until time $\tau_1$ at which the stress is increased to $x_2$ and the units will follow the CDF $F_2(t)$, once again the time at which a unit failed will be collected and the surviving units will continue until time $\tau_2$ at which the stress is increased to $x_3$ and the units will follow the CDF $F_3(t)$, but it will start at the previously accumulated fraction failed.

Thus the change in stress level from $x_1$ to $x_2$ changes the lifetime distribution at stress level $x_2$ from $F_2(t)$ to $F_2(t - \tau_1 + \hat{\tau}_1)$, also the change in stress level from $x_2$ to $x_3$ changes the lifetime distribution at stress level $x_3$ from $F_3(t)$ to $F_3(t - \tau_2 + \hat{\tau}_2)$ where

$$F_1(\tau_1) = F_2(\hat{\tau}_1).$$ \hspace{1cm} (2.5)

$$F_2(\tau_2) = F_3(\hat{\tau}_2).$$
Assuming that $\lambda_1, \lambda_2$ and $\lambda_3$ are the scale parameters associated with $F_1$, $F_2$ and $F_3$, respectively, and assuming absolute continuity of the cumulative distribution function of the lifetime, we find

\[
\hat{\tau}_1 = \frac{\lambda_1}{\lambda_2} \tau_1.
\]

\[
\hat{\tau}_2 = \frac{\lambda_2}{\lambda_3} [\tau_2 - \tau_1 + \frac{\lambda_1}{\lambda_2} \tau_1].
\]

Then, the cumulative distribution function of the model, in which there are three stress levels $x_1$, $x_2$ and $x_3$, will become

\[
G(t) = \begin{cases} 
G_1(t) = F_1(t), & \text{for } 0 < t < \tau_1 \\
G_2(t) = F_2(t - \tau_1 + \hat{\tau}_1) & \text{for } \tau_1 \leq t < \tau_2 \\
G_3(t) = F_3(t - \tau_2 + \hat{\tau}_2) & \text{for } \tau_2 \leq t < \infty
\end{cases}
\]

where

\[
F_i(t) = (1 - e^{-\lambda_i t})^\alpha, i = 1, 2, 3.
\]

The corresponding probability density function (PDF) in this case will be in the flowing form:

\[
g(t) = \begin{cases} 
g_1(t) = \alpha \lambda_1 (1 - e^{-\lambda_1 t})^{\alpha - 1} e^{-\lambda_1 t}, & 0 < t < \tau_1 \\
g_2(t) = \alpha \lambda_2 (1 - e^{-\lambda_2 (t - \tau_1 + \hat{\tau}_1)}^{\alpha - 1} e^{-\lambda_2 (t - \tau_1 + \hat{\tau}_1)} & \tau_1 \leq t < \tau_2, \\
g_3(t) = \alpha \lambda_3 (1 - e^{-\lambda_3 (t - \tau_2 + \hat{\tau}_2)}^{\alpha - 1} e^{-\lambda_3 (t - \tau_2 + \hat{\tau}_2)} & \tau_2 \leq t < \infty
\end{cases}
\]

3. Maximum likelihood estimation

In a 3-step-stress model with Type-II censoring, we start with $n$ independent and identical units placed simultaneously on a life-test. Each unit will be subjected to an initial stress level $x_1$, then the experiment will run until a fixed time $\tau_1$ when the stress level is changed to $x_2$, after that the experiment will run until a fixed time $\tau_2$ when the stress level is changed to $x_3$. The experiment is continued until a specified number of failures $r$ is observed.

Let $n_1$ be the number of units that fail before $\tau_1$, $n_2$ be the number of units that fail between $\tau_1$ and $\tau_2$ and $n_3$ is the number of units that fail after $\tau_2$, and so $r = n_1 + n_2 + n_3$. If $r$ failures occur before $\tau_1$ or $\tau_2$, then the test is terminated, otherwise the experiment continues after time $\tau_2$ until the required $r$ failures are observed. The ordered failure times that are observed will be denoted by $(t_1 < ... < t_{n_1} < \tau \leq t_{n_1+1} ... < t_{n_2} < \tau \leq t_{n_2+1} < ... < t_r)$. 
The likelihood function based on the censored data above is given by

\[
L(\alpha, \lambda_1, \lambda_2, \lambda_3; t) = \frac{n!}{r!} \left( \prod_{i=1}^{r} g(t_i) \left(1 - G(t_r)\right)^{n-r}\right)
\]  

(3.1)

where \( r = n_1 + n_2 + n_3 \) and \( t \) is the vector of observed Type-II censored data. The likelihood function of \( \alpha, \lambda_1, \lambda_2 \) and \( \lambda_3 \) is as follows:

1. If \( n_1 = 0 \):

\[
L(\alpha, \lambda_2, \lambda_3; t) = \frac{n!}{r!} \left\{ \prod_{i=1}^{n_2} g_2(y_i) \right\} \left\{ \prod_{i=n_2+1}^{r} g_3(z_i) \right\} \left(1 - G_3(z_r)\right)^{n-r}
\]

\[
= \frac{n!}{r!} \alpha^r \lambda_2^{n_2} \lambda_3^{n_3} e^{-\lambda_2 \sum_{i=1}^{n_2} y_i - \lambda_3 \sum_{i=n_2+1}^{r} z_i}
\]

\[
\times \left\{ \prod_{i=1}^{n_2} \left(1 - e^{-\lambda_2 y_i}\right)^{\alpha-1} \right\} \left\{ \prod_{i=n_2+1}^{r} \left(1 - e^{-\lambda_3 z_i}\right)^{\alpha-1} \right\}
\]

\[
\times \left(1 - (1 - e^{-\lambda_3 z_r})^{\alpha}\right)^{n-r}
\]

2. If \( n_2 = 0 \):

\[
L(\alpha, \lambda_1, \lambda_3; t) = \frac{n!}{r!} \left\{ \prod_{i=1}^{n_1} g_1(t_i) \right\} \left\{ \prod_{i=n_1+1}^{r} g_3(z_i) \right\} \left(1 - G_3(z_r)\right)^{n-r}
\]

\[
= \frac{n!}{r!} \alpha^r \lambda_1^{n_1} \lambda_3^{n_3} e^{-\lambda_1 \sum_{i=1}^{n_1} t_i - \lambda_3 \sum_{i=n_1+1}^{r} z_i}
\]

\[
\times \left\{ \prod_{i=1}^{n_1} \left(1 - e^{-\lambda_1 t_i}\right)^{\alpha-1} \right\} \left\{ \prod_{i=n_1+1}^{r} \left(1 - e^{-\lambda_3 z_i}\right)^{\alpha-1} \right\}
\]

\[
\times \left(1 - (1 - e^{-\lambda_3 z_r})^{\alpha}\right)^{n-r}
\]

3. If \( n_3 = 0 \):

\[
L(\alpha, \lambda_1, \lambda_2; t) = \frac{n!}{r!} \left\{ \prod_{i=1}^{n_1} g_1(t_i) \right\} \left\{ \prod_{i=n_1+1}^{r} g_2(y_i) \right\}
\]

\[
= \frac{n!}{r!} \alpha^r \lambda_1^{n_1} \lambda_2^{n_2} e^{-\lambda_1 \sum_{i=1}^{n_1} t_i - \lambda_2 \sum_{i=n_1+1}^{n_1+n_2} y_i}
\]

\[
\times \left\{ \prod_{i=1}^{n_1} \left(1 - e^{-\lambda_1 t_i}\right)^{\alpha-1} \right\} \left\{ \prod_{i=n_1+1}^{n_1+n_2} \left(1 - e^{-\lambda_2 y_i}\right)^{\alpha-1} \right\}
\]
(4) If \( n_i > 0 \), \( i=1,2,3 \):

\[
L(\alpha, \lambda_1, \lambda_2, \lambda_3; t) = \frac{n!}{r!} \left\{ \prod_{i=1}^{n_1} g_1(t_i) \right\} \left\{ \prod_{i=n_1+1}^{n_1+n_2} g_2(y_i) \right\} \left\{ \prod_{i=n_1+n_2+1}^{n_r} g_3(z_i) \right\} (1 - G_3(z_r))^{n-r}
\]

\[
= \frac{n!}{r!} \alpha^r \lambda_1^{n_1} \lambda_2^{n_2} \lambda_3^{n_3} e^{-\lambda_1 \sum_{i=1}^{n_1} t_i - \lambda_2 \sum_{i=n_1+1}^{n_1+n_2} y_i - \lambda_3 \sum_{i=n_1+n_2+1}^{n_r} z_i}
\times \left\{ \prod_{i=1}^{n_1} (1 - e^{-\lambda_1 y_i})^{\alpha - 1} \right\} \left\{ \prod_{i=n_1+1}^{n_1+n_2} (1 - e^{-\lambda_2 y_i})^{\alpha - 1} \right\}
\times \left\{ \prod_{i=n_1+n_2+1}^{n_r} (1 - e^{-\lambda_3 z_i})^{\alpha - 1} \right\} (1 - (1 - e^{-\lambda_3 z_r})^\alpha)^{n-r}
\]  

(3.5)

where \( y_i = t_i - t_1 + \tau_1 \), \( z_i = t_i - t_2 + \tau_2 \).

As we can see from (3.2)-(3.5) the three MLEs does not exist unless when \( n_1, n_2, n_3 > 0 \) and may be obtained by maximizing the corresponding likelihood function (3.5).

Maximizing the likelihood function for the parameters cannot be achieved analytically. The only option we have is to numerically maximize the likelihood function for the vector of parameters \((\alpha, \lambda_1, \lambda_2, \lambda_3)\). For this purpose, it is convenient to work with the log-likelihood function rather than the likelihood function in (3.5), which is given by

\[
\ell(\alpha, \lambda_1, \lambda_2, \lambda_3; t) = \log c + r \log \alpha + n_1 \log \lambda_1 + n_2 \log \lambda_2 + n_3 \log \lambda_3 - \lambda_1 \sum_{i=1}^{n_1} t_i
\]

\[
- \lambda_2 \sum_{i=n_1+1}^{n_1+n_2} y_i - \lambda_3 \sum_{i=n_1+n_2+1}^{n_r} z_i + (\alpha - 1) \left\{ \sum_{i=1}^{n_1} (1 - e^{-\lambda_1 t_i}) \right\}
\]

\[
+ (\alpha - 1) \left\{ \sum_{i=n_1+1}^{n_1+n_2} (1 - e^{-\lambda_2 y_i}) \right\} + (\alpha - 1) \left\{ \sum_{i=n_1+n_2+1}^{n_r} (1 - e^{-\lambda_3 z_i}) \right\}
\]

\[
+ (n-r) \log (1 - (1 - e^{-\lambda_3 z_r})^\alpha).
\]

The likelihood equations for the parameters \( \alpha, \lambda_1, \lambda_2 \) and \( \lambda_3 \) are given, respectively, by

\[
\frac{\partial \ell}{\partial \alpha} = \frac{r}{\alpha} + \sum_{i=1}^{n_1} (1 - e^{-\lambda_1 t_i}) + \sum_{i=n_1+1}^{n_1+n_2} (1 - e^{-\lambda_2 y_i}) + \sum_{i=n_1+n_2+1}^{n_r} (1 - e^{-\lambda_3 z_i})
\]

\[
- (n-r) \frac{(1 - e^{-\lambda_3 z_r})^\alpha \log (1 - e^{-\lambda_3 z_r})}{1 - (1 - e^{-\lambda_3 z_r})^\alpha},
\]

\[
\frac{\partial \ell}{\partial \lambda_1} = \frac{n_1}{\lambda_1} + \sum_{i=1}^{n_1} \left\{ -t_i + \frac{(\alpha - 1) t_i e^{-\lambda_1 t_i}}{1 - e^{-\lambda_1 t_i}} \right\},
\]

(3.7)

(3.8)
\[
\frac{\partial \ell}{\partial \lambda_2} = \frac{n_2}{\lambda_2} + \sum_{i=n_1+1}^{n_1+n_2} \left\{ -y_i + \frac{(\alpha - 1)y_i e^{-\lambda_2 y_i}}{1 - e^{-\lambda_2 y_i}} \right\},
\]

\[
\frac{\partial \ell}{\partial \lambda_3} = \frac{n_3}{\lambda_3} + \sum_{i=n_1+n_2+1}^{r} \left\{ -z_i + \frac{(\alpha - 1)z_i e^{-\lambda_3 z_i}}{1 - e^{-\lambda_3 z_i}} \right\} - (n - r) \frac{\alpha z_i e^{-\lambda_3 z_i} (1 - e^{-\lambda_3 z_i})^{\alpha - 1}}{1 - (1 - e^{-\lambda_3 z_i})^{\alpha}}.
\]

The maximum likelihood estimates must be derived numerically because there is no obvious solution of these four non-linear likelihood equations. We used the R software to carry out a numerical maximization on the log likelihood function and obtain the MLEs using the following algorithm:

1. Simulate \( n \) order statistics from the uniform (0,1) distribution, 
\((U_1, U_2, \ldots, U_n)\).
2. Find \( n_1 \) such that \( U_{n_1} \leq G_1(\tau_1) \leq U_{n_1+1} \).
3. For \( i \leq n_1 \), \( T_i = -\frac{1}{\lambda_1} \ln (1 - U_i^{\frac{1}{\lambda_1}}) \).
4. Find \( n_2 \) such that \( U_{n_1+n_2} \leq G_2(\tau_2) \leq U_{n_1+n_2+1} \).
5. For \( i \leq n_1 + n_2 \), \( T_i = -\frac{1}{\lambda_2} \ln (1 - U_i^{\frac{1}{\lambda_2}}) + \tau_1 - \hat{\tau}_1 \).
6. For \( n_1 + n_2 + 1 \leq i \leq r \) set \( T_i = -\frac{1}{\lambda_3} \ln (1 - U_i^{\frac{1}{\lambda_3}}) + \tau_2 - \hat{\tau}_2 \).
7. Obtain the MLEs of \((\alpha, \lambda_1, \lambda_2, \lambda_3)\) based on \((T_1, T_2, \ldots, T_{n_1}, T_{n_1+1}, \ldots, T_{n_1+n_2}, T_{n_1+n_2+1}, \ldots, T_r)\) say \( \hat{\alpha}, \hat{\lambda}_1, \hat{\lambda}_2 \) and \( \hat{\lambda}_3 \).
8. Repeat steps 2-7 1000 times.
9. Compute the MSE of the obtained estimates.

### 4. Bayesian Estimation

There is a fundamental difference between classical and Bayesian estimation. In classical estimation we consider the unknown parameter as a fixed value. But in Bayesian estimation we treat the parameter as a random variable. It is assumed that the parameters \( \alpha, \lambda_1, \lambda_2 \) and \( \lambda_3 \) are all independent and have the following prior distributions see [2-4]:
Then the joint prior density function is

\[ \pi_1(\alpha) = \frac{\mu_1^{v_1}}{\Gamma(v_1)} \alpha^{v_1-1} e^{-\mu_1 \alpha}, \quad \mu_1, v_1 > 0, \]

\[ \pi_2(\lambda_1) = \frac{\mu_2^{v_2}}{\Gamma(v_2)} \lambda_1^{v_2-1} e^{-\mu_2 \lambda_1}, \quad \mu_2, v_2 > 0, \]

\[ \pi_3(\lambda_2) = \frac{\mu_3^{v_3}}{\Gamma(v_3)} \lambda_2^{v_3-1} e^{-\mu_3 \lambda_2}, \quad \mu_3, v_3 > 0, \]

\[ \pi_4(\lambda_3) = \frac{\mu_4^{v_4}}{\Gamma(v_4)} \lambda_3^{v_4-1} e^{-\mu_4 \lambda_3}, \quad \mu_4, v_4 > 0. \]

Then the joint prior density function is

\[ \Pi(\alpha, \lambda_1, \lambda_2, \lambda_3) = \frac{\mu_1^{v_1} \mu_2^{v_2} \mu_3^{v_3} \mu_4^{v_4}}{\Gamma(v_1)\Gamma(v_2)\Gamma(v_3)\Gamma(v_4)} \alpha^{v_1-1} \lambda_1^{v_2-1} \lambda_2^{v_3-1} \lambda_3^{v_4-1} e^{-\mu_1 \alpha - \mu_2 \lambda_1 - \mu_3 \lambda_2 - \mu_4 \lambda_3} \]

(4.1)

And hence the posterior function will be as the following

\[ f(\alpha, \lambda_1, \lambda_2, \lambda_3; t) \propto \Pi(\alpha, \lambda_1, \lambda_2, \lambda_3) L(\alpha, \lambda_1, \lambda_2, \lambda_3; t) \]

\[ \propto \frac{\mu_1^{v_1} \mu_2^{v_2} \mu_3^{v_3} \mu_4^{v_4}}{\Gamma(v_1)\Gamma(v_2)\Gamma(v_3)\Gamma(v_4)} \alpha^{v_1-1} \lambda_1^{v_2-1} \lambda_2^{v_3-1} \lambda_3^{v_4-1} e^{-\mu_1 \alpha - \mu_2 \lambda_1 - \mu_3 \lambda_2 - \mu_4 \lambda_3} \]

\[ \times \frac{n!}{r!} \alpha^{n_1} \lambda_1^{n_2} \lambda_2^{n_3} \lambda_3^{n_4} \sum_{i=1}^{n_1} t_i - \lambda_1 \sum_{i=n_1+1}^{n_1+n_2} y_i - \lambda_2 \sum_{i=n_1+n_2+1}^{n_1+n_2+n_3} \sum_{i=n_1+n_2+n_3+1}^{n_1+n_2+n_3+n_4} \]

\[ \times \left\{ \prod_{i=1}^{n_1} \left(1 - e^{-\lambda_1 t_i}\right)^{\alpha-1} \right\} \left\{ \prod_{i=n_1+1}^{n_1+n_2} \left(1 - e^{-\lambda_2 y_i}\right)^{\alpha-1} \right\} \]

\[ \times \left\{ \prod_{i=n_1+n_2+1}^{n_1+n_2+n_3} \left(1 - e^{-\lambda_3 z_i}\right)^{\alpha-1} \right\} \left(1 - (1 - e^{-\lambda_3 z_r})^{\alpha} \right)^{n-r} \]

(4.3)

It is obvious from the posterior function that we are not going to be able to estimate the parameters by the traditional Bayesian methods with integration, so we are going to use one of the MCMC methods which attempt to simulate direct draws from some complex distribution of interest. MCMC approaches are so-called because one uses the previous sample values to randomly generate the next sample value. Here we are going to use the Metropolis algorithm.

Suppose you want to obtain M samples from a univariate distribution with probability density function \(f(\theta, t)\). Suppose \(\theta_i\) is the \(i - th\) sample from \(f\). To use the Metropolis algorithm, you need to have an initial value \(\theta^0\) and a symmetric proposal density \(q(\theta^{i+1} | \theta^i)\). For the \((i + 1) - th\) iteration, the algorithm generates a sample from \(q(\cdot | \cdot)\) based on the current sample \(i\), and it
makes a decision to either accept or reject the new sample. If the new sample is accepted, the algorithm repeats itself by starting at the new sample. If the new sample is rejected, the algorithm starts at the current point and repeats. The algorithm is self-repeating, so it can be carried out as long as required. The most common choice of the proposal distribution is the normal distribution \( N(\theta^i, \sigma) \) with a fixed \( \sigma \). The Metropolis algorithm can be summarized as follows:

1. Set \( i = 0 \). Choose a starting point \( \theta^0 \). This can be an arbitrary point as long as \( f(\theta^0, t) > 0 \).
2. Generate a new sample, \( \theta_{\text{new}} \), by using the proposal distribution \( q(\cdot|\theta^i) \).
3. Calculate the following quantity \( w = \min \left[ \frac{f(\theta_{\text{new}}|t)}{f(\theta^i|t)}, 1 \right] \).
4. Sample \( u \) from the uniform distribution \( U(0, 1) \).
5. Set \( \theta^{i+1} = \theta_{\text{new}} \) if \( u < w \); otherwise set \( \theta^{i+1} = \theta^i \).
6. Set \( i = i + 1 \). If \( i < M \), the number of desired samples, return to step 2. Otherwise, stop.

The number of iterations used to calculate the MCMC estimates is 50000.

The performance of the MLEs and the Bayes estimates are evaluated using a simulation study in the next section.

5. Simulation study

A simulation study was carried out for different values of \( \tau_1 \) and \( \tau_2 \) in order to examine MSE of the ML and the Bayes estimates. For each setting we simulated 1000 data sets to fit the model and estimated the desired quantities. The results are presented in Tables 1 - 4.
Table 1: Conditional Failure Probabilities (in %) for a multi-stress model under Type-II censoring when $\alpha = 3, \lambda_1 = 1.3, \lambda_2 = 0.65, \lambda_3 = 2$ and $n = 25$.

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<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$0 &lt; t &lt; \tau_1$</th>
<th>$\tau_1 &lt; t &lt; \tau_2$</th>
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Table 2: The MSE of the MCMC and ML estimates $\hat{\alpha}, \hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\lambda}_3$ based on 1000 simulations when $\alpha = 3, \lambda_1 = 1.3, \lambda_2 = 0.65, \lambda_3 = 2$ and $n = 25$.

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Table 3: Conditional Failure Probabilities (in %) for a multi-stress model under Type-II censoring when $\alpha = 3, \lambda_1 = 1.3, \lambda_2 = 0.65, \lambda_3 = 2$ and $n = 150$.

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Table 4: The MSE of the MCMC and ML estimates $\hat{\alpha}, \hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\lambda}_3$ based on 1000 simulations when $\alpha = 3, \lambda_1 = 1.3, \lambda_2 = 0.65, \lambda_3 = 2$ and $n = 150$.

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6. Concluding Remarks

We have considered multiple step-stress accelerated model when the observed failure times come from a $GE(\alpha, \lambda)$ distribution under type-II censoring. A simulation study, based on two different examples, was performed to examine the performance of the mean square error of the maximum likelihood and Bayesian estimates.

In Tables 1 and 3, we can see that for a fixed $\tau_1$ and increasing $\tau_2$ the conditional failure probabilities occurring on the first level of stress in the interval $0 < t < \tau_1$ is the same. On the meanwhile those occurring on the second and third levels of stress change. As $\tau_2$ increases the conditional failure probabilities in the interval $\tau_1 < t < \tau_2$ increase, but decrease in $\tau_2 < t < \infty$. This means that as $\tau_2$ increases, there will be more failures occurring before $\tau_2$ and less failures occurring after it, which means more information about $\lambda_2$ and less information about $\lambda_3$. We also can see that as $\tau_1$ increases the conditional failure probabilities occurring on the first level of stress in the interval $0 < t < \tau_1$ also increase.

In Tables 2 and 4, we can see that the MSEs of the Bayesian estimates of the parameters $\alpha, \lambda_1, \lambda_2$ and $\lambda_3$ doesn’t change for different values of $r, \tau_1$ or $\tau_2$, they only change for different sample sizes $n$. On the other hand any change in those values affects the MSEs of the maximum likelihood estimates. In general we can say that MCMC is a better method to estimate our model parameters in either small and large sample sizes.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements

The first author wishes to thank the Egyptian Ministry of Higher Education and Scientific Research for supporting her visiting scholar to the University of Minnesota and Galin Jones for helpful conversations about this paper.

REFERENCES


