GROUP \{1, -1, i, -i\} CORDIAL LABELING OF SUM OF $P_n$ AND $K_n$

S. ATHISAYANATHAN$^1$, R. PONRAJ$^2$ AND M.K. KARTHIK CHIDAMBARAM$^{1,*}$

$^1$Department of Mathematics, St. Xavier’s College, Palayamkottai 627 002, Tamil Nadu, India
$^2$Department of Mathematics, Sri Paramakalyani College, Alwarkurichi 627 412, Tamil Nadu, India

Abstract. Let $G$ be a $(p,q)$ graph and $A$ be a group. For $a \in A$, we denote the order of $a$ by $o(a)$. Let $f : V(G) \to A$ be a function. For each edge $uv$ assign the label 1 if $(o(u), o(v)) = 1$ or 0 otherwise. $f$ is called a group $A$ Cordial labeling if $|v_f(x) - v_f(y)| \leq 1$, $\forall x,y \in A$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(x)$ and $e_f(0)$ respectively denote the number of vertices labelled with an element $x$ and number of edges labelled with $n(n = 0, 1)$. A graph which admits a group $A$ Cordial labeling is called a group $A$ Cordial graph. In this paper we define group \{1, -1, i, -i\} Cordial graphs and prove that $P_n + K_2$ is group \{1, -1, i, -i\} Cordial for every $n$. We further characterize $P_n + K_3, P_n + K_4$ and $P_n + K_n(n \leq 30)$ that are group \{1, -1, i, -i\} Cordial.

Keywords: cordial labeling; group A cordial labeling; group \{1, -1, i, -i\} cordial labeling.

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1. Introduction

In the mathematical discipline of graph theory, a graph labeling is the assignment of labels, traditionally represented by integers, to the edges or vertices, or both, of a graph. Interest in
Graph labeling problems became prominent in the mid 1960’s from a long standing conjecture of Ringel and a paper by Rosa. Most graph labelings trace their origins to labelings presented by Alex Rosa in his 1967 paper. Rosa called a function $f$ a $\beta-$ valuation of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that, when each edge $uv$ is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. In 1980, Golomb called such labelings graceful and this is now the popular term. Ringel conjectured more than four decades ago that ” All trees are graceful” and this conjecture has been the focus of many papers related to labeling problems.

Labelled graphs have wide applications in coding theory, X-ray crystallography, radar, astronomy, circuit design and communication network addressing.

2. Preliminaries

Graphs considered here are finite, undirected and simple. Let $A$ be a group. The order of $a \in A$ is the least positive integer $n$ such that $a^n = e$. We denote the order of $a$ by $o(a)$. Cahit [3] introduced the concept of Cordial labeling. Motivated by this, we defined group A cordial labeling and investigated some of its properties. We also defined group $\{1, -1, i, -i\}$ cordial labeling and discussed that labeling for some standard graphs [1]. In this paper we prove that $P_n + K_2$ is group $\{1, -1, i, -i\}$ Cordial for every $n$. We further characterize $P_n + K_3, P_n + K_4$ and $P_n + K_n (n \leq 30)$ that are group $\{1, -1, i, -i\}$ Cordial. Terms not defined here are used in the sense of Harary[5] and Gallian [4].

The greatest common divisor of two integers $m$ and $n$ is denoted by $(m, n)$ and $m$ and $n$ are said to be relatively prime if $(m, n) = 1$. For any real number $x$, we denote by $\lfloor x \rfloor$, the greatest integer smaller than or equal to $x$ and by $\lceil x \rceil$, we mean the smallest integer greater than or equal to $x$.

A path is an alternating sequence of vertices and edges, $v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n$, which are distinct, such that $e_i$ is an edge joining $v_i$ and $v_{i+1}$ for $1 \leq i \leq n - 1$. A path on $n$ vertices is denoted by $P_n$. If $G$ is a graph on $n$ vertices in which every vertex is adjacent to every other vertex, then $G$ is called a complete graph and is denoted by $K_n$.

Given two graphs $G$ and $H$, $G + H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.
We use the following theorem:

**Theorem 1.1** [2] The Fan $F_n$ is group $\{1, -1, i, -i\}$ Cordial for all $n \leq 10$ and for $n > 10$, $F_n$ is group $\{1, -1, i, -i\}$ Cordial iff $n \equiv 0, 1, 2 \pmod 4$.

### 3. Main results

Let G be a (p,q)graph and consider the group $A = \{1, -1, i, -i\}$ with multiplication. Let $f : V(G) \to A$ be a function. For each edge $uv$ assign the label 1 if $(o(u), o(v)) = 1$ or 0 otherwise. $f$ is called a group $\{1, -1, i, -i\}$ Cordial labeling if $\left| v_f(a) - v_f(b) \right| \leq 1, \forall a, b \in A$ and $\left| e_f(0) - e_f(1) \right| \leq 1$, where $v_f(x)$ and $e_f(n)$ respectively denote the number of vertices labelled with an element $x$ and number of edges labelled with $n(n = 0, 1)$. A graph which admits a group $\{1, -1, i, -i\}$ Cordial labeling is called a group $\{1, -1, i, -i\}$ Cordial graph.

A simple example of a group $\{1, -1, i, -i\}$ Cordial graph is given in Fig. 3.1.

![Fig. 3.1](image)

We now investigate the group $\{1, -1, i, -i\}$ Cordial labeling of $P_n + K_m$ for $1 \leq m \leq 4$. $P_n + K_1$ is the Fan $F_n$ and theorem 1.1 characterizes the Fans that are group $\{1, -1, i, -i\}$ Cordial.

**Theorem 3.1.** $P_n + K_2$ is group $\{1, -1, i, -i\}$ Cordial for every $n$.

**Proof.** Let the vertices of $P_n$ be $u_1, u_2, \ldots, u_n$ and let the vertices of $K_2$ be $v_1, v_2$. Number of vertices of $P_n + K_2$ is $n+2$ and number of edges is $3n$.

**Case(1):** $n + 2 \equiv 0 \pmod 4$.

Let $r = \frac{n-2}{4}$. Label the vertices $v_1, u_1, u_2, \ldots, u_r$ with 1. Label the remaining vertices arbitrarily so that $\frac{n+2}{4}$ vertices get label $-1$, $\frac{n+2}{4}$ vertices get label $i$ and $\frac{n+2}{4}$ vertices get label $-i$. Number of edges with label 1 = $n + 2r + 1 = n + 2 \left( \frac{n-2}{4} \right) + 1 = \frac{3n}{2}$. So this is a group $\{1, -1, i, -i\}$ Cordial
labeling.

**Case (2):** \( n + 2 \equiv 1 (\text{mod} 4) \).

Let \( r = \frac{n-3}{4} \). Label the vertices \( v_1, u_1, u_2, \ldots, u_r \) with 1. Label the remaining vertices arbitrarily so that \( r+1 \) vertices get label -1, \( r+1 \) vertices get label 1 and \( r+2 \) vertices get label -i. Number of edges with label 1 = \( n + 1 + 2r = \frac{3n-1}{2} \). Also number of edges with label 0 = \( \frac{3n+1}{2} \).

**Case (3):** \( n + 2 \equiv 2 (\text{mod} 4) \).

If \( n = 4 \), a group \{1, -1, i, -i\} cordial labeling is shown in Fig 3.2.

Suppose \( n \geq 8 \). Let \( r = \frac{n}{4} \). Label the vertices \( v_1, u_2, u_3, \ldots, u_r \) with 1. Label the remaining vertices arbitrarily so that \( \frac{n}{4} \) vertices get label -1, \( \frac{n}{4} + 1 \) vertices get label 1 and \( \frac{n}{4} + 1 \) vertices get label -1. Number of edges with label 1 = \( n + 1 + 3 + 2(r-2) = n + 4 + 2(\frac{n}{4} - 2) = \frac{3n}{2} \).

![Fig. 3.2](image)

**Case (4):** \( n + 2 \equiv 3 (\text{mod} 4) \).

Let \( r = \frac{n-1}{4} \). Label the vertices \( v_1, u_1, u_2, \ldots, u_r \) with label 1. Label the remaining vertices arbitrarily so that \( r + 1 \) vertices get label -1, \( r + 1 \) vertices get label i and \( r \) vertices get label -i. Number of edges with label 1 = \( n + 1 + 2r = n + 1 + 2(\frac{n-1}{4}) = \frac{3n+1}{2} \). Also, number of edges with label 0 = \( \frac{3n-1}{2} \).

Illustration of the labeling for \( n = 6 \) is given in Fig.3.3.

![Fig 3.3](image)

**Theorem 3.2.** \( P_n + K_3 \) is group \{1, -1, i, -i\} Cordial iff \( n \not= 2, 3, 6 \).
Proof. Let the vertices of $P_n$ be $u_1, u_2, ..., u_n$ and let the vertices of $K_3$ be $v_1, v_2, v_3$. Number of vertices of $P_n + K_3$ is $n + 3$ and number of edges is $4n + 2$.

If $n = 2$, $P_2 + K_3$ has 5 vertices and 10 edges. If 1 vertex is labelled with 1, then 4 edges get label 1 and if 2 vertices are labelled with 1 then 7 edges get label 1. But only 5 edges have to get label 1. So $P_2 + K_3$ is not group $\{1, -1, i, -i\}$ Cordial.

If $n = 3$, $P_3 + K_3$ has 6 vertices and 14 edges. If 1 vertex is labelled with 1, at most 5 edges get label 1 and if 2 vertices are labelled with 1, either 8 or 9 vertices get label 1. So $P_3 + K_3$ is not group $\{1, -1, i, -i\}$ Cordial.

If $n = 6$, $P_6 + K_3$ has 9 vertices and 26 edges. It is easy to observe that there is no choice of 2 or 3 vertices so that 13 edges get label 1. Hence $P_6 + K_3$ is not group $\{1, -1, i, -i\}$ Cordial. Thus $n \neq 2, 3, 6$.

Conversely, assume $n \neq 2, 3, 6$.

We need to prove that $P_n + K_3$ is group $\{1, -1, i, -i\}$ Cordial.

Case (1): $n + 3 \equiv 0 \pmod{4}$.

Each vertex label occurs $\frac{n+3}{4}$ times and each edge label occurs $2n + 1$ times. Let $r = \frac{n-1}{4}$. Label $v_1, u_2, u_4, ..., u_{2r}$ with 1. Label the remaining vertices arbitrarily so that $r + 1$ of them get label -1, $r + 1$ of them get label i and $r + 1$ of them get label -i.

Case (2): $n + 3 \equiv 1 \pmod{4}$.

By assumption, $n \geq 10$. In this case, one vertex label occurs $\lceil \frac{n+3}{4} \rceil$ times and each of the other 3 vertex labels occur $\lfloor \frac{n+3}{4} \rfloor$ times.

Let $r = \frac{n-10}{4}$. Label $v_1, u_1, u_2, u_3, u_5, u_7, ..., u_{2r+3}$ with 1. Label the remaining vertices arbitrarily so that $r + 3$ of them get label -1, $r + 3$ of them get label i and $r + 3$ of them get label -i.

Case (3): $n + 3 \equiv 2 \pmod{4}$.

By assumption, $n \geq 7$. In this case, 2 vertex labels occur $\lceil \frac{n+3}{4} \rceil$ times and 2 other labels occur $\lfloor \frac{n+3}{4} \rfloor$ times.

Let $r = \frac{n-7}{4}$. Label $v_1, u_1, u_2, u_4, u_6, ..., u_{2r+2}$ with 1. Label the remaining vertices arbitrarily so that $r + 3$ of them get label -1, $r + 2$ of them get label i and $r + 2$ of them get label -i.

Case (4): $n + 3 \equiv 3 \pmod{4}$.

In this case, 3 vertex labels occur $\lceil \frac{n+3}{4} \rceil$ times and 1 vertex label occurs $\lfloor \frac{n+3}{4} \rfloor$ times. Let
\[ r = \frac{n+4}{4} \]. Label \( v_1, u_1, u_3, u_5, \ldots, u_{2r+1} \) with 1. Label the remaining vertices arbitrarily so that \( r + 2 \) vertices get label -1, \( r + 2 \) vertices get label i and \( r + 1 \) vertices get label -i. That \( P_n + K_3 \) is group \( \{1, -1, i, -i\} \) Cordial for \( n \neq 2, 3, 6 \) follows from Table 1.

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( v_f(1) )</th>
<th>( v_f(-1) )</th>
<th>( v_f(i) )</th>
<th>( v_f(-i) )</th>
<th>( e_f(1) )</th>
<th>( e_f(0) )</th>
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<td>( n + 3 \equiv 0 \pmod{4} )</td>
<td>( \frac{n+3}{4} )</td>
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<td>( \frac{n+3}{4} )</td>
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<td>( 2n + 1 )</td>
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<td>( n + 3 \equiv 1 \pmod{4} )</td>
<td>( \left\lceil \frac{n+3}{4} \right\rceil )</td>
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Table 1

**Theorem 3.3.** \( P_n + K_4 \) is group \( \{1, -1, i, -i\} \) Cordial iff \( n \in \{3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 17\} \).

**Proof.** Let the vertices of \( P_n \) be \( u_1, u_2, \ldots, u_n \) and let the vertices of \( K_4 \) be \( v_1, v_2, v_3, v_4 \). Number of vertices of \( P_n + K_4 \) is \( n + 4 \).

Number of edges = \( \binom{n}{2} \) + \( n - 1 \) + 4\( n = 5n + 5 \).

**Case(1):** Exactly one vertex, say \( v_1 \) is given label 1.

Now, \( \text{deg} v_i = n + 3(1 \leq i \leq 4) \), \( \text{deg} u_1 = 5 \), \( \text{deg} u_3 = 5 \), and \( \text{deg} u_i = 6(2 \leq i \leq n - 1) \).

**Subcase(i):** \( n = 4k(k \in \mathbb{Z}_+) \).

Now total number of vertices is \( 4k + 4 \) and so each vertex label should occur \( k + 1 \) times. Total number of edges is \( 5(4k) + 5 = 20k + 5 \). So one edge label should occur 10\( k \) + 3 times and another should occur 10\( k \) + 2 times. Maximum number of edges that can receive label 1 by taking \( k + 1 \) vertices is \( (4k + 3) + 5k \). So, to get a group \( \{1, -1, i, -i\} \) Cordial labeling, we need to have \( 4k + 3 + 5k = 9k + 3 \geq 10k + 2 \) and so \( k \leq 1 \). In this case \( n = 4 \). A group \( \{1, -1, i, -i\} \) Cordial labeling of \( P_4 + K_4 \) is given in Table 2.

**Subcase(ii):** \( n = 4k + 1(k \geq 0) \).

Now, number of vertices in \( P_n + K_3 \) is \( 4k + 5 \) and so one vertex label should occur \( k + 2 \) times and each of the other three vertex labels should occur \( k + 1 \) times. Total number of edges is \( 5(4k + 1) + 5 = 20k + 10 \). So each of the edge labels should occur 10\( k \) + 5 times. Maximum number of edges that can receive label 1 by taking \( k + 2 \) vertices is \( (4k + 4) + 5(k + 1) \). So, a necessary condition to get a group \( \{1, -1, i, -i\} \) Cordial labeling is \( 9k + 9 \geq 10k + 5 \) and so
When $k = 0$, $P_1 + K_4 \approx K_5$ and there is no choice of 1 vertex or 2 vertices so that 5 edges get label 1. So $k \neq 0$. Group $\{1, -1, i, -i\}$ Cordial labelings of $P_n + K_4$ when $n = 5, 9, 13$ and $n = 17$ are given in Tables 2 and 3.

**Subcase(iii):** $n = 4k + 2(k \geq 0)$.

Now, number of vertices in $P_n + K_4$ is $4k + 6$ and so 2 vertex labels should occur $k + 2$ times and 2 vertex labels should occur $k + 1$ times. Total number of edges is $5(4k + 2) + 5 = 20k + 15$. So, one edge label should occur $10k + 7$ times and another edge label should occur $10k + 8$ times. Maximum number of edges that can receive label 1 by taking $k + 2$ vertices in a group $\{1, -1, i, -i\}$ Cordial labeling is $4k + 6 + 5(k + 1)$. So, the necessary condition is, $9k + 11 \geq 10k + 8$ and so $k \leq 3$. When $k = 0, n = 2$. $P_2 + K_4 \approx K_5$ and 2 vertex labels should occur 2 times and 2 labels should occur 1 time. One edge label should occur 7 times and another edge label should occur 8 times. There is no choice of 1 vertex or 2 vertices so that 7 or 8 edges get label 1. So, $n \neq 2$. Group $\{1, -1, i, -i\}$ Cordial labelings of $P_n + K_4$ when $n = 6, 10, 14$ are given in Tables 2 and 3.

**Subcase(iv):** $n = 4k + 3(k \geq 0)$.

Now, number of vertices in $P_n + K_4$ is $4k + 7$. So 3 vertex labels should occur $k + 2$ times and 1 vertex label should occur $k + 1$ times. Total number of edges is $5(4k + 3) + 5 = 20k + 20$. So, each edge label should occur $10k + 10$ times. Maximum number of edges that can receive label 1 by taking $k + 2$ vertices in a group $\{1, -1, i, -i\}$ cordial labeling is $(4k + 7) + 5(k + 1) = 9k + 12$. So, the necessary condition is, $9k + 12 \geq 10k + 10$ i.e., $k \leq 2$. Group $\{1, -1, i, -i\}$ Cordial labelings of $P_n + K_4$ when $n = 3, 7$ or $n = 11$ are given in Tables 2 and 3.

**Case(2):** At least two vertices $v_i (1 \leq i \leq 4)$ are given label 1.

Suppose $v_1$ and $v_2$ are given label 1. Now, $(n + 3) + (n + 2) = 2n + 5$ edges get label 1.

**Subcase(i):** $n = 4k (k \geq 1)$.

Number of edges that has to receive label 1 is $10k + 2$. Minimum number of edges that can receive label 1 is $8k + 5 + (k - 1)3 = 11k + 2$. Therefore, $10k + 2 \geq 11k + 2 \Rightarrow k \leq 0$. 
Subcase(ii): $n = 4k + 1(k \geq 0)$.

Number of edges that has to receive label 1 is $10k + 5$. Minimum number of edges that can receive label 1 is $8k + 5 + (k - 1)3 = 11k + 2$. Therefore, $10k + 5 \geq 11k + 2 \Rightarrow k \leq 3$.

Subcase(iii) $n = 4k + 2(k \geq 0)$.

Number of edges that has to receive label 1 is $10k + 8$. Minimum number of edges that can receive label 1 is $8k + 5 + (k - 1)3 = 11k + 2$. So $10k + 8 \geq 11k + 2 \Rightarrow k \leq 6$. It is easy to observe that $P_{18} + K_4, P_{22} + K_4$ and $P_{26} + K_4$ are not group $\{1, -1, i, -i\}$ Cordial.

Subcase(iv): $n = 4k + 3(k \geq 0)$.

Number of edges that has to receive label 1 is $10k + 10$. Minimum number of edges that can
receive label 1 is \((8k + 5) + (k - 1) = 11k + 2\). Hence \(10k + 10 \geq 11k + 2 \Rightarrow k \leq 8\). It is easy to observe that \(P_n + K_4\) is not group \(\{1, -1, i, -i\}\) Cordial for \(3 \leq k \leq 8\).

**Case(3):** No vertex \(v_i\) is given label 1.

**Subcase(i):** \(n = 4k(k > 0)\).

Maximum number of edges that can receive label 1 is \((k + 1)6 = 6k + 6\).
So, \(6k + 6 \geq 10k + 2 \Rightarrow 4k \leq 4 \Rightarrow k \leq 1\).

**Subcase(ii):** \(n = 4k + 1(k \geq 0)\).

Maximum number of edges that can receive label 1 is \(6k + 6\). So \(6k + 6 \geq 10k + 5 \Rightarrow 4k \leq 1 \Rightarrow k = 0\).

**Subcase(iii):** \(n = 4k + 2(k \geq 0)\).

Now, \(6k + 6 \geq 10k + 8 \Rightarrow 4k \leq -2\) which is impossible.

**Subcase(iv):** \(n = 4k + 3(k \geq 0)\).

Now, \(6k + 6 \geq 10k + 10 \Rightarrow 4k \leq -4\) which is impossible.

Thus, \(P_n + K_4\) is group \(\{1, -1, i, -i\}\) Cordial iff \(n \in \{3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 17\}\).

**Theorem 3.4.** For \(n \leq 30\), \(P_n + K_n\) is group \(\{1, -1, i, -i\}\) Cordial iff \(n \in \{1, 2, 4, 5, 7, 9, 10, 11, 16, 25, 26, 27\}\).

**Proof.**

Let the vertices of \(P_n\) be labelled as \(u_1, u_2, \ldots, u_n\) and the vertices of \(K_n\) be labelled as \(v_1, v_2, \ldots, v_n\). Number of vertices in \(P_n + K_n\) is \(2n\) and the number of edges is \(\binom{n}{2} + n^2 + n - 1 = \frac{(3n-2)(n+1)}{2}\).

For \(1 \leq i \leq n\), \(\deg v_i = 2n - 1\), \(\deg u_1 = \deg u_n = n + 1\) and \(\deg u_i(1 < i < n) = n + 2\).

For \(n = 1\), the two vertices of \(P_1 + K_1 \cong K_2\) be labelled as 1,-1 respectively and this is a group \(\{1, -1, i, -i\}\) Cordial labeling.

For \(n = 2\), \(P_2 + K_2 \cong K_4\) which is group \(\{1, -1, i, -i\}\) Cordial.

For \(n = 3\), \(P_3 + K_3\) has 6 vertices and 14 edges. Two vertex labels should appear 2 times and 2 other vertex labels should appear once. Each edge label should appear 7 times. There is no choice of 1 vertex or 2 vertices so that 7 edges get label 1.

For \(n = 4\), \(f_1\) defined by \(f_1(v_1) = f_1(v_2) = 1, f_1(v_3) = f_1(v_4) = -1, f_1(u_1) = f_1(u_2) = i, f_1(u_3) = \ldots\).
For $n = 5$, $f_2$ defined by $f_2(v_1) = f_2(u_1) = f_2(u_3) = 1, f_2(v_2) = f_2(v_3) = f_2(v_4) = -1, f_2(v_5) = f_2(u_2) = i, f_2(u_4) = f_2(u_5) = -i$ is a group $\{1, -1, i, -i\}$ Cordial labeling.

For $n = 6$, $P_6 + K_6$ has 12 vertices and 56 edges. Each vertex label should occur 3 times. Each edge label should appear 28 times. There is no choice of 3 vertices so that 28 edges get label 1.

For $n = 7$, $f_3$ defined by, $f_3(v_1) = f_3(v_2) = f_3(u_1) = f_3(u_3) = 1, f_3(v_3) = f_3(v_4) = f_3(v_5) = f_3(v_6) = -1, f_3(v_7) = f_3(u_2) = f_3(u_4) = i, f_3(u_5) = f_3(u_6) = f_3(u_7) = -i$ is a group $\{1, -1, i, -i\}$ Cordial labeling.

For $n = 8$, $P_8 + K_8$ has 16 vertices and 99 edges. There is no choice of 4 vertices so that 49 or 50 edges get label 1.

For $n = 9$, $f_4$ defined by, $f_4(v_1) = f_4(v_2) = f_4(v_3) = f_4(u_1) = f_4(u_2) = 1, f_4(v_4) = f_4(v_5) = f_4(v_6) = f_4(v_7) = f_4(v_8) = -1, f_4(v_9) = f_4(u_3) = f_4(u_4) = f_4(u_5) = i, f_4(u_j) = -i$ for $6 \leq j \leq 9$ is a group $\{1, -1, i, -i\}$ Cordial labeling.

For $n = 10$, $f_5$ defined by, $f_5(v_j) = 1(1 \leq j \leq 4), f_5(u_1) = 1, f_5(v_j) = -1$, for $5 \leq j \leq 9, f_5(v_{10}) = i, f_5(u_j) = i$ for $2 \leq j \leq 5, f_5(u_j) = -i$ for $6 \leq j \leq 10$, is a group $\{1, -1, i, -i\}$ Cordial labeling.

For $n = 11$, $f_6$ defined by, $f_6(v_j) = 1$ for $1 \leq j \leq 4, f_6(u_1) = f_6(u_2) = 1, f_6(v_j) = -1$, for $5 \leq j \leq 10, f_6(v_{11}) = i, f_6(u_j) = i$ for $3 \leq j \leq 6, f_6(u_j) = -i$ for $7 \leq j \leq 11$ is a group $\{1, -1, i, -i\}$ Cordial labeling.

For $n = 12$, number of vertices of $P_{12} + K_{12}$ is 24 and number of edges is 221. There is no choice of 6 vertices so that 110 or 111 edges get label 1.

For $n = 13$, number of vertices of $P_{13} + K_{13}$ is 26 and number of edges is 259. There is no choice of 6 or 7 vertices so that 129 or 130 edges get label 1.

For $n = 14$, number of vertices of $P_{14} + K_{14}$ is 28 and number of edges is 300. There is no choice of 7 vertices so that 150 edges get label 1.

For $n = 15$, number of vertices of $P_{15} + K_{15}$ is 30 and number of edges is 344. There is no choice of 7 or 8 vertices so that 172 edges get label 1.

For $n = 16$, $f_7$ defined by, $f_7(v_j) = 1(1 \leq j \leq 6), f_7(u_2) = 1, f_7(u_4) = 1, f_7(v_j) = -1(7 \leq j \leq 14), f_7(v_{15}) = f_7(v_{16}) = i, f_7(u_1) = f_7(u_3) = i, f_7(u_j) = i(5 \leq j \leq 8) f_7(u_j) = -i(9 \leq j \leq 16)$
is a group \( \{1, -1, i, -i\} \) Cordial labeling.

For \( n = 17 \), \( P_{17} + K_{17} \) has 34 vertices and 441 edges. There is no choice of 8 or 9 vertices so that 220 or 221 edges get label 1.

For \( n = 18 \), there is no choice of 9 vertices so that 247 edges get label 1.

For \( n = 19 \), there is no choice of 9 or 10 vertices so that 225 or 226 edges get label 1.

For \( n = 20 \), there is no choice of 10 vertices so that 305 or 306 edges get label 1.

For \( n = 21 \), there is no choice of 10 or 11 vertices so that 337 edges get label 1.

For \( n = 22 \), there is no choice of 11 vertices so that 370 edges get label 1.

For \( n = 23 \), there is no choice of 11 or 12 vertices so that 404 or 405 edges get label 1.

For \( n = 24 \), there is no choice of 12 vertices so that 440 or 441 edges get label 1.

For \( n = 25 \), \( f_8 \) defined by, \( f_8(v_j) = 1(1 \leq j \leq 10), f_8(u_1) = f_8(u_3) = 1, f_8(v_j) = -1(11 \leq j \leq 23), f_8(v_{24}) = f_8(v_{25}) = i, f_8(u_2) = i, f_8(u_j) = i(4 \leq j \leq 13), f_8(u_j) = i(14 \leq j \leq 25) \) is a group \( \{1, -1, i, -i\} \) Cordial labeling.

For \( n = 26 \), \( f_9 \) defined by, \( f_9(v_j) = 1(1 \leq j \leq 10), f_9(u_1) = f_9(u_2) = f_9(u_4) = 1, f_9(v_j) = -1(11 \leq j \leq 23), f_9(v_j) = i(24 \leq j \leq 26), f_9(u_3) = i, f_9(u_j) = i(5 \leq j \leq 13), f_9(u_j) = -i(14 \leq j \leq 26) \) is a group \( \{1, -1, i, -i\} \) Cordial labeling.

For \( n = 27 \), \( f_{10} \) defined by, \( f_{10}(v_j) = 1(1 \leq j \leq 10), f_{10}(u_1) = f_{10}(u_2) = f_{10}(u_4) = 1, f_{10}(v_j) = -1(11 \leq j \leq 24), f_{10}(v_j) = i(25 \leq j \leq 27), f_{10}(u_3) = f_{10}(u_5) = i, f_{10}(u_j) = i(7 \leq j \leq 14), f_{10}(u_j) = -i(15 \leq j \leq 27) \) is a group \( \{1, -1, i, -i\} \) Cordial labeling.

For \( n = 28 \), there is no choice of 14 vertices so that 599 or 600 edges get label 1.

For \( n = 29 \), there is no choice of 14 or 15 vertices so that 643 edges get label 1.

For \( n = 30 \), there is no choice of 15 vertices so that 688 edges get label 1.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**References**


