RULED SURFACE PAIR GENERATED BY A CURVE AND ITS NATURAL LIFT IN $\mathbb{R}^3_1$

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Abstract. In this study, firstly, the Frenet vector fields $\tilde{T}, \tilde{N}, \tilde{B}$ of the natural lift $\tilde{\alpha}$ of a curve $\alpha$ are calculated in terms of those of $\alpha$ in $\mathbb{R}^3_1$. Secondly, we obtained striction lines and distribution parameters of ruled surface pair generated by the curve $\alpha$ and its natural lift $\tilde{\alpha}$. Finally, for $\alpha$ and $\tilde{\alpha}$ those notions are compared with each other.

Keywords: Natural Lift, Ruled Surface, Striction Line, Distribution Parameter.

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1. Introduction and Preliminaries

Let Minkowski 3-space $\mathbb{R}^3_1$ be the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product $g$ given by

$$g(X, X) = -x_1^2 + x_2^2 + x_3^2$$

where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ is said to be timelike if $g(X, X) < 0$, spacelike if $g(X, X) > 0$ and lightlike (or null) if $g(X, X) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(t)$ in $\mathbb{R}^3_1$ where $t$ is a pseudo-arclength parameter, can locally
be timelike, spacelike or null (lightlike), if all of its velocity vectors $\alpha'(t)$ are respectively timelike, spacelike or null (lightlike), for every $t \in I \subset \mathbb{R}$.

A lightlike vector $X$ is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$) and a timelike vector $X$ is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$). The norm of a vector $X$ is defined by [4]

$$\|X\|_{IL} = \sqrt{|g(X,X)|}.$$ 

We denote by $\{T(t), N(t), B(t)\}$ the moving Frenet frame along the curve $\alpha$. Then $T, N$ and $B$ are the tangent, the principal normal and the binormal vector of the curve $\alpha$, respectively.

Let $\alpha$ be a unit speed timelike space curve with curvature $\kappa$ and torsion $\tau$. Let Frenet vector fields of $\alpha$ be $\{T, N, B\}$. In this trihedron, $T$ is timelike vector field, $N$ and $B$ are spacelike vector fields. Then, Frenet formulas are given by [8]

$$T' = \kappa N \quad N' = \kappa T + \tau B \quad B' = -\tau N.$$ 

Let $\alpha$ be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that $T$ and $B$ are spacelike vector fields and $N$ is a timelike vector field. Then, Frenet formulas are given by [8]

$$T' = \kappa N \quad N' = \kappa T + \tau B \quad B' = \tau N.$$ 

Let $\alpha$ be a unit speed spacelike space curve with a timelike binormal. In this trihedron, we assume that $T$ and $N$ are spacelike vector fields and $B$ is a timelike vector field. Then, Frenet formulas are given by [8]

$$T' = \kappa N \quad N' = -\kappa T + \tau B \quad B' = \tau N.$$ 

**Lemma 1.1.** Let $X$ and $Y$ be nonzero Lorentz orthogonal vectors in $\mathbb{R}^3_1$. If $X$ is timelike, then $Y$ is spacelike [10].
Lemma 1.2. Let $X$ and $Y$ be positive (negative) timelike vectors in $\mathbb{R}^3_1$. Then
\[ g(X, Y) \leq \|X\| \|Y\| \]
with equality if and only if $X$ and $Y$ are linearly dependent [10].

Lemma 1.3.

i) Let $X$ and $Y$ be positive (negative) timelike vectors in $\mathbb{R}^3_1$. By the Lemma 1.2, there is unique nonnegative real number $\varphi(X, Y)$ such that
\[ g(X, Y) = \|X\| \|Y\| \cosh \varphi(X, Y) \]
the Lorentzian timelike angle between $X$ and $Y$ is defined to be $\varphi(X, Y)$.

ii) Let $X$ and $Y$ be spacelike vectors in $\mathbb{R}^3_1$ that span a spacelike vector subspace. Then we have
\[ |g(X, Y)| \leq \|X\| \|Y\|. \]
Hence, there is a unique real number $\varphi(X, Y)$ between 0 and $\pi$ such that
\[ g(X, Y) = \|X\| \|Y\| \cos \varphi(X, Y) \]
the Lorentzian spacelike angle between $X$ and $Y$ is defined to be $\varphi(X, Y)$.

iii) Let $X$ and $Y$ be spacelike vectors in $\mathbb{R}^3_1$ that span a timelike vector subspace. Then we have
\[ g(X, Y) > \|X\| \|Y\|. \]
Hence, there is a unique positive real number $\varphi(X, Y)$ between 0 and $\pi$ such that
\[ |g(X, Y)| = \|X\| \|Y\| \cosh \varphi(X, Y) \]
the Lorentzian timelike angle between $X$ and $Y$ is defined to be $\varphi(X, Y)$.

iv) Let $X$ be a spacelike vector and $Y$ be a positive timelike vector in $\mathbb{R}^3_1$. Then there is a unique nonnegative real number $\varphi(X, Y)$ such that
\[ |g(X, Y)| = \|X\| \|Y\| \sinh \varphi(X, Y) \]
the Lorentzian timelike angle between $X$ and $Y$ is defined to be $\varphi(X, Y)$ [10].

Definition 1.1. (Unit Vector $C$ of Direction $W$ for Non-null Curves):
For the curve $\alpha$ with a timelike tangent, $\theta$ being a Lorentzian timelike angle between the spacelike binormal unit $-B$ and the Frenet instantaneous rotation vector $W$, 

a) If $|\kappa| > |\tau|$, then $W$ is a spacelike vector. In this situation, from Lemma 1.3 iii) we can write

$$\kappa = ||W|| \cosh \theta, \quad \tau = ||W|| \sinh \theta$$

$$||W||^2 = g(W,W) = \kappa^2 - \tau^2 \quad \text{and} \quad C = \frac{W}{||W||} = \sinh \theta T + \cosh \theta B,$$

where $C$ is unit vector of direction $W$.

b) If $|\kappa| < |\tau|$, then $W$ is a timelike vector. In this situation, from Lemma 1.3 iv) we can write

$$\kappa = ||W|| \sinh \theta, \quad \tau = ||W|| \cosh \theta$$

$$||W||^2 = -g(W,W) = -(\kappa^2 - \tau^2) \quad \text{and} \quad C = \cosh \theta T + \sinh \theta B.$$

ii) For the curve $\alpha$ with a timelike principal normal, $\theta$ being an angle between the $B$ and the $W$, if $B$ and $W$ spacelike vectors that span a spacelike vector subspace then by the Lemma 1.3 ii) we can write

$$\kappa = ||W|| \cos \theta, \quad \tau = ||W|| \sin \theta$$

$$||W||^2 = g(W,W) = \kappa^2 + \tau^2 \quad \text{and} \quad C = \sin \theta T - \cos \theta B.$$

iii) For the curve $\alpha$ with a timelike binormal, $\theta$ being a Lorentzian timelike angle between the $-B$ and the $W$,

a) If $|\kappa| < |\tau|$, then $W$ is a spacelike vector. In this situation, from Lemma 1.3 iv) we can write

$$\kappa = ||W|| \sinh \theta, \quad \tau = ||W|| \cosh \theta$$

$$||W||^2 = g(W,W) = \tau^2 - \kappa^2 \quad \text{and} \quad C = -\cosh \theta T + \sinh \theta B.$$

b) If $|\kappa| > |\tau|$, then $W$ is a timelike vector. In this situation, from Lemma 1.3 i) we have

$$\kappa = ||W|| \cosh \theta, \quad \tau = ||W|| \sinh \theta$$

$$||W||^2 = -g(W,W) = -(\tau^2 - \kappa^2) \quad \text{and} \quad C = -\sinh \theta T + \cosh \theta B.$$

**Corollary 1.1.** Let $\alpha$ be a unit speed timelike space curve. Then the natural lift $\overline{\alpha}$ of $\alpha$ is a spacelike space curve [5].
Corollary 1.2. Let $\alpha$ be a unit speed spacelike space curve with a spacelike binormal. Then the natural lift $\overline{\alpha}$ of $\alpha$ is a timelike space curve [5].

Corollary 1.3. Let $\alpha$ be a unit speed spacelike space curve with a timelike binormal. Then the natural lift $\overline{\alpha}$ of $\alpha$ is a spacelike space curve [5].

Corollary 1.4. Let $\alpha$ be a unit speed timelike space curve and $\overline{\alpha}$ be the natural lift of $\alpha$. Then

$$T(s) = N(s), \quad \overline{N}(s) = -\frac{\kappa(s)}{\|W\|} T(s) - \frac{\tau(s)}{\|W\|} B(s), \quad \overline{B}(s) = -\frac{\tau(s)}{\|W\|} T(s) - \frac{\kappa(s)}{\|W\|} B(s) \quad [7].$$

Corollary 1.5. Let $\alpha$ be a unit speed spacelike space curve with a spacelike binormal and $\overline{\alpha}$ be the natural lift of $\alpha$. Then

$$\overline{T}(s) = N(s), \quad \overline{N}(s) = \frac{\kappa(s)}{\|W\|} T(s) + \frac{\tau(s)}{\|W\|} B(s), \quad \overline{B}(s) = \frac{\tau(s)}{\|W\|} T(s) - \frac{\kappa(s)}{\|W\|} B(s) \quad [7].$$

Corollary 1.6. Let $\alpha$ be a unit speed spacelike space curve with a timelike binormal and $\overline{\alpha}$ be the natural lift of $\alpha$. Then

$$\overline{T}(s) = N(s), \quad \overline{N}(s) = -\frac{\kappa(s)}{\|W\|} T(s) - \frac{\tau(s)}{\|W\|} B(s), \quad \overline{B}(s) = \frac{\tau(s)}{\|W\|} T(s) + \frac{\kappa(s)}{\|W\|} B(s) \quad [7].$$

Definition 1.2. Let $M$ be a hypersurface in $\mathbb{R}^3$ and let $\alpha: I \rightarrow M$ be a parametrized curve. $\alpha$ is called an integral curve of $X$ if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \quad (\text{for all } s \in I) \quad [4]$$

where $X$ is a smooth tangent vector field on $M$. We have

$$TM = \bigcup_{P \in M} T_PM = \chi(M)$$

where $T_PM$ is the tangent space of $M$ at $P$ and $\chi(M)$ is the space of vector fields on $M$. 
**Definition 1.3.** For any parametrized curve \( \alpha : I \rightarrow M , \overline{\alpha} : I \rightarrow TM \) given by

\[
\overline{\alpha}(s) = \left( \alpha(s), \alpha'(s) \right) = \alpha'(s)|_{\alpha(s)}
\]

is called the natural lift of \( \alpha \) on \( TM \) [5]. Thus, we can write

\[
\frac{d\overline{\alpha}}{ds} = \frac{d}{ds} \left( \alpha'(s)|_{\alpha(s)} \right) = D_{\alpha'(s)}\alpha'(s)
\]

where \( D \) is the Levi-Civita connection on \( \mathbb{R}^3 \).

A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation

\[
X(s,v) = \alpha(s) + ve(s),
\]

where \( \alpha(s) \) represents a space curve which is called the base curve and \( e \) is a unit vector representing the direction of a straight line.

The striction point on a ruled surface \( X \) is the foot of the common normal between two consecutive generators (or ruling). The set of striction points defines the striction curve given as

\[
\beta(s) = \alpha(s) - \frac{g(\alpha', e')}{g(e', e')} e(s) \quad [2].
\]

The distribution parameter of the ruled surface \( X \) is defined by

\[
P_e = \frac{\det(\alpha', e, e')}{\|e'\|^2}[2].
\]

The ruled surface is developable if and only if \( P_e = 0 \).

### 3. RuledSurface Pair Generated By a Curve and Its Natural Lift in \( \mathbb{R}^3 \)

Let \( \alpha \) be a unit speed timelike space curve. Then the natural lift \( \overline{\alpha} \) of \( \alpha \) is a spacelike space curve.
(i) Let $X$ and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v) = \alpha(s) + vT(s), \quad \overline{X}(s,v) = \overline{\alpha}(s) + v\overline{T}(s).$$

The striction curves of $X$ and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda T(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{T}(s)$, respectively. Then we obtain

$$\lambda = 0, \quad \mu = 0.$$

The distribution parameters of the ruled surfaces $X$ and $\overline{X}$ are defined by $P_T = \frac{\det(\alpha', T, T', \tau')}{{\|T'\|}^2}$ and $\overline{P}_T = \frac{\det(\overline{\alpha}', T', \tau', \tau')}{{\|T'\|}^2}$. Then we have

$$P_T = 0, \quad \overline{P}_T = 0.$$

**Corollary 2.1.** Let the striction curves of $X$ and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda T(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{T}(s)$, respectively. Then $\beta(s) = \alpha(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s)$.

**Corollary 2.2.** If the ruled surface $X$ is developable then the ruled surface $\overline{X}$ are also developable.

(ii) Let $X$ and $\overline{X}$ be two ruled surfaces which is given by

$$X(s,v) = \alpha(s) + vN(s), \quad \overline{X}(s,v) = \overline{\alpha}(s) + v\overline{N}(s).$$

The striction curves of $X$ and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda N(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{N}(s)$, respectively. Then we have

$$\lambda = \frac{\kappa}{\kappa^2 - \tau^2}, \quad \mu = \frac{\kappa (-\kappa^2 + \tau^2) \|W\|}{-\kappa'^2 + \tau'^2 + (-\kappa^2 + \tau^2)^2}.$$
The distribution parameters of the ruled surfaces $X$ and $\overline{X}$ are defined by $P_N = \frac{\det(\alpha',N,N')}{\|N'\|^2}$ and $\overline{P}_N = \frac{\det(\pi',\overline{N},\overline{N}')}{{\|\overline{N}'\|^2}}$. Then we obtain

$$P_N = \frac{\tau}{-\kappa^2 + \tau^2}, \quad \overline{P}_N = \frac{-\kappa^2 \tau' + \kappa \tau \kappa'}{(-\kappa'^2 + \tau'^2) + (-\kappa^2 + \tau^2)^2}.$$

**Corollary 2.3.** Let the striction curves of $X$ and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda N(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{N}(s)$, respectively.

1. If $W$ is a spacelike vector, then $\mu = \frac{-\kappa(z)^2}{(-\kappa^2 + \tau^2) + (-\frac{z}{\tau})^2}$.
2. If $W$ is a timelike vector, then $\mu = \frac{-\kappa(-\frac{z}{\tau})^2}{(-\kappa^2 + \tau^2) + (-\frac{z}{\tau})^2}$.

**Corollary 2.4.** Let the distribution parameters of the ruled surfaces $X$ and $\overline{X}$ be $P_N$ and $\overline{P}_N$, respectively.

1. If $W$ is a spacelike vector, then $\overline{P}_N = \frac{-\kappa^2 \tau' + \kappa \tau \kappa'}{(-\kappa'^2 + \tau'^2) + \tau^2}$.
2. If $W$ is a timelike vector, then $\overline{P}_N = \frac{-\kappa^2 \tau' + \kappa \tau \kappa'}{(-\kappa'^2 + \tau'^2) + \tau^2}$.

**Corollary 2.5.** If $\alpha$ is a planer curve, then the ruled surface $X$ and $\overline{X}$ are developable.

(iii) Let $X$ and $\overline{X}$ be two ruled surfaces which are given by

$$X(s,v) = \alpha(s) + vB(s), \quad \overline{X}(s,v) = \overline{\alpha}(s) + v\overline{B}(s).$$

The striction curves of $X$ and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda B(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{B}(s)$, respectively. Then we obtain

$$\lambda = 0, \quad \mu = 0.$$

The distribution parameters of the ruled surfaces $X$ and $\overline{X}$ are defined by $P_B = \frac{\det(\alpha',B,B')}{\|B'\|^2}$ and $\overline{P}_B = \frac{\det(\pi',\overline{B},\overline{B}')}{{\|\overline{B}'\|^2}}$. Then we have

$$P_B = \frac{1}{\tau}, \quad \overline{P}_B = \frac{\kappa^2 \tau' - \kappa \tau \kappa'}{\kappa'^2 - \tau'^2}.$$
Corollary 2.6. Let the striction curves of \( X \) and \( \overline{X} \) be given by \( \beta (s) = \alpha (s) - \lambda B (s) \) and \( \overline{\beta} (s) = \overline{\alpha} (s) - \mu \overline{B} (s) \), respectively. Then \( \beta (s) = \alpha (s) \) and \( \overline{\beta} (s) = \overline{\alpha} (s) \).

Corollary 2.7. Let the distribution parameters of the ruled surfaces \( X \) and \( \overline{X} \) be \( P_B \) and \( \overline{P}_{\overline{B}} \), respectively. Then \( \overline{P}_{\overline{B}} = \frac{\kappa^2 r' - \kappa (\frac{1}{r'})(\overline{r'})' \kappa'}{\kappa^2 r'^2 + r'^2} \).

Let \( \alpha \) be a unit speed spacelike space curve with a spacelike binormal. Then the natural lift \( \overline{\alpha} \) of \( \alpha \) is a timelike space curve.

(i) Let \( X \) and \( \overline{X} \) be two ruled surfaces which is given by
\[
X (s, v) = \alpha (s) + vT (s), \quad \overline{X} (s, v) = \overline{\alpha} (s) + v\overline{T} (s).
\]
The striction curves of \( X \) and \( \overline{X} \) are given by \( \beta (s) = \alpha (s) - \lambda T (s) \) and \( \overline{\beta} (s) = \overline{\alpha} (s) - \mu \overline{T} (s) \), respectively. Then we obtain
\[
\lambda = 0, \quad \mu = 0.
\]

The distribution parameters of the ruled surfaces \( X \) and \( \overline{X} \) are defined by \( P_T = \frac{\text{det} (\alpha', T, T')}{\|T'\|^2} \) and \( \overline{P}_{\overline{T}} = \frac{\text{det} (\overline{\alpha}', \overline{T}, \overline{T}')}{\|\overline{T}'\|^2} \). Then we have
\[
P_T = 0, \quad \overline{P}_{\overline{T}} = 0.
\]

Corollary 2.8. Let the striction curves of \( X \) and \( \overline{X} \) be given by \( \beta (s) = \alpha (s) - \lambda T (s) \) and \( \overline{\beta} (s) = \overline{\alpha} (s) - \mu \overline{T} (s) \), respectively. Then \( \beta (s) = \alpha (s) \) and \( \overline{\beta} (s) = \overline{\alpha} (s) \).

Corollary 2.9. If the ruled surface \( X \) is developable then the ruled surface \( \overline{X} \) are also developable.

(ii) Let \( X \) and \( \overline{X} \) be two ruled surfaces which is given by
\[
X (s, v) = \alpha (s) + vN (s), \quad \overline{X} (s, v) = \overline{\alpha} (s) + v\overline{N} (s).
\]
The striction curves of $X$ and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda N(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{N}(s)$, respectively. Then we have

$$\lambda = \frac{\kappa}{\kappa^2 + \tau^2}, \quad \mu = \frac{-\kappa (\kappa^2 + \tau^2) \| W \|}{(\kappa'{}^2 + \tau'{}^2) - (\kappa^2 + \tau^2)^2}.$$ 

The distribution parameters of the ruled surfaces $X$ and $\overline{X}$ are defined by $P_N = \frac{\det(\alpha', N, N')}{\| N' \|^2}$ and $\overline{P}_N = \frac{\det(\overline{\alpha}', \overline{N}, \overline{N}')}{\| \overline{N} \|^2}$. Then we obtain

$$P_N = \frac{\tau}{\kappa^2 + \tau^2}, \quad \overline{P}_N = \frac{-\kappa^2 \tau' + \kappa \tau' \kappa'}{(\kappa'{}^2 + \tau'{}^2) - (\kappa^2 + \tau^2)^2}.$$ 

**Corollary 2.10.** Let the striction curves of $X$ and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda N(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{N}(s)$, respectively. Then $\mu = \frac{-\kappa (\frac{\tau}{\kappa^2 + \tau^2})^2}{(\kappa^2 + \tau^2)^2 - (\frac{\tau}{\kappa^2 + \tau^2})^2}$.

**Corollary 2.11.** Let the distribution parameters of the ruled surfaces $X$ and $\overline{X}$ be $P_N$ and $\overline{P}_N$, respectively. Then $\overline{P}_N = \frac{-\kappa^2 \tau' + \kappa \tau' \kappa'}{(\kappa'{}^2 + \tau'{}^2) + (\frac{\tau}{\kappa^2 + \tau^2})^2}$.

**Corollary 2.12.** If $\alpha$ is a planer curve, then the ruled surfaces $X$ and $\overline{X}$ are developable.

(iii) Let $X$ and $\overline{X}$ be two ruled surfaces which are given by

$$X(s, v) = \alpha(s) + vB(s), \quad \overline{X}(s, v) = \overline{\alpha}(s) + v\overline{B}(s).$$

The striction curves of $X$ and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda B(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{B}(s)$, respectively. Then we obtain

$$\lambda = 0, \quad \mu = 0.$$
The distribution parameters of the ruled surfaces $X$ and $\overline{X}$ are defined by $P_B = \frac{\det(\alpha', B, B')}{\|B'\|^2}$ and $\overline{P}_B = \frac{\det(\pi', \overline{B}, \overline{B}')}{\|\overline{B}\|^2}$. Then we have

$$P_B = \frac{1}{\tau}, \overline{P}_B = \frac{-\kappa^2 \tau' + \kappa \tau' \kappa'}{\kappa^2 + \tau'^2}.$$

**Corollary 2.13.** Let the striction curves of $X$ and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda B(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu B(s)$, respectively. Then $\beta(s) = \alpha(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s)$.

**Corollary 2.14.** Let the distribution parameters of the ruled surfaces $X$ and $\overline{X}$ be $P_B$ and $\overline{P}_B$, respectively. Then $\overline{P}_B = \frac{-\kappa^2 \tau' + \kappa \tau' \kappa'}{\kappa^2 + \tau'^2}$.

Let $\alpha$ be a unit speed spacelike space curve with a timelike binormal. Then the natural lift $\overline{\alpha}$ of $\alpha$ is a spacelike space curve.

(i) Let $X$ and $\overline{X}$ be two ruled surfaces which is given by

$$X(s, v) = \alpha(s) + vT(s), \quad \overline{X}(s, v) = \overline{\alpha}(s) + v\overline{T}(s).$$

The striction curves of $X$ and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda T(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{T}(s)$, respectively. Then we obtain

$$\lambda = 0, \quad \mu = 0.$$

The distribution parameters of the ruled surfaces $X$ and $\overline{X}$ are defined by $P_T = \frac{\det(\alpha', T, T')}{\|T'\|^2}$ and $\overline{P}_T = \frac{\det(\pi', \overline{T}, \overline{T}')}{\|\overline{T}'\|^2}$. Then we have

$$P_T = 0, \quad \overline{P}_T = 0.$$

**Corollary 2.15.** Let the striction curves of $X$ and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda T(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{T}(s)$, respectively. Then $\beta(s) = \alpha(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s)$.

**Corollary 2.16.** If the ruled surface $X$ is developable then the ruled surface $\overline{X}$ are also developable.
(ii) Let $X$ and $\overline{X}$ be two ruled surfaces which is given by

$$X(s, v) = \alpha(s) + vN(s), \quad \overline{X}(s, v) = \overline{\alpha}(s) + v\overline{N}(s).$$

The striction curves of $X$ and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda N(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{N}(s)$, respectively. Then we have

$$\lambda = \frac{-\kappa}{\kappa^2 - \tau^2}, \quad \mu = \frac{\kappa(\kappa^2 + \tau^2) \|W\|}{(\kappa^2 - \tau^2) + (\kappa^2 + \tau^2)^2}.$$

The distribution parameters of the ruled surfaces $X$ and $\overline{X}$ are defined by $P_N = \frac{\det(\alpha', N, N')}{\|N'\|^2}$ and $\overline{P}_N = \frac{\det(\overline{\alpha}', \overline{N}, \overline{N}')}{\|\overline{N}'\|^2}$. Then we obtain

$$P_N = \frac{\tau}{\kappa^2 - \tau^2}, \quad \overline{P}_N = \frac{-\kappa^2 \tau' + \kappa \tau \kappa'}{(\kappa^2 - \tau^2) + (\kappa^2 + \tau^2)^2}.$$

**Corollary 2.17.** Let the striction curves of $X$ and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda N(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{N}(s)$, respectively.

1. If $W$ is a spacelike vector, then $\mu = \frac{-\kappa(\kappa^2 + \tau^2)}{(\kappa^2 - \tau^2) + (\kappa^2 + \tau^2)^2}.$
2. If $W$ is a timelike vector, then $\mu = \frac{-\kappa(\kappa^2 + \tau^2)}{(\kappa^2 - \tau^2) + (\kappa^2 + \tau^2)^2}.$

**Corollary 2.18.** Let the distribution parameters of the ruled surfaces $X$ and $\overline{X}$ be $P_N$ and $\overline{P}_N$, respectively.

1. If $W$ is a spacelike vector, then $\overline{P}_N = \frac{-\kappa^2 \tau' + \kappa(\overline{P}_N \|W\|^2) \kappa'}{(\kappa^2 - \tau^2) + (\kappa^2 + \tau^2)^2}.$
2. If $W$ is a timelike vector, then $\overline{P}_N = \frac{-\kappa^2 \tau' + \kappa(\overline{P}_N \|W\|^2) \kappa'}{(\kappa^2 - \tau^2) + (\kappa^2 + \tau^2)^2}.$

**Corollary 2.19.** If $\alpha$ is a planer curve, then the ruled surface $X$ and $\overline{X}$ are developable.

(iii) Let $X$ and $\overline{X}$ be two ruled surfaces which are given by

$$X(s, v) = \alpha(s) + vB(s), \quad \overline{X}(s, v) = \overline{\alpha}(s) + v\overline{B}(s).$$
The striction curves of $X$ and $\overline{X}$ are given by $\beta(s) = \alpha(s) - \lambda B(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{B}(s)$, respectively. Then we obtain

$$\lambda = 0, \mu = \frac{2\kappa^2 \tau ||W||}{(\kappa^2 + \tau^2) + 4\kappa^2 \tau^2}.$$ 

The distribution parameters of the ruled surfaces $X$ and $\overline{X}$ are defined by $P_B = \frac{\det(\alpha',B,B')}{||B'||^2}$ and $\overline{P}_B = \frac{\det(\overline{\alpha}',\overline{B},\overline{B}')}{||\overline{B}||^2}$. Then we have

$$P_B = -\frac{1}{\tau}, \overline{P}_B = \frac{\kappa^2 \tau' - \kappa \tau \kappa'}{(\kappa^2 + \tau^2) + 4\kappa^2 \tau^2}.$$ 

**Corollary 2.20.** Let the striction curves of $X$ and $\overline{X}$ be given by $\beta(s) = \alpha(s) - \lambda B(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \mu \overline{B}(s)$, respectively. Then $\beta(s) = \alpha(s)$ and $\overline{\beta}(s) = \overline{\alpha}(s) - \frac{2\kappa^2 \tau ||W||}{(\kappa^2 + \tau^2) + 4\kappa^2 \tau^2} \overline{B}(s)$.

**Corollary 2.21.** Let the distribution parameters of the ruled surfaces $X$ and $\overline{X}$ be $P_B$ and $\overline{P}_B$, respectively. Then $\overline{P}_B = \frac{\kappa^2 \tau' + \kappa (\frac{1}{\tau_B} \kappa')}{(\kappa^2 + \tau^2) + 4\kappa^2 \tau^2}$.

**References**


