ON UNIVALENT FUNCTIONS DEFINED BY A NEW GENERALIZED
MULTIPLIER DIFFERENTIAL OPERATOR

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Abstract: The object of this paper is to obtain some interesting properties of functions belonging to a new class $SW^m(\alpha, \beta, \gamma, \rho)$, defined by using a new generalised multiplier differential operator.

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1. Introduction.

Denote by $U$ the open unit disc of the complex plane, $U = \{ z \in \mathbb{C} ; |z| < 1 \}$. Let $H(U)$ be the space of holomorphic functions in $U$. Let $A$ denote the family of functions in $H(U)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$  

By $S$ and $K$ we denote the subclasses of functions in $A$, which are univalent and convex in $U$, respectively. Let $P$ be the well-known Caratheodory class of normalized functions with positive real part in $U$. The convolution or Hadamard product of functions $f$, given by (1.1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is defined as the power series

$$(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, z \in U.$$
We now define a new generalized multiplier differential operator.

**Definition 1.1.** Let \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \beta \geq 0, \alpha \) and \( \gamma \) are real numbers such that \( \alpha + \beta > 0 \) and \( \gamma + \beta \geq 0 \). Then for \( f \in A \), we define a new generalized multiplier operator \( I_{\alpha,\beta,\gamma}^m \) by

\[
I_{\alpha,\beta,\gamma}^0 f(z) = f(z),
\]

\[
I_{\alpha,\beta,\gamma}^1 f(z) = \frac{\alpha f(z) + \beta zf'(z) + \gamma^2 f''(z)}{\alpha + \beta},
\]

\[
\cdots,
\]

\[
I_{\alpha,\beta,\gamma}^m f(z) = I_{\alpha,\beta,\gamma} (I_{\alpha,\beta,\gamma}^{m-1} f(z)).
\]

**Remark 1.2.** If \( f(z) \) is given by (1.1), then from the definition 1.1, we obtain

\[
(1.2) \quad I_{\alpha,\beta,\gamma}^m f(z) = z + \sum_{k=2}^{\infty} A_k(\alpha,\beta,\gamma,m) a_k z^k,
\]

where

\[
(1.3) \quad A_k(\alpha,\beta,\gamma,m) = \left( \frac{\alpha + k\beta + k(k-1)\gamma}{\alpha + \beta} \right)^m.
\]

From (1.2) it follows that \( I_{\alpha,\beta,\gamma}^m f(z) \) can be written in terms of convolution as

\[
(1.4) \quad I_{\alpha,\beta,\gamma}^m f(z) = (f \ast g)(z),
\]

where

\[
(1.5) \quad g(z) = z + \sum_{k=2}^{\infty} A_k(\alpha,\beta,\gamma,m) z^k.
\]

It also follows from (1.2) that

\[
(1.6) \quad (\alpha + \beta) I_{\alpha,\beta,\gamma}^{m+1} f(z) = \alpha d_{\alpha,\beta,\gamma}^m f(z) + \beta z[I_{\alpha,\beta,\gamma}^m f(z)]' + \gamma^2 (I_{\alpha,\beta,\gamma}^m f(z))''.
\]

We note that

- \( I_{\alpha,\beta,0}^m f(z) = I_{\alpha,\beta}^m f(z) \) (See [18]).
- \( I_{\alpha-\beta,0}^m f(z) = D_{\beta}^m f(z), \beta \geq 0 \) (See Al-Oboudi [1]).
\[ I^{m}_{l+1-\beta,\beta,0}f(z) = I^{m}_{l,\beta}f(z), l > -1, \beta \geq 0 \text{(See Catas [5]).} \]

\[ I^{m}_{l,\lambda+\delta,-\delta,\delta}f(z) = D^{m}_{\lambda,\delta}f(z), \lambda \geq (\delta/(\delta + 1)), \delta \geq 0 \text{(See Raducanu et.al [14]).} \]

**Remark 1.2.**

**a)** \( I^{m}_{l+\lambda+\delta,-\delta,\delta}f(z) = D^{m}_{\lambda,\delta}f(z) \) was investigated for \( \lambda \geq \delta \geq 0 \) in [11] and [15].

So our results in this paper are improvement of corresponding results proved earlier for \( D^{m}_{\lambda,\delta}f(z) \), from \( \lambda \geq \delta \geq 0 \) to \( \lambda \geq (\delta/(\delta + 1)), \delta \geq 0 \).

**b)** \( D^{m}_{l}f(z) \) was introduced by Salagean [16] and was considered for \( m \geq 0 \) in [3].

**Definition 1.3.** Let \( m \in N_0 = N \cup \{0\} \), \( \rho \in [0,1), \beta \geq 0, \alpha \) and \( \gamma \) are real numbers such that \( \alpha + \beta > 0 \) and \( \gamma + \beta \geq 0 \). Then a function \( f \in A \) is said to be in the class \( SW^m(\alpha, \beta, \gamma, \rho) \), if it satisfies the condition

\[ \text{Re}[I^{m}_{\alpha,\beta,\gamma}f(z)] > \rho, z \in U. \]

The main object of this paper is to present a systematic investigation of the class \( SW^m(\alpha, \beta, \gamma, \rho) \). In particular; we derive an inclusion result, structural formula, extreme points and other interesting results.

**2. Preliminaries**

In order to prove our results, we will make use of the following lemmas.

**Lemma 2.1** ([13]). Let \( A \geq 0, h \in K \). Suppose that \( B(z) \) and \( D(z) \) are analytic in \( U \), with \( D(0) = 0 \) and

\[ \text{Re}(B(z)) \geq A + 4 \frac{|D(z)|}{h(0)}, z \in U. \]

If an analytic function \( p \) with \( p(0) = h(0) \) satisfies

\[ Az^2 p(z) + B(z)zp'(z) + p(z) + D(z) < h(z), z \in U, \]

then \( p(z) < h(z), z \in U \).

**Lemma 2.2** ([12]). Let \( q \) be a convex function in \( U \) and let \( h(z) = q(z) + \rho q'(z) \), where \( \rho > 0 \). If \( p \in H(U) \) with \( p(z) = q(0) + p_1 z + p_2 z^2 + \ldots \) and \( p(z) + \rho q'(z) < h(z), z \in U \), then

\[ p(z) < q(z), z \in U, \]
and this result is sharp.

**Lemma 2.3** ([17]). If $p(z)$ is analytic in $U$, $p(0) = 1$ and $\text{Re}(p(z)) > \frac{1}{2}$, then for any function $F$ analytic in $U$, the function $F^* p$ takes values in the convex hull of $F(U)$.

Note that the symbol “≺” stands for subordination throughout this paper.

### 3. Main Results.

**Theorem 3.1.** If $m \in N_0 = N \cup \{0\}$, $\rho \in [0, 1)$, $\alpha \geq 0$, $\gamma$ and $\alpha + \beta > 0$ and $\gamma + \beta \geq 0$, then $SW^{m+1}(\alpha, \beta, \gamma, \rho) \subset SW^{m}(\alpha, \beta, \gamma, \rho)$.

**Proof.** Let $f \in SW^{m+1}(\alpha, \beta, \gamma, \rho)$. By using the properties of the operator $I^{m}_{a, \beta, \gamma}$, we get

$$I_{a, \beta, \gamma}^{m+1} f(z) = \frac{\partial^{m}_{a, \beta, \gamma} f(z) + \beta \gamma (I_{a, \beta, \gamma}^{m} f(z)) + \gamma^{2} (I_{a, \beta, \gamma}^{m} f^{*}(z))^{\gamma}}{\alpha + \beta}. \quad (3.1)$$

Differentiating (3.1) with respect to $z$ and using (1.6), we obtain

$$\left(I_{a, \beta, \gamma}^{m+1} f(z) \right) = \left\{ p(z) + \left( \frac{\beta + 2 \gamma}{\alpha + \beta} \right) z \frac{\partial^{*} p(z)}{\partial z} + \left( \frac{\gamma}{\alpha + \beta} \right) z^{2} \frac{\partial^{2} p(z)}{\partial z^{2}} \right\} \quad (3.2)$$

where

$$p(z) = (I_{a, \beta, \gamma}^{m} f(z)).$$

Since $f \in SW^{m+1}(\alpha, \beta, \gamma, \rho)$, by using Definition 1.3 and (3.2), we have

$$\text{Re}\left\{ p(z) + \left( \frac{\beta + 2 \gamma}{\alpha + \beta} \right) z \frac{\partial^{*} p(z)}{\partial z} + \left( \frac{\gamma}{\alpha + \beta} \right) z^{2} \frac{\partial^{2} p(z)}{\partial z^{2}} \right\} > \rho, z \in U,$$

which is equivalent to

$$\left\{ p(z) + \left( \frac{\beta + 2 \gamma}{\alpha + \beta} \right) z \frac{\partial^{*} p(z)}{\partial z} + \left( \frac{\gamma}{\alpha + \beta} \right) z^{2} \frac{\partial^{2} p(z)}{\partial z^{2}} \right\} < \frac{1 + (2 \rho - 1)z}{1 + z} \equiv h(z).$$

From Lemma 2.1, with $A = \left( \frac{\gamma}{\alpha + \beta} \right)$, $B(z) = \left( \frac{\beta + 2 \gamma}{\alpha + \beta} \right)$, and $D(z) = 0$ we have $p(z) < h(z)$, which implies that $\text{Re}(I_{a, \beta, \gamma}^{m+1} f(z)) > \rho, z \in U$. Hence $f \in SW^{m}(\alpha, \beta, \gamma, \rho)$ and the proof of the theorem is complete.

Clearly $SW^{m}(\alpha, \beta, \gamma, \rho) \subset SW^{m-1}(\alpha, \beta, \gamma, \rho) \subset \ldots \subset SW^{0}(\alpha, \beta, \gamma, \rho) \subset S$ (see [6, 8]) and one can easily show that the set $SW^{m}(\alpha, \beta, \gamma, \rho)$ is convex (see [11]).
Theorem 3.2. Let \( q \) be convex function with \( q(0) = 1 \) and let \( h \) be a function of the form \( h(z) = q(z) + zq'(z), z \in U \). If \( f \in A \) satisfies the differential subordination 
\[
(I_{a,b,y}^m f(z))' < h(z), z \in U,
\]
then \( (I_{a,b,y}^m f(z))/z < q(z) \) and the result is sharp.

**Proof.** If we let \( p(z) = (I_{a,b,y}^m f(z))/z, z \in U \), then we obtain 
\[
(I_{a,b,y}^m f(z))' = p(z) + zp'(z).
\]
So the subordination \( (I_{a,b,y}^m f(z))' < h(z), z \in U \), becomes
\[
p(z) + zp'(z) < q(z) + zq'(z), z \in U,
\]
and hence from Lemma 2.2 we have \( (I_{a,b,y}^m f(z))/z < q(z) \). The result is sharp.

We now obtain a structural formula, extreme points and coefficient bounds for functions in \( SW^m(\alpha, \beta, \gamma, \rho) \).

Theorem 3.3. A function \( f \in A \) is in the class \( SW^m(\alpha, \beta, \gamma, \rho) \) if and only if it can be expressed as
\[
f(z) = \left[ z + \sum_{k=2}^{\infty} \frac{\lambda_k}{A_k(\alpha, \beta, \gamma, m)} z^k \right] \ast \left[ z + 2(1 - \rho) \sum_{k=2}^{\infty} (\zeta_k^2)^{k-1} \right] d\mu(\zeta),
\]
where \( A_k(\alpha, \beta, \gamma, m) \) is given by (1.3) and \( \mu \) is a positive probability measure defined on the unit circle \( E = \{ \zeta \in C : |\zeta| = 1 \} \).

**Proof.** From Definition 1.3 it follows that \( f \in SW^m(\alpha, \beta, \gamma, \rho) \) if and only if
\[
\frac{[I_{a,b,y}^m f(z)] - \rho}{1 - \rho} \in P.
\]
Using Hergoltz integral representation of functions in Caratheodory class \( P \) (see [7] and [9]), there exists a positive Borel probability measure \( \mu \) such that
\[
\frac{[I_{a,b,y}^m f(z)] - \rho}{1 - \rho} = \int_{|\zeta|=1} \frac{(1 + \zeta^2)}{(1 - \zeta^2)} d\mu(\zeta), z \in U,
\]
which is equivalent to
\[
[I_{a,b,y}^m f(z)] = \int_{|\zeta|=1} \frac{(1 + (1 - 2\rho)\zeta^2)}{1 - \zeta^2} d\mu(\zeta), z \in U.
\]
Integrating we obtain
Equality (3.3) follows now, from (1.4), (1.5) and (3.4). Since the converse of this deductive process is also true, we have proved our theorem.

**Corollary 3.4.** The extreme points of the class $SW^m(\alpha, \beta, \gamma, \rho)$ are

$$f_\zeta(z) = z + 2(1-\rho)\zeta \sum_{k=1}^\infty \frac{(\zeta^k)}{kA_k(\alpha, \beta, \gamma, \rho)}$$

$z \in U, |\zeta| = 1$.

**Proof.** Consider the functions $g_\zeta(z) = z + 2(1-\rho)\zeta \sum_{k=1}^\infty \frac{(\zeta^k)}{k}$ and $g_\mu(z) = \int_{|\zeta| = 1} g_\zeta(z) d\mu(\zeta)$.

The assertion now follows from (3.3), (making use of (1.4), (1.5) and (3.4)), since the map $\mu \rightarrow g_\mu$ is one-to-one (see [4]).

**Corollary 3.5.** If $f \in SW^m(\alpha, \beta, \gamma, \rho)$ is given by (1.1), then

$$|a_k| \leq \frac{2(1-\rho)}{kA_k(\alpha, \beta, \gamma, \rho)}, k \geq 2.$$ The result is sharp.

**Proof.** The result follows from (3.5), since the coefficient bounds are maximized at an extreme point.

**Corollary 3.6.** If $f \in SW^m(\alpha, \beta, \gamma, \rho)$, then for $|z| = r < 1$,

$$r - 2(1-\rho)r^2 \sum_{k=2}^\infty \frac{1}{kA_k(\alpha, \beta, \gamma, \rho)} \leq |f(z)| \leq r + 2(1-\rho)r^2 \sum_{k=2}^\infty \frac{1}{kA_k(\alpha, \beta, \gamma, \rho)}$$

and

$$1 - 2(1-\rho)r \sum_{k=2}^\infty \frac{1}{A_k(\alpha, \beta, \gamma, \rho)} \leq |f'(z)| \leq 1 + 2(1-\rho)r \sum_{k=2}^\infty \frac{1}{A_k(\alpha, \beta, \gamma, \rho)}.$$ 

Next, we prove the analogue of the Polya-Schoenberg conjecture for the class $SW^m(\alpha, \beta, \gamma, \rho)$.

**Theorem 3.7.** If $f \in SW^m(\alpha, \beta, \gamma, \rho)$ and $g \in K$, then $f^* g \in SW^m(\alpha, \beta, \gamma, \rho)$. 
Proof. It is known that if \( g \in \mathbb{K} \), then \( \text{Re} \left( \frac{g(z)}{z} \right) > \frac{1}{2} \) (see [12]). Making use of the convolution properties, we have

\[
\text{Re} [I_{a, \beta, \gamma}^m (f * g)(z)] = \text{Re} \left( \left( I_{a, \beta, \gamma}^m f(z) \right) \ast \frac{g(z)}{z} \right).
\]

The result now follows, by applying Lemma 2.3.

**Corollary 3.8.** The class \( SW^m (\alpha, \beta, \gamma, \rho) \) is invariant under Bernardi integral operator.

**Proof.** Let \( f \in SW^m (\alpha, \beta, \gamma, \rho) \). The Bernardi integral operator is defined as (see [2]):

\[
F_c(h)(z) = \frac{c+1}{c^c} \int_0^c t^{-1} h(t) dt, (c > -1; h \in A).
\]

It is easy to check that \( F_c(f)(z) = (f * g)(z) \) where

\[
g(z) = \sum_{k=1}^{\infty} \left( \frac{c+1}{c+k} \right)^k z^k = \frac{c+1}{c^c} \int_0^c \frac{t^c}{1-t} dt, z \in U, c > -1.
\]

Since the function \( \phi(z) = \frac{z}{1-z}, z \in U \) is convex, it follows (see [10]) that the function \( g \) is also convex. From Theorem 3.7 we obtain \( F_c(f) \in SW^m (\alpha, \beta, \gamma, \rho) \). Therefore \( F_c(SW^m (\alpha, \beta, \gamma, \rho)) \subset SW^m (\alpha, \beta, \gamma, \rho) \).

**REFERENCES**


