A GENERALIZATION OF MULTIPLICATIVE (GENERALIZED)-DERIVATIONS

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Abstract. Let $R$ be a semiprime ring and $L$ be a semigroup ideal of $R$. The main object in this paper is to study the following situations in semiprime rings: When $F$ is a multiplicative $(\alpha, 1)$-(generalized) derivation associated with a map $d$, (i) $F(xy) \pm \alpha(x)\alpha(y) = 0$ for $x, y \in L$. (ii) $F(x)F(y) \pm \alpha(x)\alpha(y) = 0$ for all $x, y \in L$. When $F$ is a multiplicative $(1, \alpha)$-(generalized) derivation associated with a map $d$, (iii) $F(xy) \pm xy = 0$ for all $x, y \in L$. (iv) $F(x)F(y) \pm xy = 0$ for all $x, y \in L$.

Keywords: semiprime ring; multiplicative derivation; multiplicative generalized derivation; multiplicative (generalized)-derivation; multiplicative $(\alpha, 1)$-(generalized) derivation; multiplicative $(1, \alpha)$-(generalized) derivation.

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1. Introduction

In this paper $R$ denotes an associative ring. A ring $R$ is called semiprime ring if $aRa = (0)$ implies that $a = 0$. A subset $L$ is called a left semigroup ideal of $R$ if $ra \in L$ for all $a \in L$ and for

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all \( r \in R \). Obviously, every left ideal is a left semigroup ideal. An additive mapping \( d : R \to R \) is called a \textit{derivation} of \( R \) if \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in R \). In 1991, Daif, M. N. [1] defined that a map \( D \) is called a \textit{multiplicative derivation} of \( R \) if \( D(xy) = D(x)y + xD(y) \) for all \( x, y \in R \). In 1997, this definition of multiplicative derivation was extended to multiplicative generalized derivation by Daif, M. N. and Tammam El-Sayid, M. S. [2] as follow: a map \( F : R \to R \) is called a \textit{multiplicative generalized derivation} if there exists a derivation \( d \) such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). In 2013, the definition of multiplicative generalized derivation was extended to multiplicative (generalized)-derivation by Dahara, B. and Ali, S. [3] as follow: a map \( F : R \to R \) is called a \textit{multiplicative (generalized)-derivation} if there exists a map \( F : R \to R \) such that \( F(xy) = F(x)y + xg(y) \) for all \( x, y \in R \) where \( g \) is any mapping on \( R \).

We introduce the notion of multiplicative two-sided \( \alpha \)-(generalized) derivation of \( R \) as follows.

A map \( F : R \to R \) is said to be a \textit{multiplicative} \((\alpha, 1)\)-(generalized) \textit{derivation} if there exists maps \( d, \alpha : R \to R \) such that

\[
F(xy) = F(x)\alpha(y) + xd(y)
\]

for all \( x, y \in R \).

Similarly, if \( F(xy) = F(x)y + \alpha(x) d(y) \) for all \( x, y \in R \) than \( F \) is called a \textit{multiplicative} \((1, \alpha)\)-(generalized) \textit{derivation}. A map \( F : R \to R \) is called a \textit{multiplicative two-sided} \( \alpha \)-(generalized) \textit{derivation} if \( F \) is a multiplicative \((\alpha, 1)\)-(generalized) derivation as well as multiplicative \((1, \alpha)\)-(generalized) derivation. It is clear that every multiplicative (generalized)-derivation is multiplicative two-sided \( \alpha \)-(generalized) derivation on \( R \). But the converse is not true. The following example justifies the fact:

\section*{Example 1.}
Let \( S \) be a ring and \( R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in S \right\} \). Define the maps \( d, \alpha, F : R \to R \) as follows:

\[
d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
and \[
F \begin{pmatrix}
0 & a & b \\
0 & 0 & c \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & bc \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Then it is easy to verify that \( F \) is a multiplicative two-sided \( \alpha \)-(generalized) derivation associated with a map \( d \) but \( F \) is not a multiplicative (generalized) derivation of \( R \).

In this connection, our aim in the present paper is to generalize the study of Dahara, B. and Ali, S. [3] in the case of a left semigroup ideal, a multiplicative \( (\alpha, 1) \)- and \( (1, \alpha) \)-(generalized) derivation and to investigate some properties satisfying certain differential identities.

Throughout this paper, \( R \) is a semiprime ring, \( L \) is a nonzero left semigroup ideal of \( R \) and \( \alpha \) is an epimorphism of \( R \).

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2. Results

**Lemma 2.1.** Let \( R \) is a semiprime ring, \( L \) is a nonzero left semigroup ideal of \( R \) and \( 0 \neq a \in R \). If \( aL = (0) \) then \( La = (0) \).

**Proof.** Since \( L \) is a semigroup ideal of \( R \), \( a(RL) = (0) \). This gives \( (La)R(La) = (0) \). Due to primeness of \( R \), \( La = (0) \). \( \square \)

**Theorem 2.1.** Let \( R \) is a semiprime ring, \( L \) is a nonzero left semigroup ideal of \( R \) and \( F \) is a multiplicative \( (\alpha, 1) \)-(generalized) derivation. If \( F(xy) \pm \alpha(x)\alpha(y) = 0 \) for all \( x, y \in L \) then \( Ld(L) = (0) \). \( F(xy) = F(x)\alpha(y) \) for all \( x, y \in L \) and \( [F(x), \alpha(x)] = 0 \) for all \( x \in L \).

**Proof.** By the hypothesis, we have

\[(1) \quad F(xy) - \alpha(x)\alpha(y) = 0 \]

for all \( x, y \in L \). Replacing \( y \) by \( yz, z \in L \) in (1), we get

\[F(xyz) - \alpha(x)\alpha(yz) = 0 \text{ for all } x, y, z \in L.\]
Since \( F(xy) = F(x)\alpha(y) + xd(y) \) for all \( x, y \in R \) and \( \alpha \) is an epimorphism of \( R \), we can rewrite the above equation

\[
0 = F(xy)\alpha(z) + xyd(z) - \alpha(x)\alpha(y)\alpha(z) = (F(xy) - \alpha(x)\alpha(y))\alpha(z) + xyd(z)
\]

for all \( x, y \in L \). By (1) that gives

\[
xd(z) = 0 \quad \text{for all} \quad x, y, z \in L.
\]

Taking \( d(z)rx, r \in R \) instead of \( y \) in the last equation, we get

\[
xd(z)rxd(z) = 0 \quad \text{for all} \quad x, y, z \in L, r \in R.
\]

In particular, \( xd(z)Rxh(z) = (0) \) for all \( x, z \in L \). Since \( R \) is a semiprime ring, the last expression forces that \( xd(z) = 0 \) for all \( x, z \in L \). That is,

\[
Ld(L) = (0).
\]

Thus \( F(xy) = F(x)\alpha(y) + xd(y) = F(x)\alpha(y) \) for all \( x, y \in L \). From the equation (1), we get

\[
0 = F(xy) - \alpha(x)\alpha(y) = F(x)\alpha(y) - \alpha(x)\alpha(y) = (F(x) - \alpha(x))\alpha(y) \quad \text{for all} \quad x, y \in L.
\]

That is,

\[
(F(x) - \alpha(x))\alpha(L) = (0) \quad \text{for all} \quad x \in L.
\]

Considering \( L \) is a left semigroup ideal of \( R \), \( \alpha \) is an epimorphism of \( R \) and \( \alpha(L) \) is a semigroup ideal of \( R \) together with Lemma 2.1, we have \( \alpha(L)(F(x) - \alpha(x)) = (0) \) for all \( x \in L \). Thus \( (F(x) - \alpha(x))\alpha(L) = (0) \) and \( \alpha(L)(F(x) - \alpha(x)) = (0) \) for all \( x \in L \), together implies

\[
[F(x) - \alpha(x), \alpha(L)] = (0) \quad \text{for all} \quad x \in L.
\]

This yields that \( [F(x), \alpha(x)] = 0 \) for all \( x \in L \).

Similarly, we can prove that the same results for

\[
F(xy) + \alpha(x)\alpha(y) = 0
\]

for all \( x, y \in L \). \( \square \)

**Corollary 2.1.** Let \( R \) is a semiprime ring, \( L \) is a nonzero left semigroup ideal of \( R \) and \( F \) is a multiplicative \((\alpha, 1)-(generalized) \) derivation. If \( F(xy) \pm \alpha(x)\alpha(y) = 0 \) for all \( x, y \in R \) then \( d = 0 \) and \( F = \pm\alpha \).
Proof. By Theorem 2.1, we get \( d = 0 \) and \( F(xy) = F(x)\alpha(y) \) for all \( x, y \in R \). From the hypothesis, \( 0 = F(x)\alpha(y) \pm \alpha(x)\alpha(y) = (F(x) \pm \alpha(x))\alpha(y) \) for all \( x, y \in R \). That is,

\[
(F(x) \pm \alpha(x))R = (0) \text{ for all } x \in R.
\]

Since \( R \) is a semiprime ring, \( F = \pm \alpha \).

\( \square \)

**Theorem 2.2.** Let \( R \) is a semiprime ring, \( L \) is a nonzero left semigroup ideal of \( R \) and \( F \) is a multiplicative \((\alpha,1)\)-(generalized) derivation. If \( F(x)F(y) \pm \alpha(x)\alpha(y) = 0 \) for all \( x, y \in L \) then \( Ld(L) = (0), \) \( F(xy) = F(x)\alpha(y) \) for all \( x, y \in L \) and \( \alpha(L) \) \( [F(x),\alpha(x)] = (0) \) for all \( x \in L \).

Proof. By the assumption, we have

\[
(2) \quad F(x)F(y) - \alpha(x)\alpha(y) = 0
\]

for all \( x, y \in L \). Replacing \( y \) by \( yz, z \in L \) in (2), we get

\[
F(x)F(yz) - \alpha(x)\alpha(yz) = 0 \text{ for all } x, y, z \in L.
\]

It holds that

\[
0 = F(x)(F(y)\alpha(z) + yd(z)) - \alpha(x)\alpha(y)\alpha(z)
\]

\[
= F(x)F(y)\alpha(z) + F(x)yd(z) - \alpha(x)\alpha(y)\alpha(z)
\]

\[
= (F(x)F(y) - \alpha(x)\alpha(y))\alpha(z) + F(x)yd(z) \text{ for all } x, y, z \in L.
\]

By (2) it reduces

\[
(3) \quad F(x)yd(z) = 0 \text{ for all } x, y, z \in L
\]

Replacing \( x \) with \( ux, u \in L \), we obtain \( F(ux)yd(z) = 0 \) for all \( u, x, y, z \in L \). It follows that

\[
0 = (F(u)\alpha(x) + ud(x))yd(z)
\]

\[
= F(u)\alpha(x)yd(z) + ud(x)yd(z)
\]

Since \( L \) is a left semigroup ideal of \( R \) and by using (3) it gives

\[
(4) \quad ud(x)yd(z) = 0 \text{ for all } x, y, z \in L
\]

Replacing \( y \) by \( ry, r \in R \) in (4), we get \( ud(x)ryd(z) = 0 \) for all \( u, x, y, z \in L \) and \( r \in R \). This implies that \( ud(x)Ryd(z) = (0) \) for all \( u, x, y, z \in L \). Taking \( y = u \) and \( z = x \), we obtain \( yd(x)Ryd(x) = (0) \) for all \( x, y \in L \). Since \( R \) is a semiprime ring, we have \( yd(x) = 0 \) for all \( x, y \in L \). Namely,
Therefore \( F(xy) = F(x)\alpha(y) + yd(z) = F(x)\alpha(y) \) for all \( x, y \in L \). Replacing \( x \) by \( xy \) in (2), we get

\[
F(x)\alpha(y)F(y) - \alpha(x)\alpha(y)^2 = 0
\]

for all \( x, y \in L \). Equation (2) multiplied by \( \alpha(y) \) from right, we get

\[
F(x)F(y)\alpha(y) - \alpha(x)\alpha(y)^2 = 0
\]

for all \( x, y \in L \). Substracting (5) from (6), we get

\[
F(x)[F(y), \alpha(y)] = 0
\]

for all \( x, y \in L \). Replacing \( x \) by \( xz, z \in L \) in (7), we get

\[
F(x)\alpha(z)[F(y), \alpha(y)] = 0
\]

for all \( x, y, z \in L \). This implies

\[
\alpha(L)[F(x), \alpha(x)]R\alpha(L)[F(x), \alpha(x)] = (0).
\]

Since \( R \) is a semiprime ring, it implies that \( \alpha(L)[F(x), \alpha(x)] = (0) \) for all \( x \in L \).

Similar way, we can prove that same conclusion for \( F(xy) + \alpha(x)\alpha(y) = 0 \) for all \( x, y \in L \).

\[\square\]

**Corollary 2.2.** Let \( R \) is a semiprime ring, \( L \) is a nonzero left semigroup ideal of \( R \) and \( F \) is a multiplicative \((\alpha, 1)\)-(generalized) derivation. If \( F(xy) = F(x)\alpha(y) = 0 \) for all \( x, y \in R \) then \( d = 0 \) and \( F(xy) = F(x)\alpha(y) \) for all \( x, y \in R \).

**Proof.** Using Theorem 2.2, we come to a conclusion \( d = 0 \) and \( F(xy) = F(x)\alpha(y) \) for all \( x, y \in R \).

\[\square\]

**Theorem 2.3.** Let \( R \) is a semiprime ring, \( L \) is a nonzero left semigroup ideal of \( R \) and \( F \) is a multiplicative \((1, \alpha)\)-(generalized) derivation. If \( F(xy) + \alpha(x)\alpha(y) = 0 \) for all \( x, y \in L \) then \( \alpha(L)d(L) = (0) \), \( F(xy) = F(x)y \) for all \( x, y \in L \) and \( F \) is a commuting map on \( L \).

\[\square\]
Proof. Assume that

\[(8) \quad F(xy) - xy = 0 \text{ for all } x, y \in L.\]

Taking \(yz, z \in L\) instead of \(y\) in (8), \(F(xyz) - xyz = 0\) for all \(x, y, z \in L\). Since \(F(xy) = F(x)y + \alpha(x)d(y)\) for all \(x, y \in R\) and \(\alpha\) is an epimorphism of \(R\), it follows that

\[
0 = F(xy)z + \alpha(xy)d(z) - xyz = (F(xy) - xy)z + \alpha(x)\alpha(y)d(z)
\]

for all \(x, y \in L\). By (8) it holds that

\[
\alpha(x)\alpha(y)d(z) = 0 \text{ for all } x, y, z \in L.
\]

Replacing \(y\) with \(rx, r \in R\), we get \(\alpha(x)\alpha(rx)d(z) = 0\). Since \(\alpha\) is an epimorphism of \(R\), it holds \(\alpha(x)R\beta(x)d(z) = (0)\) for all \(x, z \in L\). This implies

\[
\alpha(x)d(z)R\alpha(x)d(z) = (0) \text{ for all } x, y, z \in L.
\]

Since \(R\) is a semiprime ring, \(\alpha(x)d(z) = 0\) for all \(x, z \in L\). That is,

\[
\alpha(L)d(L) = (0).
\]

So, we obtain \(F(xy) = F(x)y + \alpha(y)d(z) = F(x)y\) for all \(x, y \in L\). Using (8), one obtains

\[
0 = F(xy) - xy = F(x)y - xy = (F(x) - x)y \text{ for all } x, y \in L.\]

In particular

\[
(F(x) - x)L = (0) \text{ for all } x \in L.
\]

Since \(L\) is a left semigroup ideal of \(R\). By Lemma 2.1, we have

\[
L(F(x) - x) = (0) \text{ for all } x \in L.
\]

Thus \((F(x) - x)L = (0)\) and \(L(F(x) - x) = (0)\) for all \(x \in L\), together implies

\[
[F(x) - x, L] = (0) \text{ for all } x \in L.
\]

This yields that \([F(x), x] = 0\) for all \(x \in L\). Thus, \(F\) is a commuting map on \(L\).

In a similarly, we can prove that to achieve the same results for \(F(xy) + xy = 0\) for all \(x, y \in L\).

□
Corollary 2.3. Let $R$ is a semiprime ring, $L$ is a nonzero left semigroup ideal of $R$ and $F$ is a multiplicative $\ (1, \alpha)\ -(generalized)$ derivation. If $F(xy) \pm xy = 0$ for all $x, y \in R$ then $d = 0$, $F(x) = \pm x$ and $F$ is a commuting map on $R$.

Proof. By Theorem 2.3 we have $d = 0$ and $F(xy) = F(x)y$ for all $x, y \in R$. From the hypothesis, we obtain $F(xy) \pm xy = 0$ for all $x, y \in R$. Since $F(xy) = F(x)y$, it implies that $(F(x) \pm x)y = 0$ for all $x, y \in R$. That is,

$$(F(x) \pm x)R = (0) \quad \text{for all } x \in R.$$  

Since $R$ is a semiprime ring, it follows that $F(x) = \pm x$ for all $x \in R$. \hfill \Box

Theorem 2.4. Let $R$ is a semiprime ring, $L$ is a nonzero left semigroup ideal of $R$ and $F$ is a multiplicative $\ (1, \alpha)\ -(generalized)$ derivation. If $F(xy) \pm xy = 0$ for all $x, y \in L$ then $\alpha(L)d(L) = (0)$, $F(xy) = F(x)y$ for all $x, y \in L$ and $L[F(x), x] = (0)$ for all $x \in L$.

Proof. First we consider that

$$(9) \quad F(x)F(y) - xy = 0$$

for all $x, y \in L$. Substituting $yz, z \in L$ for $y$ in $(9)$, we get $F(x)F(yz) - xyz = 0$ for all $x, y, z \in L$. Since $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in R$, it follows that

$$0 = F(x)(F(y)z + \alpha(y)d(z)) - xyz$$
$$= F(x)F(y)z + F(x)\alpha(y)d(z) - xyz$$
$$= (F(x)F(y) - xy)z + F(x)\alpha(y)d(z)$$

By $(9)$ it gives

$$(10) \quad F(x)\alpha(y)d(z) = 0 \quad \text{for all } x, y, z \in L$$

Replacing $x$ with $ux, u \in L$, we get $F(ux)\alpha(y)d(z) = 0$ for all $u, x, y, z \in L$. Since $F(xy) = F(xy) + \alpha(x)d(y)$ for all $x, y \in R$, it follows that $0 = (F(u)x + \alpha(u)d(x))\alpha(y)d(z) = F(ux)\beta(y)d(z) +$
\(\alpha(u)d(x)\alpha(y)d(z)\). Since \(L\) is a left semigroup ideal of \(R\) and \(\alpha\) is an epimorphism of \(R\), \(\alpha(L)\) is a left semigroup ideal of \(R\). By using (10), it gives

\[\alpha(u)d(x)\alpha(y)d(z) = 0\]

for all \(u, x, y, z \in L\). Replacing \(y\) by \(ry\), \(r \in R\) in (11), we get \(\alpha(u)d(x)\alpha(ry)d(z) = 0\) for all \(u, x, y, z \in L\) and \(r \in R\). Since \(\alpha\) is an epimorphism of \(R\), it implies that \(\alpha(u)d(x)\alpha(y)d(z) = 0\) for all \(u, x, y, z \in L\). Taking \(y = u\) and \(z = x\). We obtain

\[\alpha(y)d(x)\alpha(y)d(x) = 0\] for all \(x, y \in L\).

Since \(R\) is a semiprime ring, we have \(\alpha(y)d(x) = 0\) for all \(x, y \in L\). That is, \(\alpha(L)d(L) = 0\).

Thus \(F(xy) = F(x)y + \alpha(y)d(z) = F(x)y\) for all \(x, y \in L\). Replacing \(x\) by \(xy\) in (9), we get

\[F(xy)G(y) - xy^2 = 0\]

for all \(x, y \in L\). (9) multiplied by \(\alpha(y)\) from right, we get

\[F(x)F(y) - xy^2 = 0\]

for all \(x, y \in L\). Substracting (12) from (13), we get

\[F(x)[F(y), y] = 0\]

for all \(x, y \in L\). Replacing \(x\) by \(xz\), \(z \in L\) in (14), we get

\[F(x)z[F(y), y] = 0\]

for all \(x, y, z \in L\). This implies \(L[F(x), x]R[L[F(x), x] = 0\). Since \(R\) is a semiprime ring, it follows that \(L[F(x), x] = 0\) for all \(x \in L\).

In the same way, we can prove the same results for \(F(x)F(y) + xy = 0\) for all \(x, y \in L\).

\[\square\]

**Corollary 2.4.** Let \(R\) is a semiprime ring, \(L\) is a nonzero left semigroup ideal of \(R\) and \(F\) is a multiplicative \((1, \alpha)-(generalized)\) derivation. If \(F(xy)F(y) \pm xy = 0\) for all \(x, y \in R\) then \(d = 0\), \(F(xy) = F(x)y\) for all \(x, y \in R\) and \(F\) is a commuting map on \(R\)

**Proof.** By using Theorem 2.4, we conclude that \(d = 0\), \(F(xy) = F(x)y\) for all \(x, y \in R\) and \(F\) is a commuting map on \(R\).

\[\square\]
Conflict of Interests

The authors declare that there is no conflict of interests.

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