OSCILLATORY SOLUTIONS FOR DYNAMIC EQUATIONS WITH NON-MONOTONE ARGUMENTS

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Abstract. Consider the first-order delay dynamic equation

\[ x^\Delta(t) + p(t)x(\tau(t)) = 0, \quad t \in [t_0, \infty)_T \]

where \( p \in C_{rd}([t_0, \infty)_T, \mathbb{R}^+) \), \( \tau \in C_{rd}([t_0, \infty)_T, \mathbb{T}) \) is non-monotone, and \( \tau(t) \leq t, \lim_{t \to \infty} \tau(t) = \infty \). Under the assumption that the \( \tau \) is non-monotone, we present sufficient conditions for the oscillation of first-order delay dynamic equations on time scales. An example illustrating the result is also given.

Keywords: dynamic equations; time scale; non-monotone argument; retarded argument; oscillatory solutions.

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1. Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential/difference and dynamic equations have been the subject of many investigations. See, for example, [1–32] and the references cited therein. Consider the first-order delay dynamic
equation

\[ (E) \quad x^\Delta(t) + p(t)x(\tau(t)) = 0, \quad t \in [t_0, \infty)_T \]

where \( T \) is a time scale unbounded above with \( t_0 \in T \), \( p \) is rd-continuous and nonnegative, the delay function \( \tau : T \to T \) is non-monotone and satisfies

\[ (1.1) \quad \tau(t) \leq t \text{ for all } t \in T, \quad \lim_{t \to \infty} \tau(t) = \infty, \]

and \( \sup T = \infty \).

First we give a short review on the time scales calculus extracted from [3]. A time scale, which inherits the standard topology on \( \mathbb{R} \), is a nonempty closed subset of reals. Here and later throughout this paper, a time scale will be denoted by the symbol \( T \), and the intervals with a subscript \( T \) are used to denoted the intersection of the usual interval with \( T \). For \( t \in T \), we define the forward jump operator \( \sigma : T \to T \) by \( \sigma := \inf(t, \infty)_T \) while the backward jump operator \( \rho : T \to T \) is defined by \( \rho := \sup(-\infty, t)_T \), and the graininess function \( \mu : T \to \mathbb{R}^+_0 \) is defined to be \( \mu(t) := \sigma(t) - t \). A point \( t \in T \) is called right-dense if \( \sigma(t) = (t) \) and/or equivalently \( \mu(t) = 0 \) holds; otherwise it is called right-scattered, and similarly left-dense and left scattered points are defined with respect to the backward jump operator. We also need the set \( T^\kappa \) as follows: If \( T \) has a left-scattered maximum \( m \), then \( T^\kappa = T - \{m\} \). Otherwise, \( T^\kappa = T \). A function \( f : T \to \mathbb{R} \) is said to be \( \Delta \)-differentiable at the point \( t \in T^\kappa \) provided that there exists \( f^\Delta(t) \) such that for every \( \varepsilon > 0 \) there exists a neighborhood \( U \) of \( t \) such that

\[
\left| [f(\sigma(t)) - f(s)] - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in U.
\]

We shall mean the \( \Delta \)-derivative of a function when we only say derivative unless otherwise is specified. A function \( f : T \to \mathbb{R} \) is called rd-continuous provided it is continuous at right-dense points in \( T \), and its left-sided limits exist (finite) at left-dense points in \( T \). The set of rd-continuous functions \( f : T \to \mathbb{R} \) will be denoted by \( C_{rd}(T, \mathbb{R}) \).

The set of functions \( f : T \to \mathbb{R} \) that are differentiable and whose derivative is rd-continuous is denoted by \( C^1_{rd}(T, \mathbb{R}) \). For \( s, t \in T \) and a function \( f \in C_{rd}(T, \mathbb{R}) \), the \( \Delta \)-integral of is defined
by
\[ \int_{s}^{t} f(\eta) \Delta(\eta) = F(t) - F(s) \]
where \( F \in C_{rd}^{1}(\mathbb{T}, \mathbb{R}) \) is an anti-derivative of \( f \), i.e., \( F^\Delta = f \) on \( \mathbb{T}^\kappa \). Every rd-continuous function has an antiderivative. In particular, if \( t_0 \in \mathbb{T} \) then \( F \) defined by
\[ F(t) = \int_{s}^{t} f(\eta) \Delta(\eta) \quad \text{for} \quad t \in \mathbb{T} \]
is an antiderivative of \( f \). And, for \( t \in \mathbb{T}^\kappa \)
\[ \int_{t}^{\sigma(t)} f(\eta) \Delta(\eta) = \mu(t)f(t) \]
It is obvious that if \( f^\Delta \geq 0 \), then \( f \) is nondecreasing.

A function \( f \in C_{rd}(\mathbb{T}, \mathbb{C}) \) is called regressive if \( 1 + f\mu \neq 0 \) on \( \mathbb{T}^\kappa \), and \( f \in C_{rd}(\mathbb{T}, \mathbb{C}) \) is called positively regressive if \( 1 + f\mu > 0 \) on \( \mathbb{T}^\kappa \). The set of regressive functions and the set of positively regressive functions are denoted by \( \mathcal{R}(\mathbb{T}, \mathbb{C}) \) and \( \mathcal{R}^+(\mathbb{T}, \mathbb{R}) \), respectively, \( \mathcal{R}^-(\mathbb{T}, \mathbb{R}) \) is defined similarly. For simplicity, we denote by \( \mathcal{R}_c(\mathbb{T}, \mathbb{C}) \) the set of regressive constants, and similarly we define the sets \( \mathcal{R}^+_c(\mathbb{T}, \mathbb{R}) \) and \( \mathcal{R}^-_c(\mathbb{T}, \mathbb{R}) \).

A function \( x : \mathbb{T} \to \mathbb{R} \) is called a solution of the equation (E), if \( x(t) \) is delta differentiable for \( t \in \mathbb{T}^\kappa \) and satisfies equation (E) for \( t \in \mathbb{T} \). We say that a solution \( x \) of equation (E) has a generalized zero at \( t \) if \( x(t) = 0 \) or if \( \mu(t) > 0 \) and \( x(t)x(\sigma(t)) < 0 \). Let \( \text{sup} \mathbb{T} = \infty \) and then a nontrivial solution \( x \) of equation (E) is called oscillatory on \( [t, \infty) \) if it has arbitrarily large generalized zeros in \( [t, \infty) \).

Next, let us recall some known oscillation results on this subject. For \( \mathbb{T} = \mathbb{R} \) and \( \mathbb{T} = \mathbb{Z} \), equation (E) reduces to
\[ x'(t) + p(t)x(\tau(t)) = 0, \quad t \in \mathbb{R}^+_0 \]
and
\[ \Delta x(n) + p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}^+_0, \]
respectively.
In 1972, Ladas, Lakshmikantham and Papadakis [20] proved that if $\tau(t)$ is nondecreasing and

$$\limsup_{t \to \infty} \int_{\tau(t)}^{t} p(s) \, ds > 1,$$

then all solutions of (1.2) oscillate.

In 1982, Koplatadze and Canturija [19] established the following result.

If $\tau(t)$ is non-monotone or nondecreasing, and

$$\liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \, ds > \frac{1}{e},$$

then all solutions of (1.2) oscillate.

Assume that the argument $\tau(t)$ is non-monotone. Set

$$h(t) := \sup_{s \leq t} \tau(s), \quad t \geq 0.$$  

Clearly, $h(t)$ is nondecreasing, and $\tau(t) \leq h(t)$ for all $t \geq 0$.

In 2011, Braverman and Karpuz [5], proved that, if $\tau(t)$ is non-monotone and

$$\limsup_{t \to \infty} \int_{h(t)}^{t} p(s) \exp \left\{ \int_{\tau(s)}^{h(t)} p(\xi) \, d\xi \right\} \, ds > 1,$$

then all solutions of (1.2) oscillate.

Very recently, Chatzarakis and Öcalan [9], proved that, if $\tau(t)$ is non-monotone and

$$\liminf_{t \to \infty} \int_{h(t)}^{t} p(s) \exp \left\{ \int_{\tau(s)}^{h(t)} p(\xi) \, d\xi \right\} \, ds > \frac{1}{e},$$

then all solutions of (1.2) oscillate.

In 1998, Zhang and Tian [30], studied the equation (1.3) and proved that, if $(\tau(n))$ is non-monotone, and

$$\limsup_{n \to \infty} p(n) > 0 \quad \text{and} \quad \liminf_{n \to \infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e},$$

then all solutions of (1.3) oscillate.
In 2006, Chatzarakis, Koplatadze and Stavroulakis [6,7], when \( \tau(n) \) is non-monotone or nondecreasing, studied the equation (1.3) and proved that, if one of the following conditions

\[
\limsup_{n \to \infty} \sum_{j=h(n)}^{n} p(j) > 1, \quad \text{where} \quad h(n) = \max_{0 \leq s \leq n} \tau(s), \, n \geq 0,
\]

or

\[
\limsup_{n \to \infty} \sum_{j=\tau(n)}^{n-1} p(j) < \infty \quad \text{and} \quad \liminf_{n \to \infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}
\]

is satisfied, then all solutions of (1.3) oscillate.

Assume that the argument \( \tau(n) \) is non-monotone. Set

\[
h(n) := \max_{s \leq n} \tau(s), \quad n \geq 0.
\]

Clearly, \( h \) is nondecreasing, and \( \tau(n) \leq h(n) \leq n - 1 \) for all \( n \geq 0 \).

In 2016, Öcalan [26], proved that, if \( \tau(n) \) is non-monotone and

\[
\liminf_{n \to \infty} \sum_{j=\tau(n)}^{n-1} p(j) \left( \frac{j - \tau(j) + 1}{j - \tau(j)} \right)^{j - \tau(j) + 1} > 1,
\]

then all solutions of (1.3) oscillate.

In 2011, Braverman and Karpuz [5], proved that, if \( \tau(n) \) is non-monotone and

\[
\limsup_{n \to \infty} \sum_{j=h(n)}^{n} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1 - p(i)} > 1,
\]

then all solutions of (1.3) oscillate.

Very recently, Chatzarakis and Öcalan [8], proved that, if \( \tau(n) \) is non-monotone and

\[
\liminf_{n \to \infty} \sum_{j=h(n)}^{n-1} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1 - p(i)} > \frac{1}{e},
\]

then all solutions of (1.3) oscillate.

For Equation (E), in 2002, Zhang and Deng [31], proved the following result by the help of cylinder transforms.

Define

\[
\alpha = \limsup_{t_0 \to \infty} \sup_{\lambda \in E} \{ \lambda \exp - \lambda \rho(\tau(t), t) \}
\]
where
\[
\exp_{-\lambda p}(\tau(t),t) = \exp \int_{\tau(t)}^{t} \xi_{\mu(s)}(-\lambda p(s))\Delta s,
\]
\[
E = \{\lambda : \lambda > 0, \ 1 - \lambda p(t)\mu(t) > 0\}, \text{ and}
\]
\[
\xi_{h}(z) = \begin{cases} 
\log(1+hz), & \text{if } h \neq 0 \\
z, & \text{if } h = 0
\end{cases}
\]
If \(\alpha < 1\), then all solutions of equation (E) are oscillatory.

In 2005, Bohner [4], proved that, using exponential functions notation for any time scale \(\mathbb{T}\), if Eq. (E) has an eventually positive solution, then \(\alpha\) defined by (1.16) satisfies \(\alpha \geq 1\).

In 2005, Zhang et al. [32], and in 2006, Şahiner and Stavroulakis [28], using by different technique, obtained that if \(\tau(t)\) is nondecreasing and
\[
\limsup_{t \to \infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1,
\]
then all solutions of equation (E) are oscillatory.

2. Main results

In this section, we present a new sufficient condition for the oscillation of all solutions of (E), under the assumption that the argument \(\tau(t)\) is non-monotone. Set
\[
h(t) := \sup_{s \leq t} \tau(s), \quad t \geq 0.
\]
Clearly, \(h(t)\) is nondecreasing, and \(\tau(t) \leq h(t)\) for all \(t \geq 0\).

The following lemma was given in [28].

**Lemma 2.1.** Assume that \(f : \mathbb{T} \to \mathbb{R}\) is rd-continuous, \(g : \mathbb{T} \to \mathbb{R}\) is nonincreasing and \(\tau : \mathbb{T} \to \mathbb{T}\) is nondecreasing. If \(b < u\), then
\[
\int_{b}^{\sigma(u)} f(s)g(\tau(s))\Delta s \geq g(\tau(u)) \int_{b}^{\sigma(u)} f(s)\Delta s.
\]
Theorem 2.2. Assume that (1.1) holds. If \( \tau(t) \) is non-monotone and

\[
\limsup_{t \to \infty} \int_{h(t)}^{\sigma(t)} p(s) \Delta s > 1,
\]

where \( h(t) \) is defined (2.1), then all solutions of (E) oscillate.

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution \( x(t) \) of (E). Since \( -x(t) \) is also a solution of (E), we can confine our discussion only to the case where the solution \( x(t) \) is eventually positive. Then there exists a \( t_1 > t_0 \) such that \( x(t), x(\tau(t)), x(h(t)) > 0 \), for all \( t \geq t_1 \). Thus, from (E) we have

\[
x^\Delta(t) = -p(t)x(\tau(t)) \leq 0, \quad \text{for all } t \geq t_1,
\]

which means that \( x(t) \) is an eventually nonincreasing function of positive numbers. In view of this, and taking into account that \( \tau(t) \leq h(t) \leq t \) and \( h(t) \) is nondecreasing, (E) gives

\[
x^\Delta(t) + p(t)x(h(t)) \leq 0, \quad t \geq t_1.
\]

Integrating (2.4) from \( h(t) \) to \( \sigma(t) \) and using Lemma 2.1, we obtain

\[
x(\sigma(t)) - x(h(t)) + \int_{h(t)}^{\sigma(t)} p(s)x(h(s)) \Delta s \leq 0
\]

and

\[
-x(h(t)) + x(h(t)) \int_{h(t)}^{\sigma(t)} p(s) \Delta s \leq 0
\]

or

\[
x(h(t)) \left[ \int_{h(t)}^{\sigma(t)} p(s) \Delta s - 1 \right] \leq 0.
\]

Consequently,

\[
\limsup_{t \to \infty} \int_{h(t)}^{\sigma(t)} p(s) \Delta s \leq 1,
\]

which contradicts (2.3). The proof of the theorem is complete.

We remark that if \( \tau(t) \) is nondecreasing, then we have \( \tau(t) = h(t) \) for all \( t \geq 0 \), and the condition (2.3) reduce to

\[
\limsup_{t \to \infty} \int_{\tau(t)}^{\sigma(t)} p(s) \Delta s > 1,
\]

which implies that it is condition (1.17).
Lemma 2.3. Assume that (2.1) holds and \( m > 0 \). Then, we have

\[
m = \liminf_{t \to \infty} \int_{h(t)}^{t} p(s) \Delta s = \liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s,
\]

where \( h(t) \) is defined (2.1).

**Proof.** Clearly \( h(t) \geq \tau(t) \) and so

\[
\int_{h(t)}^{t} p(s) \Delta s \leq \int_{\tau(t)}^{t} p(s) \Delta s.
\]

Hence

\[
\liminf_{t \to \infty} \int_{h(t)}^{t} p(s) \Delta s \leq \liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s.
\]

If (2.5) does not hold, then there exists a \( m' > 0 \) and a sequence \( \{t_n\} \) \( (t_n \in T, n \in \mathbb{N}) \) such that \( t_n \to \infty \) as \( n \to \infty \) and

\[
\lim_{n \to \infty} \int_{h(t_n)}^{t_n} p(s) \Delta s \leq m' < m.
\]

By definition, \( h(t_n) = \sup_{s \leq t_n} \tau(s) \), and hence there exists a \( t'_n \leq t_n \) such that \( h(t_n) = \tau(t'_n) \). Hence

\[
\int_{h(t_n)}^{t_n} p(s) \Delta s = \int_{\tau(t'_n)}^{t_n} p(s) \Delta s \geq \int_{\tau(t'_n)}^{t'_n} p(s) \Delta s.
\]

It follows that \( \left\{ \int_{\tau(t'_n)}^{t'_n} p(s) \Delta s \right\}_{n=1}^{\infty} \) is a bounded sequence having a convergent subsequence, say \( \int_{\tau(t'_{n_k})}^{t'_{n_k}} p(s) \Delta s \to c \leq m' \), as \( k \to \infty \)

which implies that

\[
\liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s \leq m' < m
\]

contradicting (2.5).
Theorem 2.4. Assume that (1.1) holds. If \( \tau(t) \) is non-monotone or nondecreasing and

\[
\liminf_{t \to \infty} \int_{\tau(t)}^t p(s) \Delta s > \frac{1}{e},
\]

then all solutions of (E) oscillate.

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution \( x(t) \) of (E). Since \( -x(t) \) is also a solution of (E), we can confine our discussion only to the case where the solution \( x(t) \) is eventually positive. Then there exists a \( t_1 > t_0 \) such that \( x(t), x(\tau(t)) > 0 \), for all \( t \geq t_1 \). Thus, from (E) we have

\[
x^\Delta(t) = -p(t)x(\tau(t)) \leq 0, \quad \text{for all } t \geq t_1,
\]

which means that \( x(t) \) is an eventually nonincreasing function of positive numbers.

Since \( \tau(t) \leq h(t) \leq t \) and \( h(t) \) is nondecreasing for all \( t \geq 0 \), from Eq. (E), we have

\[
x^\Delta(t) + p(t)x(h(t)) \leq 0, \quad t \geq t_1.
\]

Integrating (2.7) from \( h(t) \) to \( t \), we have

\[
x(t) - x(h(t)) + \int_{h(t)}^t p(s)x(h(s)) \Delta s \leq 0, \quad \text{for all } t \geq t_1
\]

or

\[
x(t) - x(h(t)) + x(h(t)) \int_{h(t)}^t p(s) \Delta s \leq 0, \quad \text{for all } t \geq t_1
\]

From (2.8) dividing by \( x(h(t)) \), we have

\[
\frac{x(t)}{x(h(t))} - 1 + \int_{h(t)}^t p(s) \Delta s \leq 0
\]

Using by Lemma 2.3 and from (2.5) it follows that there exists a constant \( c > 0 \) such that

\[
\int_{h(t)}^t p(s) \Delta s \geq c > \frac{1}{e}, \quad t \geq t_2 > t_1.
\]

Combining the inequalities (2.9) and (2.10), we obtain

\[
\frac{x(t)}{x(h(t))} - 1 + c \leq 0, \quad t \geq t_2
\]
or

\[
\frac{x(t)}{x(h(t))} \leq 1 - c, \quad t \geq t_2
\]

Thus, we have \( c < 1 \) and

\[
\frac{x(h(t))}{x(t)} \geq \frac{1}{1 - c}, \quad t \geq t_2.
\]

Repeating the above procedure, it follows by induction that for any positive integer \( k \),

\[
(2.11) \quad \frac{x(h(t))}{x(t)} \geq \left(\frac{1}{1 - c}\right)^k, \quad \text{for sufficiently large } t,
\]

where \( c < 1 \).

Now, in view of (2.10), and for all large \( t \), there exists a real number \( t^* \in [h(t), t] \), \( t^* \in \mathbb{T} \), such that

\[
(2.12) \quad \int_{h(t)}^{t^*} p(s) \Delta s \geq \frac{c}{2} \quad \text{and} \quad \int_{t^*}^{t} p(s) \Delta s \geq \frac{c}{2}.
\]

Integrating (2.7) from \( t^* \) to \( t \), and using the fact that the function \( x(t) \) is nonincreasing and the function \( h(t) \) is nondecreasing, we obtain

\[
x(t) - x(t^*) + \int_{t^*}^{t} p(s)x(h(s)) \Delta s \leq 0,
\]

and using (2.12), we obtain

\[
-x(t^*) + x(h(t)) \int_{t^*}^{t} p(s) \Delta s \leq 0
\]

or

\[
(2.13) \quad x(t^*) - x(h(t)) \frac{c}{2} \geq 0.
\]

Integrating (2.7) from \( h(t) \) to \( t^* \), and using the same arguments we have

\[
x(t^*) - x(h(t)) + \int_{h(t)}^{t^*} p(s)x(h(s)) \Delta s \leq 0,
\]

or

\[
-x(h(t)) + x(h(t^*)) \int_{h(t)}^{t^*} p(s) \Delta s \leq 0
\]
and
\begin{equation}
(2.14) \quad x(h(t)) - x(h(t^*)) \frac{c}{2} \geq 0.
\end{equation}

Combining the inequalities (2.13) and (2.14), we obtain
\begin{equation}
x(t^*) \geq x(h(t)) \frac{c}{2} \geq x(h(t^*)) \left( \frac{c}{2} \right)^2,
\end{equation}
or
\begin{equation}
\frac{x(h(t^*))}{x(t^*)} \leq \left( \frac{2}{c} \right)^2 < +\infty
\end{equation}
i.e., \( \liminf_{t \to \infty} \frac{x(h(t))}{x(t)} \) exists. This contradicts (2.11).

The proof of the theorem is complete.

**Example 2.5.** For \( T = \mathbb{R} \), consider the retarded differential equation
\begin{equation}
(2.15) \quad x'(t) + (0.37) x(\tau(t)) = 0, \quad t \geq 0,
\end{equation}
where
\begin{equation}
\tau(t) = \begin{cases}
t - 1, & \text{if } t \in [3k, 3k + 1] \\
-3t + 12k + 3, & \text{if } t \in [3k + 1, 3k + 2], \quad k \in \mathbb{N}_0.
\end{cases}
\end{equation}

By (2.1), we see that
\begin{equation}
h(t) := \sup_{s \leq t} \tau(s) = \begin{cases}
t - 1, & \text{if } t \in [3k, 3k + 1] \\
3k, & \text{if } t \in [3k + 1, 3k + 2], \quad k \in \mathbb{N}_0.
\end{cases}
\end{equation}

(For figure of \( \tau(t) \) and \( h(t) \), see Example 1 in [5]). Computing, we get
\begin{equation}
\liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \, ds = 0.37 > \frac{1}{e^1},
\end{equation}
that is, condition (2.6) of Theorem 2.4 is satisfied and therefore all solutions of (2.15) oscillate.
Observe, however, that

\[
\sigma(t) \int_{h(t)}^{t} p(s) ds = \int_{h(t)}^{3k+2.6} p(s) ds = 3k+2.6
\]

and therefore

\[
\limsup_{t \to \infty} \sigma(t) \int_{h(t)}^{3k+2.6} p(s) ds = 0.962 < 1,
\]

that is, condition (2.3) of Theorem 2.2 is not satisfied.

Conflict of Interests

The authors declare that there is no conflict of interests.

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