BERTRAND CURVES OF AW(k)-TYPE IN THREE DIMENSIONAL LIE GROUPS

SEZAI KIZILTUĞ, SEZER ÇAKAL*

Erzincan University, Department of Mathematics, Faculty of Arts and Sciences, Erzincan, Turkey

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Abstract. In this paper, we consider curves of AW(k)-type (1 ≤ k ≤ 3) in Three Dimensional Lie Groups. We give harmonic curvature conditions of AW(k)-type curves. Furthermore, we investigate that under what conditions AW(k)-type curves are helix. Besides, considering AW(k)-type curves, we investigate Bertrand curves and we show that there are Bertrand curves of AW(2), AW(3) and weak AW(2)−types.

Keywords: Lie Groups; Aw(k)-type curve; Bertrand curves; helix.

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1 Introduction

In the curve theory of Euclidean space, the most important subject is to obtain a characterization for a regular curves. These characterizations can be given for a single curve or for a curve pair. Helix, slant helix, plane curve, spherical curve, etc. especially the helices, are used in many applications [2, 3, 19]. Similarly, by considering two curves, some special curve pairs such as involute evolute curves, Bertrand curves, Mannheim curves have been defined and studied so

*Corresponding author

E-mail address: sezercakal123@gmail.com

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Accordingly, Bertrand mates represent particular examples of offset curves which are used in computer-aided design (CAD) and computer-aided manufacture (CAM). The distance between a Bertrand curve and its mate measured along the principal normal is known to be constant. We can see helical structures in nature and mechanic tools.

As a matter of fact, it is the simplest of the three-dimensional spirals. One of the most interesting spirals is referred to as the k-Fibonacci spirals which appears naturally from studying the k-Fibonacci numbers and the related hyperbolic k-Fibonacci function. Fibonacci numbers and the related Golden Mean or Golden Section appear very often in theoretical physics and physics of the high energy particles (see [7, 8, 9]). Besides, in the field of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or design of highways [18]. Also we can see the helix curve or helical structure in fractal geometry, for instance hyperhelices. In differential geometry; a curve of constant slope or general helix in Euclidean 3-space \( E^3 \) is defined by the property that its tangent vector field makes a constant angle with a fixed straight line (the axis of the general helix).

Çöken and Çiftçi have studied the degenerate semi-Riemannian geometry of Lie group [6]. They obtained a naturally reductive homogeneous semi-Riemannian space using the Lie group. Later, some of subjects given above have been considered in three dimensional Lie groups and some characterizations for these curves have been obtained in a three dimensional Lie group [15, 16]. Also, Çiftçi[5] defined general helices in three dimensional Lie groups with a bi-invariant metric and obtained a generalization of Lancret’s theorem and gave a relation between the geodesics of the so-called cylinders and general helices.

Recently, many interesting results on curves of AW(k)-type have been obtained by many mathematicians (see [12, 13, 17]). For example, Özgür and Gezgin studied a Bertrand curve of AW(k)-type and they showed that there was no such Bertrand curve of AW(1)-type and \( \alpha \) was of AW(3)-type if and only if it was a right circular helix. In addition they studied weak AW(2)-type and AW(3)-type conical geodesic curves in \( E^3 \). Külahci, Bektaş and Ergüt give curvature conditions of a AW(k)-type Frenet curve in Lorentzian space.
In this paper, we have done a study on Bertrand curves of $AW(k)$-type. However, to the best of author’s knowledge, Bertrand curves of $AW(k)$-type have not been presented in Three Dimensional Lie Groups. Thus, the study is proposed to serve such a need.

2 Preliminaries

Let $G$ be a Lie group with a bi-invariant metric $\langle , \rangle$ and $D$ be the Levi-Civita connection of Lie group $G$. If $g$ denotes the Lie algebra of $G$ then we know that $g$ is isomorphic to $T_eG$ where $e$ is neutral element of $G$. If $\langle , \rangle$ is a bi-invariant metric on $G$ then we have

\begin{align}
\langle X, [Y, Z] \rangle &= \langle [X, Y], Z \rangle \\
\end{align}

and

\begin{align}
D_xY &= \frac{1}{2} [X, Y] \\
\end{align}

for all $X, Y$ and $Z \in g$.

Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed curve and $\{X_1, X_2, ..., X_n\}$ be an orthonormal basis of $g$. In this case, we write that any two vector fields $W$ and $Z$ along the curve $\alpha$ as $W = \sum_{i=1}^{n} w_iX_i$ and $Z = \sum_{i=1}^{n} z_iX_i$ where $w_i : I \to \mathbb{R}$ and $z_i : I \to \mathbb{R}$ are smooth functions. Also the Lie bracket of two vector fields $W$ and $Z$ is given

$[W, Z] = \sum_{i=1}^{n} w_i z_j [X_i, X_j]$ 

and the covariant derivative of $W$ along the curve $\alpha$ with the notation $D_\alpha W$ is given as follows

\begin{align}
D_\alpha W &= \dot{W} + \frac{1}{2} [T, W] \\
\end{align}

where $T = \alpha'$ and $\dot{W} = \sum_{i=1}^{n} \dot{w}_iX_i$ or $\dot{W} = \sum_{i=1}^{n} \frac{dw_i}{dt}X_i$. Note that if $W$ is the left-invariant vector field to the curve $\alpha$ then $\dot{W} = 0$ (For see detail [4]).
Let $G$ be a three dimensional Lie group and $(T, N, B, \kappa, \tau)$ denote the Frenet apparatus of the curve $\alpha$, and calculate $\kappa = \|\dot{T}\|$.

**Definition 1.** Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be a parametrized curve with the Frenet apparatus $(T, N, B, \kappa, \tau)$ then

$$\tau_G = \frac{1}{2} \langle [T, N], B \rangle$$

or

$$\tau_G = \frac{1}{2\kappa^2\tau} \langle \dot{T}, [T, \dot{T}] \rangle + \frac{1}{4\kappa^2\tau} \| [T, \dot{T}] \|^2$$

(see [4]).

**Definition 2.** Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be an arc length parametrized curve with the Frenet apparatus $(T, N, B, \kappa, \tau)$. Then the harmonic curvature function of the curve $\alpha$ is defined by

$$H = \frac{\tau - \tau_G}{\kappa}$$

where $\tau_G = \frac{1}{2} \langle [T, N], B \rangle$.

**Theorem 3.** Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be an arc length parametrized curve with the Frenet apparatus $(T, N, B, \kappa, \tau)$. If the curve $\alpha$ is a general helix if and only if

$$H = \text{const.}$$

(see [5]).

**Theorem 4.** Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be an arc length parametrized curve with the Frenet apparatus $(T, N, B, \kappa, \tau)$. Then $\alpha$ is a slant helix if and only if

$$\sigma = \frac{\kappa \left(1 + H^2\right)^{\frac{3}{2}}}{H} = \tan \theta$$

is a constant where $H$ is a harmonic curvature function of the curve $\alpha$ and $\theta \neq \frac{\pi}{2}$ is a constant [16].

**Proposition 5.** Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be an arc-length parametrized curve with the Frenet apparatus $\{T, N, B\}$. Then the following equalities
\[ [T,N] = \langle [T,N], B \rangle B = 2\tau_G B \]
\[ [T,B] = \langle [T,B], N \rangle N = -2\tau_G N \]

hold [16].

**Remark 6.** Let $G$ be a Lie group with a bi-invariant metric $\langle , \rangle$. Then the following equalities can be given in different lie groups [4].

i) If $G$ is abelian group then $\tau_G = 0$

ii) If $G$ is $SO^3$ then $\tau_G = \frac{1}{2}$

iii) If $G$ is $SU^2$ then $\tau_G = 1$

3 Aw(k)-type curves in Three Dimensional Lie Groups

In this section, harmonic curvature of curves of AW($k$)-type are considered. We give some theorems and corollaries.

Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-length parametrized unit speed curve in three dimensional Lie groups. The curve $\alpha$ is called a Frenet curve of osculating order 3 if its derivatives $\alpha'(s), \alpha''(s), \alpha'''(s), \alpha''''(s)$ are linearly dependent and $\alpha'(s), \alpha''(s), \alpha'''(s), \alpha''''(s)$ are no longer linearly independent for all $s \in I$. To each Frenet curve of order 3 one can associate an orthonormal 3–frame \{\(T(s), N(s), B(s)\)\} along $\alpha$ such that $(\alpha'(s) = T(s))$ called the Frenet frame and functions $\kappa, \tau : I \to \mathbb{R}$ called the Frenet curvatures, such that the Frenet formulas in three dimensional Lie groups are defined

\[
\begin{align*}
D_T T(s) &= \kappa(s) N(s) \\
D_T N(s) &= -\kappa(s) T(s) + (\tau - \tau_G)(s) B(s) \\
D_T B(s) &= (\tau_G - \tau)(s) N(s)
\end{align*}
\]

where $D$ is the Levi-Civita connections of Lie group $G$ and $\tau_G = \frac{1}{2} \langle [T,N], B \rangle$ [16].
Proposition 7. Let $\alpha : I \subset \mathbb{R} \to G$ be a Frenet curve in three dimensional Lie groups, then we have
\[
\begin{align*}
\alpha'(s) &= T(s) \\
\alpha''(s) &= \kappa(s)N(s) \\
\alpha'''(s) &= -\kappa^2(s)T(s) + \kappa'(s)N(s) + \kappa^2(s)H(s)B(s) \\
\alpha''''(s) &= (-3\kappa(s)\kappa'(s))T(s) + (\kappa''(s) - \kappa^3(s)(1 - H^2(s)))N(s) + (2\kappa'(s)\kappa(s)H(s) + (\kappa(s)H(s))' \, B(s).
\end{align*}
\]

Proof. From Frenet formulas in three dimensional Lie groups (3.1) and by using $H = \frac{\tau - \tau_G}{\kappa}$, we have the results. $\square$

Notation. Let us write
\[
\begin{align*}
(3.2) & \quad N_1(s) = \kappa(s)N(s) \\
(3.3) & \quad N_2(s) = \kappa'(s)N(s) + \kappa^2(s)H(s)B(s) \\
(3.4) & \quad N_3(s) = (\kappa''(s) - \kappa^3(s)(1 - H^2(s)))N(s) + (3\kappa'(s)\kappa(s)H(s) + \kappa^2(s)H'(s))B(s)
\end{align*}
\]

Remark 8. $\alpha'(s), \alpha''(s), \alpha'''(s), \alpha''''(s)$ are linearly dependent if and only if $N_1(s), N_2(s), N_3(s)$ are linearly dependent.

As the definition of Aw($k$) type curves in [1], we have

Definition 9. Frenet curves (of osculating order 3) in three dimensional Lie groups are

(i) of type weak Aw(2) if they satisfy
\[
(3.5) \quad N_3(s) = \langle N_3(s), N_2^*(s) \rangle N_2^*(s),
\]

(ii) of type weak Aw(3) if they satisfy
\[
(3.6) \quad N_3(s) = \langle N_3(s), N_1^*(s) \rangle N_1^*(s)
\]

where
\[
N_1^*(s) = \frac{N_1(s)}{||N_1(s)||}, N_2^*(s) = \frac{N_2(s) - \langle N_2(s), N_1^*(s) \rangle N_1^*(s)}{||N_2(s) - \langle N_2(s), N_1^*(s) \rangle N_1^*(s)||}.
\]
**Proposition 10.** Let \( \alpha \) be a Frenet curve (of osculating order 3) in three dimensional Lie groups. If \( \alpha \) is of type weak Aw(2) then

\[
\kappa''(s) - \kappa^3(s)(1 - H^2(s)) = 0.
\]

**Corollary 11.** Let \( \alpha \) be a Frenet curve of type weak Aw(2). If \( \alpha \) is plane curve then

\[
\kappa(s) = \pm \frac{\sqrt{2}}{s + c}
\]

where \( c \) is constant.

**Proof.** Suppose that \( \alpha \) is a Frenet curve of type weak Aw(2). Then the Eq. (3.7) hold on \( \alpha \). Since \( \alpha \) is a plane curve, we have

\[
H(s) = 0.
\]

Substituting (3.9) in (3.7), we get

\[
\kappa''(s) - \kappa^3(s) = 0.
\]

So the solution of the last equation gives us (3.8). Hence, the proof is completed. \( \square \)

**Proposition 12.** Let \( \alpha \) be a Frenet curve (of osculating order 3) in three dimensional Lie groups. If \( \alpha \) is of type weak Aw(3) then

\[
3\kappa'(s)\kappa(s)H(s) + \kappa^2(s)H'(s) = 0.
\]

**Definition 13.** Frenet curves (of osculating order 3) in three dimensional Lie groups are

(i) of type Aw(1) if they satisfy \( N_3(s) = 0 \),

(ii) of type Aw(2) if they satisfy

\[
\|N_2(s)\|^2 N_3(s) = \langle N_3(s), N_2(s) \rangle N_2(s).
\]

(iii) of type Aw(3) if they satisfy

\[
\|N_1(s)\|^2 N_3(s) = \langle N_3(s), N_1(s) \rangle N_1(s).
\]
Theorem 14. Let $\alpha$ be a Frenet curve (of osculating order 3) in three dimensional Lie groups. Then $\alpha$ is of type $\text{Aw}(1)$ if and only if

\begin{equation}
\kappa''(s) - \kappa^3(s)(1 - H^2(s)) = 0
\end{equation}

and

\begin{equation}
3\kappa'(s)\kappa(s)H(s) + \kappa^2(s)H'(s) = 0
\end{equation}

Proof. Since $\alpha$ is a curve of type $\text{Aw}(1)$, we have $N_3(s) = 0$. Then from Eq. (3.4), we have

$$(\kappa''(s) - \kappa^3(s)(1 - H^2(s)))N(s) + (3\kappa'(s)\kappa(s)H(s) + \kappa^2(s)H'(s))B(s) = 0.$$ 

Furthermore, since $N$ and $B$ are linearly independent, we get

$$\kappa''(s) - \kappa^3(s)(1 - H^2(s)) = 0$$

and

$$3\kappa'(s)\kappa(s)H(s) + \kappa^2(s)H'(s) = 0.$$ 

The converse statement is trivial. Hence our theorem is proved.

Corollary 15. Let $\alpha$ be a Frenet curve (of osculating order 3) in three dimensional Lie groups. Then there is no (circular or general) helix of type $\text{Aw}(1)$.

Proof. Assume that $\alpha$ be a helix. Then by the Theorem $H(s)$ is constant. So, $H'(s) = 0$. Therefore the equations (3.13) and (3.14) can be written as follows:

$$\kappa''(s) - \kappa^3(s)(1 - H^2(s)) = 0$$

and

$$3\kappa'(s)\kappa(s)H(s) = 0.$$ 

Since the solution of above differential equations does not exist, there are not circular and general helix of type $\text{Aw}(1)$.

Theorem 16. Let $\alpha$ be a Frenet curve (of osculating order 3) in three dimensional Lie groups. Then $\alpha$ is of type $\text{Aw}(2)$ if and only if

\begin{equation}
3(\kappa'(s))^2\kappa(s)H(s) + \kappa'(s)\kappa^2(s)H'(s) - \kappa''(s)\kappa^2(s)H(s) + \kappa^5(s)H(s)(1 - H^2(s)) = 0.
\end{equation}
Proof. Suppose that $\alpha$ is a Frenet curve of order 3, then from (3.3) and (3.4), we can write

\begin{align}
N_2(s) &= \gamma(s)N(s) + \beta(s)B(s), \\
N_3(s) &= \eta(s)N(s) + \delta(s)B(s),
\end{align}

where $\gamma$, $\beta$, $\eta$ and $\delta$ are differentiable functions. Since $N_2(s)$ and $N_3(s)$ are linearly dependent, coefficients determinant is equal to zero and hence one can write

\begin{align}
\begin{vmatrix}
\gamma(s) & \beta(s) \\
\eta(s) & \delta(s)
\end{vmatrix}
= 0.
\end{align}

Here,

\begin{align*}
\gamma(s) &= \kappa'(s), \quad \beta(s) = \kappa^2(s)H(s), \\
\eta(s) &= \kappa''(s) - \kappa^3(s)(1 - H^2(s)), \\
\delta(s) &= 3\kappa'(s)\kappa(s)H(s) + \kappa^2(s)H'(s).
\end{align*}

Substituting these into (3.18), we obtain (3.15).

Conversely if the equation (3.15) holds it is easy to show that $\alpha$ is of type Aw(2). This completes the proof. \qed

**Corollary 17.** If a Frenet curve of order 3 is a general helix of type Aw(2), then one can have

\begin{equation}
3(\kappa'(s))^2 - \kappa''(s)\kappa(s) + \kappa^4(s)(1 - H^2(s)) = 0.
\end{equation}

**Theorem 18.** Let $\alpha$ be a general helix in three dimensional Lie groups. If $\alpha$ is of type Aw(2), then

\begin{equation}
\kappa(s) = \frac{1}{\sqrt{-A\kappa^2 + Bs + C}} \quad \text{and} \quad (\tau - \tau_G)(s) = \sqrt{1 - A\kappa(s)}
\end{equation}

where $A = 1 - H^2(s)$, $B$ and $C$ are real constants.

**Proof.** Suppose that $\alpha$ is a general helix of type Aw(2). Then Eq.(3.19) holds. If we substitute $\kappa(s) = x$ in (3.19), we get

\begin{equation}
x \frac{d^2x}{ds^2} - 3 \left( \frac{dx}{ds} \right)^2 = Ax^4, \quad A = 1 - H^2(s).
\end{equation}
Let us take $x = y^p$ and differentiating it twice we obtain

\begin{equation}
\frac{dx}{ds} = py^{p-1}y',
\end{equation}

\begin{equation}
\frac{d^2x}{ds^2} = p(p-1)y^{p-2}\left(\frac{dy}{ds}\right)^2 + py^{p-1}\frac{d^2y}{ds^2}.
\end{equation}

Now, the substitution of (3.22) and (3.23) into (3.21), we get

\[ y^p \left[ py^{p-1}\frac{d^2y}{ds^2} + p(p-1)y^{p-2}\left(\frac{dy}{ds}\right)^2 \right] - 3p^2y^{2p-2}\left(\frac{dy}{ds}\right)^2 = Ay^{4p}, \]

\[ py^{2p-1}\frac{d^2y}{ds^2} + p(p-1)y^{2p-2}\left(\frac{dy}{ds}\right)^2 - 3p^2y^{2p-2}\left(\frac{dy}{ds}\right)^2 = Ay^{4p}. \]

Putting $p(p-1) = 3p^2$ (i.e. $p = -\frac{1}{2}$) into the last equation we get

\[ py^{2p-1}\frac{d^2y}{ds^2} = Ay^{4p}. \]

So,

\[ \frac{d^2y}{ds^2} = -2A. \]

Now, we solve this last equation. Since $\frac{dy}{ds} = -2As + B$, we get

\[ y = -As^2 + Bs + C. \]

Furthermore, use of $x = y^{-\frac{1}{p}}$ we obtain

\[ x = (-As^2 + Bs + C)^{\frac{1}{p}}. \]

Since $H(s) = \frac{(\tau - \tau_0)(s)}{\kappa(s)}$, we have the result. 

**Theorem 19.** Let $\alpha$ be a Frenet curve (of osculating order 3) in three dimensional Lie groups. Then $\alpha$ is of type $Aw(3)$ if and only if

\begin{equation}
3\kappa'(s)\kappa(s)H(s) + \kappa^2(s)H'(s) = 0.
\end{equation}

**Proof.** Suppose that $\alpha$ is a Frenet curve of order 3 which is of type $Aw(3)$. If substituting (3.2) and (3.4) in (3.12), we get (3.24).

The converse statement is trivial. Hence our proposition is proved.
Theorem 20. Let be $\alpha$ a general helix of osculating order 3. Then $\alpha$ is of type $Aw(3)$ if and only if $\alpha$ is a circular helix.

Proof. Suppose that $\alpha$ is a general helix, then by the Theorem (3) $H'(s) = 0$. So, the equation (3.24) becomes $\kappa'(s)\kappa(s)H(s) = 0$. Since $H(s)$ is none zero, $\kappa'(s) = 0$. By the general helix $(\tau - \tau_G)(s)$ must be constant. So, $\alpha$ is a circular helix. The converse statement is trivial. Hence our theorem is proved. 

4 AW(k)-type Bertrand Curves in Three Dimensional Lie Groups G

This section characteries the curvatures of AW(k)-type Bertrand curves in G. We obtain some theorems and results about these curves in three dimensional Lie groups.

Definition 21. A curve $\alpha : I \subset \mathbb{R} \rightarrow G$ with $\kappa(s) \neq 0$ is called a Bertrand curve if there exist a curve $\tilde{\alpha} : I \subset \mathbb{R} \rightarrow G$ such that the principal normal lines of $\alpha$ and $\tilde{\alpha}$ at $s \in I$ are equal. In this case $\tilde{\alpha}$ is called a Bertrand mate of $\alpha$ [15].

Theorem 22. Let $\alpha \subset G$ be a Bertrand curve. A Bertrand mate of $\alpha$ is as follows:

$$\tilde{\alpha}(s) = \alpha(s) + \lambda N(s)$$ (4.1)

where $\lambda$ is constant [15].

Corollary 23. If $\tilde{\alpha}$ is a Bertrand mate of $\alpha$, then

$$\left(\tilde{\alpha}(s)\right)' = (1 - \lambda \kappa(s))T(s) + (\lambda \kappa(s)H(s))B(s).$$ (4.2)

Proof. Since $(\alpha, \tilde{\alpha})$ is a Bertrand mate, then the Eq.(4.1) hold on $\alpha$. Differentiating (4.1) with respect to $s$, by using Frenet formulas in three dimensional Lie groups (3.1) and $H = \frac{\tau - \tau_G}{\kappa}$, then (4.2) is obtained. 

Theorem 24. Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be unit speed curve. If $\tilde{\alpha}$ is a Bertrand mate of $\alpha$, then angle measurement of this curve between tangent vectors at corresponding points is constant.
Proof. If $\langle \tilde{T}(s), T(s) \rangle' = 0$, then the proof is complete.

\begin{equation}
\langle \tilde{T}(s), T(s) \rangle' = \langle \left(\tilde{T}(s)\right)', T(s) \rangle + \langle \tilde{T}(s), (T(s))' \rangle
\end{equation}

\begin{equation}
= \langle \kappa(s)\tilde{N}(s), T(s) \rangle + \langle \tilde{T}(s), \kappa(s)N(s) \rangle
\end{equation}

\begin{equation}
= \kappa(s)\langle \tilde{N}(s), T(s) \rangle + \kappa(s)\langle \tilde{T}(s), N(s) \rangle
\end{equation}

Since $\tilde{N}(s)$ is parallel to $N(s)$ and $N(s) \perp T(s)$, then

\begin{equation}
\langle \tilde{N}(s), T(s) \rangle = 0.
\end{equation}

Since $\tilde{N}(s)$ is parallel to $N(s)$ and $\tilde{T}(s) \perp \tilde{N}(s)$, then

\begin{equation}
\langle \tilde{T}(s), N(s) \rangle = 0.
\end{equation}

Substituting (4.6) and (4.7) in (4.5), we have

\begin{equation}
\langle \tilde{T}(s), T(s) \rangle' = 0.
\end{equation}

Hence, the proof is completed.

Proposition 25. Let $\alpha$ be a Frenet curve (of osculating order 3) in three dimensional Lie groups. For $\kappa(s) \neq 0$, $\alpha$ is a Bertrand curve if and only if there exists a linear relation

\begin{equation}
\lambda \kappa(s) + \mu \kappa(s)H(s) = 1.
\end{equation}

where $\lambda$, $\mu$ are non-zero constants and $H$ is the harmonic curvature function of the curve $\alpha$ [13].

Corollary 26. Suppose that $\kappa(s) \neq 0$ and $(\tau - \tau_G)(s) \neq 0$. Then $\alpha$ is a Bertrand curve if and only if there exist a nonzero real number $\lambda$ such that

\begin{equation}
\lambda (\kappa'(s)\kappa(s)H(s) - \kappa(s)(\kappa(s)H(s))') - (\kappa(s)H(s))' = 0.
\end{equation}

Proof. By the proposition (25), $\alpha$ is a Bertrand curve if and only if there exist real numbers $\lambda \neq 0$ and $\mu$ such that $\lambda \kappa(s) + \mu \kappa(s)H(s) = 1$. This is equivalent to the condition that there exists a real number $\lambda \neq 0$ such that $\frac{1-\lambda \kappa(s)}{\kappa H(s)}$ is constant. Differentiating both sides of the last equality, we get (4.9). The converse assertion is also true.
**Proposition 27.** Let \( \alpha : I \subset \mathbb{R} \to G \) be a Bertrand curve with \( \kappa(s) \neq 0 \) and \( (\tau - \tau_G)(s) \neq 0 \). Then \( \alpha \) is of AW(2)-type if and only if there is a non zero real number \( \lambda \) such that

\[
3(\kappa'(s))^2 H(s) + \kappa'^2(s) \frac{\lambda \kappa'(s) H(s)}{\lambda \kappa(s) - 1} - \kappa^2(s) H(s) (3\kappa'(s) H(s) + \kappa(s) H'(s)) = 0. \tag{4.10}
\]

**Proof.** Since \( \alpha \) is of AW(2)-type, Eq.(3.15) holds and since \( \alpha \) is a Bertrand curve, Eq.(4.9) holds. If both of these equations are considered, (4.10) is obtained. \( \square \)

**Theorem 28.** Let \( \alpha : I \subset \mathbb{R} \to G \) be a Bertrand curve with \( \kappa(s) \neq 0 \) and \( (\tau - \tau_G)(s) \neq 0 \). If \( \alpha \) is of type Aw(3), then \( \alpha \) is a circular helix.

**Proof.** Suppose that \( \alpha : I \subset \mathbb{R} \to G \) is a Bertrand curve of AW(3)-type with \( \kappa(s) \neq 0 \) and \( (\tau - \tau_G)(s) \neq 0 \). Then the Eqs.(3.24) and (4.9) hold on \( \alpha \), we get

\[
H'(s) (2\lambda \kappa^2(s) - \kappa^2(s)) = 0. \tag{4.11}
\]

Since \( \kappa(s) \neq 0 \), from Eq.(4.11) \( H'(s) = 0 \). Thus, \( H(s) \) is constant, then \( \alpha \) is a circular helix. Hence our theorem is proved. \( \square \)

**Proposition 29.** Let \( \alpha : I \subset \mathbb{R} \to G \) be a Bertrand curve with \( \kappa(s) \neq 0 \) and \( (\tau - \tau_G)(s) \neq 0 \). If \( \alpha \) is of weak AW(2)-type, then

\[
H'(s) (\lambda \kappa^2(s) - \kappa(s)) + H'(s) (2\lambda \kappa(s) \kappa'(s) - 2\kappa'(s)) - \kappa^2(s) H(s) (1 - H^2(s)) = 0. \tag{4.12}
\]

**Proof.** Since \( \alpha \) is of weak Aw(2)-type, From Eq.(3.7) we have

\[
\kappa''(s) - \kappa^3(s) (1 - H^2(s)) = 0. \tag{4.13}
\]

Since \( \alpha \) is a Bertrand curve, Eq.(4.9) holds

\[
H'(s) (\lambda \kappa^2(s) - \kappa(s)) = \kappa'(s) H(s). \tag{4.14}
\]

Differentiating above equation(4.14), we get

\[
\kappa''(s) = \frac{H''(s)(\lambda \kappa^2(s) - \kappa(s)) + H'(s)(2\lambda \kappa(s) \kappa'(s) - 2\kappa'(s))}{H(s)}. \tag{4.15}
\]

If equation (4.13) is substituted in (4.15), then (4.12) is obtained. \( \square \)
Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES
