STABILITY OF A MIXED TYPE CUBIC AND QUARTIC FUNCTIONAL EQUATION IN FUZZY BANACH SPACES

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Abstract. In this paper, we generalized Ulam-Hyers stability of the mixed type cubic and quartic functional equation in fuzzy Banach space.

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1. Introduction

In 1940, Ulam [13] posed the first stability problem concerning group homomorphisms. In the next year, Hyers [7] gave an affirmative answer to the question of Ulam in Banach spaces. Aoki [14] generalized Hyers result for additive mappings. For additive mapping involving different powers of norms [18,20]. This stability is also investigates by Park [6]. In 1984, Katsaras [1] constructed a fuzzy vector topological structure on the linear space. Later, some mathematicians considered some other type fuzzy norms and some properties of fuzzy normed linear spaces [5,15]. Recently, several various fuzzy versions stability problem concerning quadratic,
cubic and quartic function equation have been considered [2,3].

Hyers [8] was the first person to point out the direct method for studying the stability of functional equation. In 2003, Radu [19] proposed the fixed point alternative method to solve the Ulam problem. Subsequently, Mihet [9] applied the fixed point alternative method to solve fuzzy stability of Jensen functional equation in fuzzy normed space.

Jun and Kim [12] introduced cubic function equation and they established the solution of Hyers-Ulam-Rassias stability for the functional equation (1.1), and this equation is called the cubic function equation, if the cubic function \( f(x) = cx^3 \) satisfies (1.1). The quartic functional equation was introduced by Rassias [19] in 2000

\[
f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y)
\]  

(1.2)

It is easy to show that the function \( f(x) = cx^4 \) satisfies the functional equation (1.2). In this paper, we establish a fuzzy version stability for following functional equation:

\[
f(x+2y) + f(x-2y) = 4(f(x+y) + 4f(x-y)) - 24f(y) - 6f(x) + 3f(2y)
\]  

(1.3)

in fuzzy Banach space and the function \( f(x) = ax^3 + bx^4 \) is a solution of the functional equation (1.3). We using the fixed point alternative method to establish fuzzy stability.

2. Preliminaries

We start our works with basic definition using in this paper.

**Definition 2.1.** [16] Let \( X \) be a real linear space. A fuzzy subset \( N \) of \( X \times \mathbb{R} \) is called a fuzzy norm on \( X \) if and only if

(N1) For all \( t \in \mathbb{R} \) with \( t \leq 0 \), \( N(x,t) = 0 \);

(N2) For all \( t \in \mathbb{R} \) with \( t > 0 \), \( N(x,t) = 1 \) if and only if \( x = 0 \);

(N3) For all \( \lambda \in \mathbb{R} \) with \( \lambda \neq 0 \), \( N(\lambda x,t) = N(x,t/|\lambda|) \);

(N4) For all \( s,t \in \mathbb{R} \), \( N(x+y,s+t) \geq \min\{N(x,s),N(y,t)\} \);

(N5) \( N(x,\cdot) \) is a non-decreasing function on \( \mathbb{R} \) and \( \lim_{t \to \infty} N(x,t) = 1 \);

(N6) For \( x \neq 0 \), \( N(x,\cdot) \) is (upper semi) continuous on \( \mathbb{R} \).

Then \((X,N)\) is called a fuzzy normed linear space.
Example 2.2.[4] Let \((X, \| \cdot \|)\) be a normed space. For every \(x \in X\), we define

\[
N(x, t) = \begin{cases} 
\frac{t}{t + \|x\|}, & \text{when } t > 0, \\
0, & \text{when } t \leq 0.
\end{cases}
\]

Then \((X, N)\) is a fuzzy normed linear space.

A sequence \(\{x_n\}\) in \(X\) is called Cauchy if for each \(\varepsilon > 0\) and each \(t > 0\) there exists \(n_0\) such that for all \(n \geq n_0\) and all \(p > 0\), we have \(N(x_{n+p} - x_n, t) > 1 - \varepsilon\). If every Cauchy sequence is convergent, then the fuzzy normed space space is called a fuzzy Banach space.

3. Fuzzy Stability of Cubic and Quartic Function Equation Using Direct Method

In this section, for given \(f : X \to Y\), we define operator \(Df : X \times X \to Y\) by

\[
Df(x, y) = f(x + 2y) + f(x - 2y) - 4[f(x + y) + f(x - y)] - 3f(2y) + 24f(y) + 6f(x)
\]

Theorem 3.1. (The fixed point alternative theorem,[17]) Let \((\Omega, d)\) be a complete generalized metric space and \(T : \Omega \to \Omega\) be a strictly contractive mapping with Lipschitz constant \(L\), that is

\[
d(Tx, Ty) \leq Ld(x, y), \quad \forall x, y \in \Omega.
\]

Then for each given \(x \in \Omega\), either

\[
d(T^n x, T^{n+1} y) = \infty, \quad \forall n \geq 0,
\]

or there exists a natural number \(n_0\) such that

1. \(d(T^n x, T^{n+1} y) < \infty, \quad \forall n \geq 0,\)

2. The sequence \(T^n\) is convergent to a fixed point \(y^*\) of \(T\).

3. \(y^*\) is the unique fixed point of \(T\) in the set \(\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}\).

4. \(d(y, y^*) \leq \frac{1}{1-L}d(y, Ty) \) for all \(y \in \Delta\).

Theorem 3.2. Let \(X\) be a linear space, \((Y, N)\) and \((Z, N')\) be a fuzzy Banach space and a fuzzy normed linear space respectively. Suppose that \(\alpha\) is a constant satisfies \(0 < |\alpha| < 16\), \(\varphi\) is a
mapping from $X \times X \to Z$ such that

$$N'(\varphi(2x, 2y), t) \geq N'(\alpha \varphi(x, y), t)$$

for all $x \in X, t > 0$, and

$$\lim_{k \to \infty} N'(\varphi(2^k x, 2^k y), 16^k t) = 1$$

for all $x, y \in X, t > 0, k \geq 0$. If $f : X \to Y$ be an even function and $f(0) = 0$. In the sense that

$$N(Df(x, y), t) \geq N'(\varphi(x, y), t)$$

(3.1)

for all $x, y \in X, t > 0$. Then there exists a unique quartic mapping $C : X \to Y$ such that

$$N(C(x) - f(x), t) \geq N'(\varphi(0, x), (16 - \alpha)t)$$

for all $x \in X, t > 0$. Moreover,

$$C(x) = \lim_{n \to \infty} \frac{f(2^nx)}{16^n}$$

for all $x \in X$.

**Proof.** We assume that $0 < \alpha < 16$. Let

$$\Omega = \{g : g : X \to Y, g(0) = 0\}$$

and introduce the generalized metric $d$ on $\Omega$ by

$$d(g, h) = \inf \{\beta \in (0, \infty) : N(g(x) - h(x), \beta t) \geq N'(\varphi(0, x), 16t)\}$$

We know that $(\Omega, d)$ is complete generalized metric on $\Omega$. We now defined a mapping $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{16}g(2x)$$

We now prove $T$ is a strictly contractive mapping with the Lipschitz constant $\frac{\alpha}{16}$. Given $g, h \in \Omega$, set $\varepsilon \in (0, \infty)$ be an arbitrary constant with $d(g, h) < \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq N'(\varphi(0, x), 16t), \forall x \in X, t > 0.$$
Therefore

\[ N(Tg - Th, \frac{\alpha \varepsilon t}{16}) = N\left( \frac{1}{16}g(2x) - \frac{1}{16}h(2x), \frac{\alpha \varepsilon t}{16} \right) \]

\[ = N(g(2x) - h(2x), \alpha \varepsilon t) \]

\[ \geq N' (\varphi(0, 2x), 16\alpha t) \]

\[ \geq N' (\alpha \varphi(0, x), 16\alpha t) \]

\[ = N' (\varphi(0, x), 16t) \]

Hence, we can conclude that

\[ d(Tg, Th) \leq \frac{\alpha \varepsilon}{16} \]

Hence

\[ d(g, h) < \varepsilon \Rightarrow d(Tg, Th) \leq \frac{\alpha \varepsilon}{16}, g, h \in \Omega. \]

That is

\[ d(Tg, Th) \leq \frac{\alpha}{16}d(g, h) \]

Put \( x = 0 \) in (3.1), then replace \( y \) by \( x \), we obtain

\[ N\left( \frac{f(2x)}{16} - f(x), t \right) \geq N' (\varphi(0, x), 16t) \]

for all \( x \in X, t > 0 \), it follows that \( d(Tf, f) \leq 1 \). From the fixed point alternative theorem, we can conclude that, there exists a fixed point \( C \) of \( T \) in \( \Omega \) such that

\[ C(2x) = 16C(x), \forall x \in X. \]

Moreover, we have \( \lim_{n \to \infty} d(T^n f, C) \to 0 \), which implies

\[ N \left( \lim_{n \to \infty} \frac{f(2^n x)}{16^n} - C(x), t \right) = 0. \]

By the fixed point alternative, we conclude that

\[ d(f, C) \leq \frac{1}{1 - L} d(Tf, f) \]

Then

\[ d(f, C) \leq \frac{16}{16 - \alpha} \]

This means that

\[ N(C(x) - f(x), t) \geq N' (\varphi(0, x), (16 - \alpha)t) \]
for all \( x \in X, t > 0 \). The uniqueness of \( C \) follows from the fact that \( C \) is the unique fixed point of \( T \) with the property that there exists \( k \in (0, \infty) \) such that
\[
N(C(x) - f(x), kt) \geq N'(\varphi(0, x), t), \quad \forall x \in X, t > 0.
\]
This completes the proof of this theorem.

**Corollary 3.3.** Let \((X, \| \cdot \|)\) be a normed space, \((Y, N)\) be a fuzzy Banach space and \((Z, N')\) be a fuzzy normed space, \(u, v, \gamma, s\) be non-negative real numbers satisfies \(u + v, \gamma, s < 4\). If \( f : X \to Y \) be a mapping such that for some \( u_0 \in Z \)
\[
N(Df(x, y), t) \geq N'((\|x\|^u \|y\|^v + \|x\|^\gamma + \|y\|^s)u_0, t)
\]
for all \( x, y \in X, t > 0 \). Then there exists a unique quartic mapping \( C : X \to Y \) such that
\[
N(f(x) - C(x), t) \geq N'((\|x\|^s u_0, (16 - \alpha)t)
\]

**Proof.** We define \( \varphi : X \times X \to Z \) by
\[
\varphi(x, y) = (\|x\|^u \|y\|^v + \|x\|^\gamma + \|y\|^s)u_0.
\]
for all \( x, y \in X \). It follows the conditions of Theorem 3.2, then completes the proof.

**Theorem 3.4.** Let \( X \) be a linear space, \((Y, N)\) and \((Z, N')\) be a fuzzy Banach space and a fuzzy normed linear space respectively. Suppose that \( \alpha \) is a constant satisfies \( 0 < |\alpha| < 8 \), \( \varphi \) is a mapping from \( X \times X \to Z \) such that
\[
N'((\alpha \varphi(x, y), t)
\]
for all \( x \in X, t > 0 \), and
\[
\lim_{k \to \infty} N'(\varphi(2^k x, 2^k y), 8^k t) = 1
\]
for all \( x, y \in X, t > 0, k \geq 0 \). If \( f : X \to Y \) be an odd function and \( f(0) = 0 \). In the sense that
\[
N(Df(x, y), t) \geq N'(\varphi(x, y), t)
\]
(3.2)
for all \( x, y \in X, t > 0 \). Then there exists a unique cubic mapping \( C : X \to Y \) such that
\[
N(C(x) - f(x), t) \geq N'(\varphi(0, x), 3(8 - \alpha)t)
\]
for all $x \in X, t > 0$. Moreover,
\[ C(x) = \lim_{n \to \infty} \frac{f(2^n x)}{8^n} \]
for all $x \in X$.

**Proof.** Similar to the proof of Theorem 3.2. We can assume that $0 < \alpha < 8$. Let
\[ \Omega = \{ g : g : X \to Y, g(0) = 0 \} \]
and introduce the generalized metric $d$ on $\Omega$ by
\[ d(g, h) = \inf \{ \beta \in (0, \infty) : N(g(x) - h(x), \beta t) \geq N'(\varphi(0, x), 24t) \} \]
We know that $(\Omega, d)$ is complete generalized metric on $\Omega$. We now defined a mapping $T : \Omega \to \Omega$ by
\[ Tg(x) = \frac{1}{8} g(2x) \]
We now prove $T$ is a strictly contractive mapping with the Lipschitz constant $\frac{\alpha}{8}$. Given $g, h \in \Omega$, set $\varepsilon \in (0, \infty)$ be an arbitrary constant with $d(g, h) < \varepsilon$. Then
\[ N(g(x) - h(x), \varepsilon t) \geq N'(\varphi(0, x), 24t), \forall x \in X, t > 0 \]
Therefore
\[ N(Tg - Th, \frac{\alpha \varepsilon t}{8}) = N\left(\frac{1}{8} g(2x) - \frac{1}{8} h(2x), \frac{\alpha \varepsilon t}{8}\right) \]
\[ = N(g(2x) - h(2x), \alpha \varepsilon t) \]
\[ \geq N'(\varphi(0, 2x), 24\alpha t) \]
\[ \geq N'(\alpha \varphi(0, x), 24\alpha t) \]
\[ = N'(\varphi(0, x), 24t) \]
Hence we can conclude that
\[ d(Tg, Th) \leq \frac{\alpha \varepsilon}{8} \]
Hence
\[ d(g, h) < \varepsilon \Rightarrow d(Tg, Th) \leq \frac{\alpha \varepsilon}{8}, g, h \in \Omega. \]
That is
\[ d(Tg, Th) \leq \frac{\alpha}{8} d(g, h) \]
Put \( x = 0 \) in (3.2), then replace \( y \) by \( x \), we obtain

\[
N\left(\frac{f(2x)}{8} - f(x), t\right) \geq N'(\varphi(0,x), 24t)
\]

for all \( x \in X, t > 0 \), it follows that \( d(Tf, f) \leq 1 \). From the fixed point alternative theorem, we can conclude that, there exists a fixed point \( C \) of \( T \) in \( \Omega \) such that

\[
C(2x) = 8C(x), \forall x \in X
\]

Moreover, we have \( \lim_{n \to \infty} d(T^n f, C) \to 0 \), which implies

\[
N\left(\lim_{n \to \infty} \frac{f(2^n x)}{8^n} - C(x), t\right) = 0.
\]

By the fixed point alternative, we can conclude that

\[
d(f, C) \leq \frac{1}{1-L} d(Tf, f)
\]

Then

\[
d(f, C) \leq \frac{8}{8 - \alpha}
\]

This means that

\[
N(C(x) - f(x), t) \geq N'(\varphi(0,x), 3(8 - \alpha)t)
\]

for all \( x \in X, t > 0 \). The uniqueness of \( C \) follows from the fact that \( C \) is the unique fixed point of \( T \). This completes the proof of this theorem.

**Corollary 3.5.** Let \((X, \| \cdot \|)\) be a normed space, \((Y, N)\) be a fuzzy Banach space and \((Z, N')\) be a fuzzy normed space, \(u, v, \gamma, s\) be non-negative real numbers satisfies \( u + v, \gamma, s < 3 \). If \( f : X \to Y \) be a mapping such that for some \( u_0 \in Z \)

\[
N(Df(x,y), t) \geq N'(\|x\|^u\|y\|^v + \|x\|^\gamma + \|y\|^s)u_0, t)
\]

for all \( x, y \in X, t > 0 \). Then there exists a unique cubic mapping \( C : X \to Y \) such that

\[
N(f(x) - C(x), t) \geq N'(\|x\|^s u_0, 3(8 - \alpha)t)
\]

**Proof.** Similar with the proof of Corollary 3.3.
Theorem 3.6. Let $X$ be a linear space, $(Y, N)$ and $(Z, N')$ be a fuzzy Banach space and a fuzzy normed linear space respectively. Suppose that $\alpha$ is a constant satisfies $0 < |\alpha| < 8$, $\varphi$ is a mapping from $X \times X \to Z$ such that

$$N'(\varphi(2x, 2y), t) \geq N'(\alpha \varphi(x, y), t)$$

for all $x \in X, t > 0$, and

$$\lim_{k \to \infty} N'(\varphi(2^k x, 2^k y), 8^k t) = 1$$

for all $x, y \in X, t > 0, k \geq 0$. If $f: X \to Y$ be a function such that $f(0) = 0$. In the sense that

$$N(Df(x, y), t) \geq N'(\varphi(x, y), t)$$

for all $x, y \in X, t > 0$. Then there exists a unique cubic mapping $C: X \to Y$ and a unique quartic mapping $Q: X \to Y$ such that

$$N(f(x) - C(x) - Q(x), t) \geq \begin{cases} N'(\varphi(0, x), \frac{(16 - \alpha)}{2} t), & 0 < \alpha \leq 4, \\ N'(\varphi(0, x), \frac{3(8 - \alpha)}{2} t), & 4 < \alpha < 8. \end{cases}$$

for all $x \in X, t > 0$. Moreover,

$$C(x) = \lim_{n \to \infty} \frac{f(2^n x)}{8^n}, Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{16^n}$$

for all $x \in X$.

Proof. We assume that $0 < \alpha < 8$. Let $f_0(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$. Then $f_0(0) = 0, f_0(-x) = -f_0(x)$ and

$$N(D(f_0(x, y), t) \geq \min \{N'(\varphi(x, y), t), N'(\varphi(-x, -y), t)\}$$

Let $f_1(x) = \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$. Then $f_1(0) = 0, f_1(-x) = f_1(x)$ and

$$N(D(f_1(x, y), t) \geq \min \{N'(\varphi(x, y), t), N'(\varphi(-x, -y), t)\}$$

Using the proof Theorem 3.2 and 3.4, we get unique cubic mapping $C$ and unique quartic mapping $Q$ satisfying

$$N(f_0(x) - C(x)) \geq N'(\varphi(0, x), 3(8 - \alpha)t), N(f_1(x) - Q(x)) \geq N'(\varphi(0, x), (16 - \alpha)t).$$
Therefore,
\[ N(f(x) - C(x) - Q(x), t) \geq \min \{ N(f_0(x) - C(x), \frac{t}{2}), N(f_1(x) - Q(x), \frac{t}{2}) \} \]
\[ \geq \min \{ N'(\varphi(0,x), \frac{3(8 - \alpha)}{2} t), N'(\varphi(0,x), \frac{(16 - \alpha)}{2} t) \} . \]

This means that
\[ N(f(x) - C(x) - Q(x), t) \geq \begin{cases} 
N'(\varphi(0,x), \frac{(16 - \alpha)}{2} t), & 0 < \alpha \leq 4, \\
N'(\varphi(0,x), \frac{3(8 - \alpha)}{2} t), & 4 < \alpha < 8.
\end{cases} \]

This completes the proof of this theorem.

Conflict of Interests
no conflict of interest.

References