Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 6, 1721-1733
ISSN: 1927-5307

ON WEIGHTED CLASSES OF ANALYTIC FUNCTION SPACES

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Abstract. In this paper, we introduce a general class of analytic functions which extend the generalized Hardy space. We investigate the continuity of the point evaluations on this space.

2000 AMS Subject Classification: 30H05; 46E15

1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\Delta$ its boundary and $H(\Delta)$ the space of all analytic function on the unit disk. For an analytic function f on the unit disk and $0 < r < 1$, we define the delay function f_r by $f_r(e^{i\theta}) = f(re^{i\theta})$. It is easy to see that the functions f_r are continuous on $\partial\Delta$ for each r.

The theory of harmonic functions motivates the following classes of analytic functions, determined by their limiting behavior as their arguments approach to the boundary $\partial\Delta$. For $0 < p < \infty$, the Hardy space H^p is defined as the set of analytic functions $f : \Delta \to \mathbb{C}$ such that

$$\|f\|^p_{H^p} = \sup_{0<r<1} \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

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Received April 14, 2012

By the Littlewood Subordination Theorem (see [1]), we see that the supremum in the above definition of H^p is actually a limit, that is,

$$\|f\|_{H^p}^p = \lim_{r\to 1}\int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

It should be mentioned that the function $\|.\|_{H^p}^p : H^p \to \mathbb{R}^+$ is a norm on H^p, and makes H^p into a Banach space for $1 \le p < \infty$ (see [2]). For more studies on Hardy space, we refer to [2, 5, 6].

Recently Fatehi [4], introduced the following definition

Definition 1. *Let $F : H(\Delta) \to H(\Delta)$ be a linear operator such that $F(f) = 0$ if and only if $f = 0$, that is, F is $1-1$. For $1 \le p < \infty$, the generalized Hardy space $H_{F,p}(\Delta) = H_{F,p}$ is defined to be the collection of all analytic functions f on Δ for which*

$$\sup_{0<r<1}\int_0^{2\pi} |(F(f))_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

Denote the pth root of this supremum by $\|f\|_{H_{F,p}}$. Since, $|F(f)|^p$ is a subharmonic function, so by [1], we have

$$\|f\|_{H_{F,p}}^p = \lim_{r\to 1^-}\int_0^{2\pi} |F(f)_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

Therefore, $f \in H_{F,p}$ if and only if $F(f) \in H^p$ and

$$\|F(f)\|_p^p = \|f\|_{H_{F,p}}^p = \lim_{r\to 1^-}\int_0^{2\pi} |F(f)_r(e^{i\theta})|^p \frac{d\theta}{2\pi}.$$

It is easy to see that $H_{F,p}$ is a normed space with the norm $\|.\|_{H_{F,p}}$.

For $0 < p < \infty$, the Bergman space A^p is the set of all $f \in H(\Delta)$ such that

$$\int_\Delta |f(z)|^p dA(z) < \infty,$$

where $dA(z) = dx\,dy = r\,dr\,d\theta$ is the Lebegue area measure. We mention [3] as general reference for the theory of Bergman spaces.

Throughout this paper, P denotes the set of all analytic polynomials and for a function F, R_F denotes the range of F.

We assume from now on that $\Psi : [0,1] \to [0,\infty)$ to appear in this paper is right-continuous and nondecreasing functions such that the integral

$$\int_0^1 \Psi(1-\rho))\rho d\rho < \infty.$$

We can define an auxiliary function as follows:

$$\varphi_\Psi(s) = \sup_{0<t\leq 1} \frac{\Psi(st)}{\Psi(t)}, \quad 0 < s < \infty, \tag{1}$$

we assume that

$$\int_0^1 \varphi_\Psi(s)\frac{ds}{s} < \infty. \tag{2}$$

From now on we suppose that the above weight function Ψ satisfies the following properties:

(a) Ψ is nondecreasing on $[0,1]$,

(b) Ψ is twice differentiable on $(0,1)$,

(c) $\int_0^1 \Psi(1-r)rdr < \infty$,

(d)$\Psi(t) = \Psi(1) > 0$, $t \geq 1$ *and*

(e)$\Psi(st) \approx \Psi(t), \;\; t \geq 0$.

We will need the following condition in the sequel.

$$\int_0^1 (1-r^2)^{q-2}\Psi(1-r)dr < \infty \quad \text{where} \;\; 0 < q < \infty. \tag{3}$$

Throughout this paper, P denotes the set of all analytic polynomials and for a function F, R_F denotes the range of F.

For $p, q \in (0,\infty)$, the weighted Bergman space $A^p_{\Psi,q}$ is the set of all $f \in H(\Delta)$ such that

$$\|f\|_{A^p_{\Psi,q}} = \sup_{0<\rho<1} \int_0^1 \int_0^{2\pi} |f_\rho(e^{i\theta})|^p (1-r^2)^{q-2}\Psi(1-r)\, d\theta\, dr < \infty. \tag{4}$$

The above formula defines a norm that turns $A^2_{\Psi,q}$ into a Hilbert space whose inner product is given by

$$\langle f, g\rangle_{A^2_{\Psi,q}} = \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} = \int_0^{2\pi} \big(f_r(e^{i\theta})\big)\big(\overline{g_r(e^{i\theta})}\big)\, r\, d\theta\, dr \tag{5}$$

for each $f, g \in A^2_{\Psi,q}$.

Remark 1. *By using known technique, it not hard to prove that* $(A^p_{\Psi,q}, \|.\|_{A^p_{\Psi,q}})$ *is a Banach space, that is, the norm* $\|.\|_{A^p_{\Psi,q}}$ *is complete.*

1. (F,Ψ)-Bergman spaces

Definition 2. *Let* $F : H(\Delta) \to H(\Delta)$ *be a linear operator such the* $F(f) = 0$ *if and only if* $f = 0$*, that is,*F *is* $1-1$*. Suppose that* $\Psi : [0,1] \to [0,\infty)$ *is a nondecreasing and right-continuous function. For* $p, q \in (0,\infty)$*, the* (F,Ψ)*-Bergman space* $A^p_{F,\Psi,q}(\Delta) = A^p_{F,\Psi,q}$ *is defined to be the collection of all analytic function* f *on* Δ *for which*

$$\|f\|_{A^p_{F,\Psi,q}} = \sup_{0<\rho<1} \int_0^1 \int_0^{2\pi} |F(f_\rho(e^{i\theta}))|^p (1-r^2)^{q-2} \Psi(1-r) d\theta\, dr < \infty. \tag{6}$$

The importance of this definition is that it contains some known classes of analytic function spaces like Bergman and Hardy classes as we mention in the following remark:

Remark 2. *We note that if* $\int_0^1 (1-r^2)^{q-2}\Psi(1-r) r\, dr = 1$*, then we obtain the generalized Hardy space as defined and studied in* [4]*. Also, if* $\Psi(1-r) = 1$*,* $q = 0$*, and* $F(f_\rho(e^{i\theta})) = f(z)$*, then we obtain the Bergman space* A^p*.*

Theorem 1. *Let* $0 < p, q < \infty$ *and* $P \subseteq R_F$*. Then* $A^p_{\Psi,q}$ *is a subspace of* R_F *if and only if* $A^p_{F,\Psi,q}$ *is a Banach space.*

Proof. Suppose that $A^p_{\Psi,q} \subseteq R_F$. Since $A^p_{F,\Psi,q}$ is a normed space, it suffices to show that it is complete. Let $\{f_n\}$ be Cauchy sequence in $A^p_{F,\Psi,q}$ and set $F(f_n) = g_n$. Then $\{g_n\}$ is a Cauchy sequence in $A^p_{\Psi,q}$. Since $A^p_{\Psi,q}$ is complete, there is a $g \in A^p_{\Psi,q}$ such that

$$\|g_n - g\|_{A^p_{\Psi,q}} \to 0, \text{ as } n \to \infty.$$

Since $A^p_{\Psi,q} \subseteq R_F$, there is an $f \in A(\Delta)$ such that $F(f) = g$. Now we show that this f is the $A^p_{F,\Psi,q}$-limit of $\{f_n\}$. We have

$$\|f_n - f\|_{A^p_{F,\Psi,q}} = \|g_n - g\|^p_{\Psi,q} \to 0, \quad \text{as} \quad n \to \infty.$$

Hence $f_n \to f \in A^p_{F,\Psi,q}$ for sufficiently large positive integer n, which implies that $f \in A^p_{F,\Psi,q}$. So $f_n \to f$ in $A^p_{F,\Psi,q}$ as $n \to \infty$.

Conversely, suppose that $A^p_{F,\Psi,q}$ is a Banach space. If $A^p_{\Psi,q} \subseteq R_F$, then there is a $g \in A^p_{\Psi,q}$

such that g is not in R_f. Since the polynomials are dense in $A^p_{\Psi,q}$, there is a sequence $\{p_n\}$ in P such that $\|p_n - g\|_{A^p_{\Psi,q}} \to 0$ as $n \to \infty$. Let $q_n = F^{-1}(p_n)$. Then $\{q_n\}$ is a Cauchy sequence in $A^p_{F,\Psi,q}$ and so there is a $q \in A^p_{F,\Psi,q}$ such that $\|q_n - q\|_{A^p_{F,\Psi,q}} \to 0$ as $n \to \infty$. Hence $\|F(q_n) - F(q)\|_{A^p_{\Psi,q}} \to 0$ as $n \to \infty$. On the other hand, $\|F(q_n) - g\|_{A^p_{\Psi,q}} \to 0$ as $n \to \infty$. This shows that $g = F(q)$ which is a contradiction.

Proposition 1. *Let $A^2_{\Psi,q} \subseteq R_F$, and suppose that*

$$J(\Psi, q) = \int_0^1 (1 - r^2)^{q-2}\Psi(1 - r)\, dr < \infty, \tag{7}$$

then $A^2_{F,\Psi,q}$ is a Hilbert space.

Proof. We define the scalar product on $A^2_{F,\Psi,q}$ by

$$\begin{aligned}\langle f, g\rangle_{A^2_{F,\Psi,q}} &= \int_0^1 \int_0^{2\pi} F(f_\rho(e^{i\theta}))\overline{F(g_r(e^{i\theta}))}(1 - r^2)^{(q-2)}\Psi(1 - r)d\theta\, dr \\ &\leq C \int_0^{2\pi} F(f_\rho(e^{i\theta}))\overline{F(g_r(e^{i\theta}))}\, d\theta = \langle F(f), F(g)\rangle_{H^2}.\end{aligned}$$

It is not hard to show that this scalar product defines an inner product on $A^2_{F,\Psi,2}$.

There is a Banach space $A^p_{\Psi,q}$, such that it does not satisfy the conditions of Theorem 2.1. For example, let $1 \leq p, q < \infty$, $F(f)(z) = zf(z)$ for each $f \in H(\Delta)$. Then $1 \not\ni R_F$. By the following proposition, we see that although $A^p_{\Psi,q} \subseteq R_F$, $A^p_{F,\Psi,q}$ is a Banach space.

Proposition 2. *Suppose that $1 \leq p < \infty$, $0 < q < \infty$, $h(z) \in H(\Delta)$, $h \neq 0$ and $F(f) = fh$ for every $f(z) \in H(\Delta)$. Then $A^p_{F,\Psi,q}$ is a Banach space.*

Proof. If $A^p_{\Psi,p} \subseteq R_F$, then by Theorem 2.1, the proposition holds. Otherwise, let $\{f_n\}$ be a Cauchy sequence in $A^p_{F,\Psi,q}$. Setting $F(f_n) = g_n$, so $\{g_n\}$ is a Cauchy sequence in $A^p_{\Psi,q}$. Therefore, there is a $g \in A^p_{\Psi,q}$ such that $\|g_n - g\|_{A^p_{\Psi,q}} \to 0$ as $n \to \infty$. If $g \in R_F$, then the proof is similar to the proof of Theorem 2.1.

Now suppose that g is not in R_F. Then there are $z_0 \in \Delta$, $m_1 \geq 0$, and $m_2 > m_1$ such that

$$g(z) = (z - z_0)^{m_1} g_0(z),$$

$$h(z) = (z - z_0)^{m_2} h_0(z),$$

where $h_0(z), g_0(z) \in H(\Delta)$; $g_0(z_0) \neq 0$ and $h_0(z_0) \neq 0$. Therefore, we have

$$\begin{aligned}\|g_n - g\|_{A^p_{\Psi,q}} &= \|hf_n - g\|_{A^p_{\Psi,q}} \\ &= \int_0^1 \int_0^{2\pi} \Lambda_1(f_n, h_n, r, \theta)(1-r^2)^{q-2}\Psi(1-r)d\theta\, dr,\end{aligned}$$

where

$$\left|\Big((\rho\, e^{i\theta} - z_0)^{m_2} h_0(\rho\, e^{i\theta}) f_n - (\rho\, e^{i\theta} - z_0)^{m_1} g_0(\rho\, e^{i\theta})\Big)\right|^p = \Lambda_1(f_n, h_n, r, \theta).$$

Since $\|g_n - g\|_{A^p_{\Psi,q}} \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int_0^1 \int_0^{2\pi} \Lambda(f_n, h_n, r, \theta)(1-r^2)^{q-2}\Psi(1-r)d\theta\, dr = 0. \tag{8}$$

where $\left|\Big((\rho\, e^{i\theta} - z_0)^{m_2} h_0(\rho\, e^{i\theta}) f_n - (\rho\, e^{i\theta} - z_0)^{m_1} g_0\Big)(\rho e^{i\theta})\right|^p = \Lambda(f_n, h_n, r, \theta)$.
Hence, $\|(z - z_0)^{m_2} h_0 f_n - (z - z_0)^{m_1} g_0\|_{A^p_{\Psi,q}} \to 0$ as $n \to \infty$. Since the point evaluation at z_0 is a bounded linear functional on $A^p_{\Psi,q}$, we obtain

$$(z_0 - z_0)^{m_2} h_0 f_n(z_0) - (z_0 - z_0)^{m_1} g_0(z_0) \to 0, \;\; n \to \infty. \tag{9}$$

So $g_0(z_0) = 0$, which is a contradiction. The proof of Proposition 2 is therefore established.

In the following proposition, we will find a dense subset in $A^p_{F,\Psi,q}$, whenever $P \subseteq R_F$.

Proposition 3. *Suppose that* $1 \le p < \infty,\ 0 < q < \infty$, *and* $P \subseteq R_F$. *Then*

$$\{\overline{F^{-1}(p) : p \in P}\} = A^p_{F,\Psi,q}.$$

Proof. It is clear that $\{F^{-1}(p) : p \in P\} \subseteq A^p_{F,\Psi,q}$. Suppose that $f \in A^p_{F,\Psi,q}$. Then there is a sequence $\{h_n\}$ in P such that $\|h_n - F(f)\|_{A^p_{\Psi,q}} \to 0$ as $n \to \infty$. Setting $f_n = F^{-1}(h_n)$, we have

$$\|f_n - f\|_{A^p_{F,\Psi,q}} = \|h_n - F(f)\|_{A^p_{\Psi,q}}, \tag{10}$$

so the result follows.

Corollary 1. *Suppose that* $1 \le p < \infty,\ 0 < q < \infty,$, $P \subseteq R_F$, *and* $F^{-1}(p) \in P$ *for each* $p \in P$. *Then* $\overline{P \cap A^p_{F,\Psi,q}} = A^p_{F,\Psi,q}$.

2. Point Evaluations

Let e_ω be the point evaluation at ω, that is, $e_\omega(f) = f(\omega)$. Let $\omega \in \Delta$ and H be a Hilbert space of analytic functions on Δ. If e_ω is a bounded linear functional on H, then the Riesz Representation Theorem implies that there is a function (which is usually called K_ω) in H that induces this linear functional, that is, $e_\omega(f) = \langle f, K_\omega \rangle$. It is well known that point evaluations at the point of Δ are all continuous.

In this section, we investigate the continuity of the point evaluations on $A^p_{F,\Psi,q}$. Next, we prove that an analytic function f on the unit disk with Hadamard gaps, that is, $f(z)$ satisfying $\frac{n_{k+1}}{n_k} \geq c > 1$ for all $k \in \mathbb{N}$ belongs to the space $A^p_{F,K,q}$.

Theorem 2. *Let $0 < q < \infty$ and $1 \leq p < \infty$. Suppose that Ψ satisfies the following condition*

$$\int_0^1 r^{2^n-p+1}\left(\log\frac{1}{r}\right)^{\frac{2q-p-3}{2}}\Psi(1-r)dr < \infty. \tag{11}$$

Also, suppose that

$$f(z) = \sum_{j=1}^{\infty} b_j z^{n_j - 1},$$

is in the Hadamard gap class, then $f \in A^p_{F,\Psi,q}$ if

$$\sum_{j=1}^{\infty} |b_j|^p < \infty. \tag{12}$$

Proof. First assume that condition (12) holds. We write $z = re^{i\theta}$ in polar form and observe that

$$|f(z)| \leq \sum_{j=1}^{\infty} |b_j| r^{n_j - 1}.$$

Then by Theorem 2.1, letting $F(f) = g$, we obtain

$$\begin{aligned}
\|f\|_{A^p_{F,\Psi,q}} &= \int_0^1 \int_0^{2\pi} |F(f(re^{i\theta}))|^p (1-r^2)^{q-2}\Psi(1-r) d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} |g(re^{i\theta})|^p (1-r^2)^{q-2}\Psi(1-r) d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} \left(\sum_{j=1}^{\infty} |b_j| r^{n_j - 1} \right)^p (1-r^2)^{q-2}\Psi(1-r) d\theta\, dr \\
&= 2\pi \int_0^1 r^{-p+1} \left[\sum_{j=1}^{\infty} |b_j| r^{n_j} \right]^p (1-r^2)^{q-2}\Psi(1-r)\, dr.
\end{aligned}$$

Using Cauchy-Schwarz inequality to produce

$$\begin{aligned}
\left[\sum_{j=1}^{\infty} |b_j| r^{n_j} \right]^p &= \left[\sum_{n=0}^{\infty} \sum_{n_j \in I_n} |b_j| r^{n_j} \right]^p \leq \left[\sum_{n=0}^{\infty} \sum_{n_j \in I_n} |b_j| r^{2^n} \right]^p \\
&\leq \left[\sum_{n=0}^{\infty} (2^{n/2} r^{2^n})^{1-1/p} (r^{2^n} 2^{(1-p)n/2})^{1/p} \sum_{n_j \in I_n} |b_j| \right]^p \\
&\leq \left[\sum_{n=0}^{\infty} r^{2^n} 2^{((1-p)/2)n} \left(\sum_{n_j \in I_n} |b_j| \right)^p \right] \left[\sum_{n=0}^{\infty} 2^{n/2} r^{2^n} \right]^{p-1} \\
&\leq C \left(\log \frac{1}{r} \right)^{-(p-1)/2} \sum_{n=0}^{\infty} r^{2^n} 2^{((1-p)/2)n} \left(\sum_{n_j \in I_n} |b_j| \right)^p,
\end{aligned}$$

where $I_n = \{j : 2^n \leq j < 2^{n+1}, j \in \mathbb{N}\}$. To this end, we combine the elementary estimates:

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^{\frac{n}{2}} r^{2^n} &= \sqrt{2} \sum_{n=0}^{\infty} \int_{2^n}^{2^{n+1}} t^{-\frac{1}{2}} r^{\frac{t}{2}} dt \\
&\leq \sqrt{2} \int_0^{\infty} t^{-\frac{1}{2}} r^{\frac{t}{2}} dt \\
&\leq 2\Gamma(\frac{1}{2}) \left(\log \frac{1}{r} \right)^{-\frac{1}{2}}.
\end{aligned}$$

This very useful tool can now be applied to the calculation above to obtain

$$\|f\|_{A^p_{F,K,q}} \leq C \sum_{n=0}^{\infty} (2^n)^{\frac{1-p}{2}} \left[\sum_{n_j \in I_n} |b_j| \right]^p \int_0^1 r^{2^n - p + 1} \left(\log \frac{1}{r} \right)^{\frac{2q-p-3}{2}} \Psi(1-r) dr \tag{13}$$

where $(1-r^2) \leq 2\log\frac{1}{r}$. This together with (11), imply that

$$\begin{aligned}\|f\|_{A^p_{F,\Psi,q}} &\leq C\sum_{n=0}^{\infty}\Big[\sum_{n_j\in I_n}|b_j|\Big]^p\left(\frac{1}{2^n}\right)^{\frac{p-1}{2}}\\ &\leq C\sum_{n=0}^{\infty}\Big[\sum_{n_j\in I_n}|b_j|\Big]^p\left(\frac{1}{2^n}\right)^{\frac{p-1}{2}} \qquad (14)\end{aligned}$$

If $n_j \in I_n$, then $n_j < 2^n < 2^{n+1}$. It follows that

$$\left(\frac{1}{2^n}\right)^{\frac{p-1}{2}} < n_j^{\frac{p-1}{2}}.$$

Combining this with (14), we obtain

$$\|f\|_{A^p_{F,K,q}} \lesssim \sum_{n=0}^{\infty}\Big[\sum_{n_j\in I_n}|b_j|\Big]^p n_j^{\frac{p-1}{2}}. \qquad (15)$$

Since f is in the Hadamard gap class, there exists a constant c such that $n_{j+1} \geq cn_j$ for all $j \in \mathbb{N}$. Hence, the Taylor series of $f(z)$ has at most $([\log_c 2]+1)$ terms $a_j z^{n_j}$ such that $n_j \in I_n$. By (15) and Hölder's inequality, we deduce that

$$\|f\|_{A^p_{F,K,q}} \lesssim (\log_c 2+1)^{\frac{p-1}{2}}\sum_{n=0}^{\infty}\sum_{n_j\in I_n}|b_j|^p.$$

Then, $f \in A^p_{F,\Psi,q}$

Lemma 1. *If $f \in A^p_{K,q}(0<p,q<\infty)$, then*

$$\begin{aligned}&\lim_{\rho\to 1}\int_0^1\int_0^{2\pi}|F(f(\rho\, e^{i\theta}))|^p(1-r^2)^{q-2}\Psi(1-r)r\,d\theta\,dr\\ &\quad=\int_0^1\int_0^{2\pi}|F(f(e^{i\theta}))|^p(1-r^2)^{q-2}\Psi(1-r)r\,d\theta\,dr\end{aligned}$$

and

$$\lim_{\rho\to 1}\int_0^1\int_0^{2\pi}|F(f(\rho\, e^{i\theta}))-F(f(e^{i\theta}))|^p(1-r^2)^{q-2}\Psi(1-r)d\theta\,dr=0.$$

Proof. First let us prove

$$\lim_{\rho\to 1}\int_0^1\int_0^{2\pi}|F(f_\rho(e^{i\theta}))-F(f(e^{i\theta}))|^p(1-r^2)^{q-2}\Psi(1-r)d\theta\,dr=0$$

for $p=2$. If $F(f(z))=\sum b_j^p(f(z))^n$ is in $A^2_{F,\Psi,q}$, then $\sum_{j=1}^{\infty}|b_j|^p<\infty$.
But by Fatou's lemma, we have

$$\begin{aligned}
&\int_0^1\int_0^{2\pi}|F(f_\rho(e^{i\theta}))-F(f(e^{i\theta}))|^2(1-r^2)^{q-2}\Psi(1-r)\,d\theta\,dr\\
&\leq \liminf_{\rho\to 1}\int_0^1\int_0^{2\pi}|F(f_\rho(e^{i\theta}))-F(f(\rho e^{i\theta}))|^2(1-r^2)^{q-2}\Psi(1-r)\,d\theta\,dr\\
&=\sum_{n=1}^{\infty}\int_0^1\int_0^{2\pi}\left|b_jf(\rho\, e^{i\theta})-b_j\, f(e^{i\theta})\right|^2(1-r^2)^{q-2}\Psi(1-r)\,d\theta\,dr\\
&=\sum_{n=1}^{\infty}|b_j|^2K\left(\frac{1}{n_j}\right)\int_0^1\int_0^{2\pi}|f(\rho\, e^{i\theta})-f(e^{i\theta})|^2(1-r^2)^{q-2}\Psi(1-r)d\theta\,dr
\end{aligned}$$

which tends to zero as $\rho\to 1$. Now, we proof

$$\begin{aligned}
&\lim_{\rho\to 1}\int_0^1\int_0^{2\pi}|F(f(\rho\, e^{i\theta}))|^p(1-r^2)^{q-2}\Psi(1-r)\,d\theta\,dr\\
&\quad=\int_0^1\int_0^{2\pi}|F(f(e^{i\theta}))|^p(1-r^2)^{q-2}\Psi(1-r)\,d\theta\,dr
\end{aligned}$$

in the case $p=2$, If $f\in A^p_{F,\Psi,q}$ $(0<q<\infty)$, we use the factorization $f=B\,g$ where $B(z)$ is a Blaschke product and g(z) is an $A^p_{F,\Psi,q}$. Since $(g(z))^{\frac{p}{2}}\in A^2_{F,\Psi,q}$, it follows from what we have just proved that

$$\begin{aligned}
&\int_0^1\int_0^{2\pi}|F(f(\rho\, e^{i\theta}))|^p(1-r^2)^{q-2}\Psi(1-r)\,d\theta\,dr\\
&\leq\int_0^1\int_0^{2\pi}|F(g(\rho\, e^{i\theta}))|^p(1-r^2)^{q-2}\Psi(1-r)r\,d\theta\,dr.
\end{aligned}$$

Then,

$$\int_0^1\int_0^{2\pi}|F(g(e^{i\theta}))|^p(1-r^2)^{q-2}\Psi(1-r)r\,d\theta\,dr=\int_0^1\int_0^{2\pi}|F(f(e^{i\theta}))|^p(1-r^2)^{q-2}\Psi(1-r)r\,d\theta\,dr.$$

This together with Fatou's lemma complete the proof.

Theorem 3. *Let $\Psi:[0,1]\to[0,\infty)$ be a non-decreasing and right-continuous function. Suppose that $\omega\in\Delta$ and $A^p_{\Psi,q}\subseteq R_F$. For $1\leq p<2,\ 0<q<\infty$ and $\sum_{j=0}^{\infty}\overline{F^{-1}(z^j)(\omega)}\,z^j\in H^\infty$. If for each $0<\rho<1$, $f\in A^1_{F,\Psi,q}$, and $(F(f))_\rho=F(f_\rho)$, then e_ω is continuous on $A^p_{F,\Psi,q}$.*

Proof. Let $f \in A^1_{F,\Psi,q}$. Then for each $0 < \rho < 1$, $f_\rho \in A^2_{F,\Psi,q}$ and then

$$\begin{aligned} f_\rho(\omega) &= \langle f_\rho, K_\omega \rangle_{A^2_{F,\Psi,q}} \\ &= \langle F(f_\rho), F(K_\omega) \rangle_{A^2_{\Psi,q}} \\ &= \int_0^1 \int_0^{2\pi} F(f_\rho(e^{i\theta}))\overline{F(K_\omega)}(e^{i\theta})(1-r^2)^{q-2}\Psi(1-r)r\, d\theta\, dr. \end{aligned}$$

Also by Lemma 1, we have $\|(F(f))_\rho - F(f)\|_{A^1_{F,\Psi,q}} \to 0$ as $\rho \to 1$.
Hence, using Hölder's inequality and the fact that $F(K_\omega) = \sum_{j=0}^{\infty} \overline{F^{-1}(z^j)(\omega)}\, z^j$, we obtain

$$\begin{aligned} &\left| \int_0^1 \int_0^{2\pi} \big(F((f))_\rho - F(f)(e^{i\theta})\big)\overline{F(K_\omega)}(e^{i\theta})(1-r^2)^{q-2}\Psi(1-r) r d\theta\, dr \right| \\ \le\ & \|F(K_\omega)\|_\infty \int_0^1 \int_0^{2\pi} \left| F(f_\rho(e^{i\theta})) - F(f(e^{i\theta})) \right| (1-r^2)^{q-2}\Psi(1-r)r\, d\theta\, dr \\ \le\ & \|F(K_\omega)\|_\infty \|(F(f))_\rho - F(f)\|_{A^1_{F,\Psi,q}} \to 0 \ \text{ as } \ \rho \to 1, \end{aligned}$$

so we obtain

$$\begin{aligned} f(\omega) &= \lim_{\rho\to 1} f_\rho(\omega) \\ &= \int_0^1 \int_0^{2\pi} F(\lim_{\rho\to 1} f_\rho(\rho\, e^{i\theta}))\overline{F(K_\omega)}(e^{i\theta})(1-r^2)^{q-2}\Psi(1-r)r d\theta\, dr \\ &= \int_0^1 \int_0^{2\pi} F(f(e^{i\theta}))\overline{F(K_\omega)}(e^{i\theta})(1-r^2)^{q-2}\Psi(1-r)r d\theta\, dr. \end{aligned}$$

Hence,

$$\begin{aligned} |f(\omega)| &= \left| \int_0^1 \int_0^{2\pi} F(f(e^{i\theta}))\overline{F(K_\omega)}(e^{i\theta})(1-r^2)^{q-2}\Psi(1-r)r d\theta\, dr \right| \\ &\le \|F(K_\omega)\|_\infty \|f\|_{A^1_{F,\Psi,q}} \end{aligned}$$

for each $f \in A^1_{F,\Psi,q}$. Now let $1 \le p < 2$. If $f \in A^p_{F,\Psi,q}$, then

$$|f(w)| \le \|F(K_\omega)\|_\infty \|f\|_{A^1_{F,\Psi,q}} \le \|F(K_\omega)\|_\infty \|f\|_{A^p_{F,\Psi,q}},$$

so, the result follows.

Theorem 4. *Let* $\Psi : [0,1] \to [0,\infty)$ *be a non-decreasing and right-continuous function satisfying (7) and let* $1 \le p < \infty$, $0 < q < \infty$, $\omega \in \Delta$, $h \in H(\Delta)$, $h \ne 0$. *For each* $f \in H(\Delta)$, $F(f) = fh$. *Then* e_ω *is continuous on* $A^p_{F,K,q}$.

Proof. We break the proof in to two parts.

(1) Let $h(w) \neq 0$. If $|\omega| < \rho < 1$ and Γ_ρ is the circle of radius ρ with center at the origin, then the Cauchy formula shows that for any f in $A^p_{F,\Psi,q}$,

$$\begin{aligned} f(\omega)h(\omega) &= \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{f(\zeta)h(\zeta)}{\zeta-\omega}d\zeta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\rho\, e^{i\theta})h(\rho\, e^{i\theta})}{\rho\, e^{i\theta}-\omega}\rho\, ie^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\rho\, e^{i\theta})h(\rho\, e^{i\theta})\frac{\rho}{\rho-\omega e^{-i\theta}}d\theta, \end{aligned}$$

Then,

$$\int_0^1 f(\omega)h(\omega)(1-r^2)^{q-2}\Psi(1-r)rdr = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \frac{f(re^{i\theta})h(\rho\, e^{i\theta})}{\rho-\omega e^{-i\theta}}(1-\rho^2)^{q-2}\Psi(1-\rho)r\,\rho\, d\theta\, dr.$$

By Hölder's inequality, it follows that

$$|f(\omega)||h(\omega)|\int_0^1 (1-r^2)^{q-2}\Psi(1-r)r\,dr \le \frac{1}{2\pi}\|(fh)_\rho\|_{A^p_{\Psi,q}}\Big\|\frac{\rho}{\rho-\omega e^{-i\theta}}\Big\|_{p^*} \tag{16}$$

where $\frac{1}{p}+\frac{1}{p^*}=1$. Now if r tends to 1, $\left|\frac{\rho}{(\rho-\omega e^{-i\theta})}\right|$ converges uniformly to the bounded function $|1-\omega e^{i\theta}|^{-1}$ and

$$\|(fh)_\rho\|_{A^p_{\Psi,q}} \le \|fh\|_{A^p_{\Psi,q}}.$$

Hence there in an $M = \frac{\|\rho/(\rho-\omega e^{-i\theta})\|}{2\pi J(\Psi,q)} < \infty$ such that

$$|f(\omega)| \le \frac{M}{|h(\omega)|}\|f\|_{A^p_{F,\Psi,q}},$$

and the result follows.

(2) Let $h(\omega)=0$. Then $h(z)=(z-\omega)^m h_0(z)$, where $m\in\mathbb{N}$, $h_0(z)\in H(\Delta)$, and $h_0(\omega)\neq 0$. Let $F_1(f)=fh_0$ for each $f\in H(\Delta)$, it is easy to see that $A^p_{F,\Psi,q}\subseteq A^p_{F_1,\Psi,q}$. Then by the

preceding part, there is a constant $0 < C < \infty$ such that

$$\begin{aligned}
|f(\omega)|^p &\leq C\|fh_0\|_{A^p_{\Psi,q}} \\
&= C\int_0^1\int_0^{2\pi}\left|f(\rho\, e^{i\theta})\right|^p\left|h_0(e^{i\theta})\right|^p\frac{|e^{i\theta}-\omega|^{mp}}{|e^{i\theta}-\omega|^{mp}}(1-r^2)^{q-2}\Psi(1-r)rd\theta\, dr \\
&\leq \frac{C}{(1-|\omega|)^{mp}}\int_0^1\int_0^{2\pi}\left|f(\rho\, e^{i\theta})\right|^p\left|h(e^{i\theta})\right|^p(1-r^2)^{q-2}\Psi(1-r)rd\theta\, dr \\
&= \frac{C}{(1-|\omega|)^{mp}}\|f\|_{A^p_{F,\Psi,q}}
\end{aligned}$$

for each $f \in A^p_{F,\Psi,q}$. So e_ω is continuous on $A^p_{F,\Psi,q}$.

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