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DUAL COMBINATION-COMBINATION MULTI SWITCHING ANTI SYNCHRONIZATION OF CHAOTIC SYSTEMS

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Abstract. This paper presents the dual combination-combination multi switching anti synchronization between two pairs of drive chaotic systems and two pairs of response chaotic systems. The multiple combination of chaotic systems and multi switching results in a complex dynamic behaviour, which is interesting to study. Using Lyapunov stability theory, sufficient conditions are achieved and suitable controllers are designed to realize the desired synchronization among eight chaotic systems. Corresponding theoretical analysis is presented and numerical simulations performed to demonstrate the effectiveness of the proposed scheme.

Keywords: chaos synchronization; dual synchronization; multi-switching synchronization; combination combination synchronization; nonlinear control.

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1. Introduction

The chaos synchronization problem, because of its interdisciplinary nature, has received interest from researchers across the academic fields since it was first introduced by Pecora and

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Caroll [1]. The potential applications of chaos synchronization to engineering systems, information processing, secure communications, and biomedical science amongst many others has led to a vast variety of research studies in this topic of nonlinear science [2–5]. Various kinds of synchronization have been reported and presented in a chaotic systems using many effective methods such as complete synchronization, anti synchronization, projective synchronization, active control, adaptive control, backstepping control, and so on [6–11].

In this paper, the authors have combined the idea of multi switching [12, 13] with dual synchronization [14, 15] and extended it to combination combination [16] anti synchronization of four chaotic systems. The novel scheme, dual combination combination multi switching anti synchronization involves eight chaotic systems. This work is a significant improvement and extension of existing multi switching synchronization schemes. Using Lyapunov stability theory, sufficient conditions have been achieved to realise the desired synchronization. To demonstrate the effectiveness of the proposed method numerical simulations have been performed.

2. Formulation of dual combination combination multi switching synchronization

In this section, we formulate the synchronization scheme involving eight chaotic systems. Let the first two drive systems be described as

$$\dot{x_1} = f_1(x_1) \tag{1}$$

$$\dot{x_2} = f_2(x_2) \tag{2}$$

where $x_1 = (x_{11}, x_{12}, ..., x_{1n})^T$, $x_2 = (x_{21}, x_{22}, ..., x_{2n})^T$, f_1, f_2: $R^n \to R^n$ are known continuous vector functions. Linear combination of the states of two drive systems (1) and (2) gives a resultant signal of the form

$$\begin{aligned} S_1 =& [a_{11}x_{11}, a_{12}x_{12}, ..., a_{1n}x_{1n}, a_{21}x_{21}, a_{22}x_{22}, ..., a_{2n}x_{2n}]^T \\ =& \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax \end{aligned} \tag{3}$$

where $A_1 = diag(a_{11}, a_{12}, ..., a_{1n})$, and $A_2 = diag(a_{21}, a_{22}, ..., a_{2n})$ are two known matrices and a_{1i}, a_{2j} are not all zero at the same time $(i, j = 1, 2, ..., n)$.

Next two drive systems are written as

$$\dot{y_1} = g_1(y_1) \tag{4}$$

$$\dot{y_2} = g_2(y_2) \tag{5}$$

where $y_1 = (y_{11}, y_{12}, ..., y_{1n})^T$, $y_2 = (y_{21}, y_{22}, ..., y_{2n})^T$, g_1, g_2: $R^n \to R^n$ are known continuous vector functions. Hence, the linear combination of the states of two drive systems (4) and (5) gives a resultant signal of the form

$$\begin{aligned} S_2 =& [b_{11}y_{11}, b_{12}y_{12}, ..., b_{1n}y_{1n}, b_{21}y_{21}, b_{22}y_{22}, ..., b_{2n}y_{2n}]^T \\ =& \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = By \end{aligned} \tag{6}$$

where $B_1 = diag(b_{11}, b_{12}, ..., b_{1n})$, and $B_2 = diag(b_{21}, b_{22}, ..., b_{2n})$ are two known matrices and b_{1i}, b_{2j} are not all zero at the same time $(i, j = 1, 2, ..., n)$.

Let the first two response systems be given by

$$\dot{z_1} = h_1(z_1) + u_1 \tag{7}$$

$$\dot{z_2} = h_2(z_2) + u_2 \tag{8}$$

where $z_1 = (z_{11}, z_{12}, ..., z_{1n})^T$, $z_2 = (z_{21}, z_{22}, ..., z_{2n})^T$, h_1, h_2: $R^n \to R^n$ are known continuous vector functions, and $u_1 = (u_{11}, u_{12}, ..., u_{1n})$, $u_2 = (u_{21}, u_{22}, ..., u_{2n})$ are the controllers to be designed. By linear combination of the states of two response systems (7) and (8) a resultant signal is obtained of the form

$$\begin{aligned} S_3 =& [c_{11}z_{11}, c_{12}z_{12}, ..., c_{1n}z_{1n}, c_{21}z_{21}, c_{22}z_{22}, ..., c_{2n}z_{2n}]^T \\ =& \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Cz \end{aligned} \tag{9}$$

where $C_1 = diag(c_{11}, c_{12}, ..., c_{1n})$, and $C_2 = diag(c_{21}, c_{22}, ..., c_{2n})$ are two known matrices and c_{1i}, c_{2j} are not all zero simultaneously $(i, j = 1, 2, ..., n)$.

Let the next two response systems be described as

$$\dot{w_1} = k_1(w_1) + u_3 \tag{10}$$

$$\dot{w_2} = k_2(w_2) + u_4 \tag{11}$$

where $w_1 = (w_{11}, w_{12}, ..., w_{1n})^T$, $w_2 = (w_{21}, w_{22}, ..., w_{2n})^T$, k_1, k_2: $R^n \to R^n$ are known continuous vector functions, and $u_3 = (u_{31}, u_{32}, ..., u_{3n})$, $u_4 = (u_{41}, u_{42}, ..., u_{4n})$ are the controllers to be designed. Linear combination of the states of two response systems (10) and (11) gives a resultant signal of the form

$$\begin{aligned} S_4 =& [d_{11}w_{11}, d_{12}w_{12}, ..., d_{1n}w_{1n}, d_{21}w_{21}, d_{22}w_{22}, ..., d_{2n}w_{2n}]^T \\ =& \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = Dw \end{aligned} \tag{12}$$

where $D_1 = diag(d_{11}, d_{12}, ..., d_{1n})$, and $D_2 = diag(d_{21}, d_{22}, ..., d_{2n})$ are two known matrices and d_{1i}, d_{2j} are not all zero at the same time $(i, j = 1, 2, ..., n)$.

The error signal for dual combination combination synchronization is

$$\begin{aligned} e =& S_1 + S_2 + S_3 + S_4 \\ =& Ax + By + Cz + Dw \\ =& \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ =& \begin{bmatrix} A_1x_1 + B_1y_1 + C_1z_1 + D_1w_1 \\ A_2x_2 + B_2y_2 + C_2z_2 + D_2w_2 \end{bmatrix} \end{aligned} \tag{13}$$

Definition 2.1. If there exist four constant diagonal matrices $A, B, C, D \in R^{2n\times 2n}$ and $C \neq 0$ or $D \neq 0$ such that

$$\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \|Ax + By + Cz + Dw\| = 0, \tag{14}$$

where $\|.\|$ is the vector norm, then the drive systems (1), (2), (4), and (5) realise dual combination combination anti synchronization with the response systems (7), (8), (10), and (11).

Remark 2.2. The diagonal matrices A, B, C, and D are called the scaling matrices and can be extended to functional matrices of state variables x, y, z, and w.

Comment 2.3. From (14) we get that dual combination combination anti synchronization is achieved when

$$\lim_{t\to\infty}\|e\| = \lim_{t\to\infty}\|Ax+By+Cz+Dw\| = 0,$$

which is equivalent to say that

$$\lim_{t\to\infty}\|e_1\| = \lim_{t\to\infty}\|A_1x_1+B_1y_1+C_1z_1+D_1w_1\| = 0$$
$$\lim_{t\to\infty}\|e_2\| = \lim_{t\to\infty}\|A_2x_2+B_2y_2+C_2z_2+D_2w_2\| = 0$$

where $e=(e_1,e_2)^T$. This can be further written as

$$\lim_{t\to\infty} e_{1m} = \lim_{t\to\infty} a_{1m}x_{1m}+b_{1m}y_{1m}+c_{1m}z_{1m}+d_{1m}w_{1m} = 0$$
$$\lim_{t\to\infty} e_{2m} = \lim_{t\to\infty} a_{2m}x_{2m}+b_{2m}y_{2m}+c_{2m}z_{2m}+d_{2m}w_{2m} = 0$$

where $e_1=(e_{11},e_{12},...,e_{1n})$, $e_2=(e_{21},e_{22},...,e_{2n})$, and $m=1,2,...,n$.

Comment 2.4. Let us rewrite the components of e_1, and e_2 as

$$\begin{cases} e_{1m_{(ijlm)}} = a_{1i}x_{1i}+b_{1j}y_{1j}+c_{1l}z_{1l}+d_{1m}w_{1m} \\ e_{2m_{(ijlm)}} = a_{2i}x_{2i}+b_{2j}y_{2j}+c_{2l}z_{2l}+d_{2m}w_{2m} \end{cases} \tag{15}$$

where $i,j,l,m=1,2,...,n$ and the subscript $(ijlm)$ denotes i^{th} component of x_1 and x_2, j^{th} component of y_1 and y_2, l^{th} component of z_1 and z_2, and m^{th} component of w_1 and w_2. In relation to Definition 1, the indices $(ijlm)$ of the error states $e_{1m_{(ijlm)}}$, and $e_{2m_{(ijlm)}}$ are strictly chosen to satisfy $i=j=l=m$ $(i,j,l,m=1,2,...,n)$.

Definition 2.5. If the indices of the error states $e_{1m_{(ijlm)}}$, and $e_{2m_{(ijlm)}}$ are redefined such that $i=j=l\neq m$ or $i=j=m\neq l$ or $i=l=m\neq j$ or $j=l=m\neq i$; or $i=j\neq l=m$ or $i=l\neq j=m$ or $i=m\neq j=l$; or $i=j\neq l\neq m$ or $i=l\neq j\neq m$ or $i=m\neq l\neq j$ or $i\neq j=l\neq m$ or $i\neq j\neq l=m$ or $i\neq l\neq j=m$; or $i\neq j\neq l\neq m$ and

$$\lim_{t\to\infty}\|e\| = \lim_{t\to\infty}\|Ax+By+Cz+Dw\| = 0, \tag{16}$$

where $i,j,l,m = 1,2,...,n$ and $\|.\|$ is the vector norm, then the drive systems (1), (2), (4), and (5) are said to be in dual combination combination multi switching anti synchronization with response systems (7), (8), (10), and (11).

Remark 2.6. If $A_1 = B_1 = C_1 = D_1 = 0$, or $A_2 = B_2 = C_2 = D_2 = 0$, then dual combination combination multi switching anti synchronization changes to multi switching combination combination anti synchronization problem of chaotic systems.

Remark 2.7. If $C_1 = C_2 = 0$, or $D_1 = D_2 = 0$, then dual combination combination multi switching anti synchronization changes to dual combination multi switching anti synchronization of chaotic systems.

Remark 2.8. If $A_1 = B_1 = C_1 = D_1 = 0$, and $C_2 = 0$ or $D_2 = 0$, or $A_2 = B_2 = C_2 = D_2 = 0$, and $C_1 = 0$ or $D_1 = 0$, then dual combination combination multi switching anti synchronization changes to multi switching combination anti synchronization of chaotic systems.

Remark 2.9. Using suitable values for the scaling factors A_1, A_2, B_1, B_2, C_1, C_2, D_1, and D_2, multi switching dual projective anti synchronization and multi switching projective anti synchronization may also be obtained by the proposed scheme.

3. Synchronization Theory

In this section we achieve the dual combination combination multi switching anti synchronization among four chaotic drive systems and four chaotic response systems. Let the control functions be defined as

$$(17)\qquad \begin{cases} U_{1m} = -a_{1i}f_{1i} - b_{1j}g_{1j} - c_{1l}h_{1l} - d_{1m}k_{1m} - e_{1m_{(ijlm)}}, & (i,j,l,m = 1,2,...,n) \\ U_{2m} = -a_{2i}f_{2i} - b_{2j}g_{2j} - c_{2l}h_{2l} - d_{2m}k_{2m} - e_{2m_{(ijlm)}}, & (i,j,l,m = 1,2,...,n) \end{cases}$$

where

$$(18)\qquad \begin{cases} U_{1m} = c_{1l}u_{1l} + d_{1m}u_{3m}, & (l,m = 1,2,...,n) \\ U_{2m} = c_{2l}u_{2l} + d_{2m}u_{4m}, & (l,m = 1,2,...,n) \end{cases}$$

and $f_1 = (f_{11}, f_{12}, ..., f_{1n})^T$, $f_2 = (f_{21}, f_{22}, ..., f_{2n})^T$, $g_1 = (g_{11}, g_{12}, ..., g_{1n})^T$, $g_2 = (g_{21}, g_{22}, ..., g_{2n})^T$, $h_1 = (h_{11}, h_{12}, ..., h_{1n})^T$, $h_2 = (h_{21}, h_{22}, ..., h_{2n})^T$, $k_1 = (k_{11}, k_{12}, ..., k_{1n})^T$, and $k_2 = (k_{21}, k_{22}, ..., k_{2n})^T$.

Theorem 3.1. The drive systems (1), (2), (4), and (5) achieve dual combination combination multi switching anti synchronization with response systems (7), (8), (10), and (11) if the control functions are chosen as given in (17).

Proof Using (13) the error dynamical system can be written as

$$\dot{e} = \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} A_1\dot{x}_1 + B_1\dot{y}_1 + C_1\dot{z}_1 + D_1\dot{w}_1 \\ A_2\dot{x}_2 + B_2\dot{y}_2 + C_2\dot{z}_2 + D_2\dot{w}_2 \end{bmatrix} \tag{19}$$

which can be further written as

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} A_1 f_1 + B_1 g_1 + C_1(h_1 + u_1) + D_1(k_1 + u_3) \\ A_2 f_2 + B_2 g_2 + C_2(h_2 + u_2) + D_2(k_2 + u_4) \end{bmatrix} \tag{20}$$

From this we obtain

$$\begin{cases} \dot{e}_{1m_{(ijlm)}} = a_{1i}f_{1i} + b_{1j}g_{1j} + c_{1l}(h_{1l} + u_{1l}) + d_{1m}(k_{1m} + u_{3m}), \\ \qquad\qquad (i,j,l,m = 1,2,...,n) \\ \dot{e}_{2m_{(ijlm)}} = a_{2i}f_{2i} + b_{2j}g_{2j} + c_{2l}(h_{2l} + u_{2l}) + d_{2m}(k_{2m} + u_{4m}), \\ \qquad\qquad (i,j,l,m = 1,2,...,n) \end{cases} \tag{21}$$

where the indices (ijlm) satisfies one of the generic conditions given in Definition 2.

Let the Lyapunov function be defined as

$$\begin{aligned} V &= \frac{1}{2}e^T e \\ &= \frac{1}{2}\sum_{m=1}^{n}(e_{1m_{(ijlm)}})^2 + \frac{1}{2}\sum_{m=1}^{n}(e_{2m_{(ijlm)}})^2 \end{aligned}$$

The derivative $\dot{V}$ is obtained as

$$\dot{V} = \sum_{m=1}^{n} e_{1m_{(ijlm)}}\dot{e}_{1m_{(ijlm)}} + \sum_{m=2}^{n} e_{2m_{(ijlm)}}\dot{e}_{2m_{(ijlm)}} \tag{22}$$

Using (17) and (21) in the above equation we get

$$\begin{aligned}
\dot{V} &= \sum_{m=1}^{n} e_{1m_{(ijlm)}}[a_{1i}f_{1i} + b_{1j}g_{1j} + c_{1l}h_{1l} + d_{1m}k_{1m} + U_{1m}] \\
&\quad + \sum_{m=1}^{n} e_{2m_{(ijlm)}}[a_{2i}f_{2i} + b_{2j}g_{2j} + c_{2l}h_{2l} + d_{2m}k_{2m} + U_{2m}] \\
&= \sum_{m=1}^{n} e_{1m_{(ijlm)}}(-e_{1m_{(ijlm)}}) + \sum_{m=1}^{n} e_{2m_{(ijlm)}}(-e_{2m_{(ijlm)}}) \quad (Using(17)) \\
&= -e^T e
\end{aligned}$$

Thus we see that $\dot{V}$ is negative definite. Using Lyapunov stability theory, we get $\lim_{t\to\infty}\|e\| = 0$, which gives us $\lim_{t\to\infty}\|e_1\| = 0$ and $\lim_{t\to\infty}\|e_2\| = 0$. This means that the drive systems (1), (2), (4), and (5) achieve dual combination combination multi switching synchronization with response systems (7), (8), (10), and (11).

The following corollaries are easily obtained from Theorem 1 and their proofs are omitted here.

Corollary 3.2. (i) If $a_{1i} = b_{1j} = c_{1l} = d_{1m} = 0,\ i,j,l,m = 1,2,...,n$ then the drive systems (2) and (5) achieve multi switching combination combination anti synchronization with the response systems (8) and (11) provided the control function is chosen as

$$U_{2m} = -a_{2i}f_{2i} - b_{2j}g_{2j} - c_{2l}h_{2l} - d_{2m}k_{2m} - e_{2m_{(ijlm)}}, \quad (i,j,l,m = 1,2,...,n)$$

(ii) If $a_{2i} = b_{2j} = c_{2l} = d_{2m} = 0,\ i,j,l,m = 1,2,...,n$ then the drive systems (1) and (4) achieve multi switching combination combination anti synchronization with the response systems (7) and (10) provided the control function is chosen as

$$U_{1m} = -a_{1i}f_{1i} - b_{1j}g_{1j} - c_{1l}h_{1l} - d_{1m}k_{1m} - e_{1m_{(ijlm)}}, \quad (i,j,l,m = 1,2,...,n)$$

Corollary 3.3. (i) If $c_{1l} = c_{2l} = 0,\ l = 1,2,...,n$ then the drive systems (1), (2), (4) and (5) achieve dual combination multi switching anti synchronization with the response systems (10) and (11) provided the control functions are chosen as

$$u_{3m} = -d_{1m}^{-1}a_{1i}f_{1i} - d_{1m}^{-1}b_{1j}g_{1j} - k_{1m} - d_{1m}^{-1}e_{1m_{(ijlm)}}, \quad (i,j,l,m = 1,2,...,n)$$
$$u_{4m} = -d_{2m}^{-1}a_{2i}f_{2i} - d_{2m}^{-1}b_{2j}g_{2j} - k_{2m} - d_{2m}^{-1}e_{2m_{(ijlm)}}, \quad (i,j,l,m = 1,2,...,n)$$

(ii) If $d_{1m} = d_{2m} = 0$, $m = 1,2,...,n$ then the drive systems (1), (2), (4) and (5) achieve dual combination multi switching anti synchronization with the response systems (7) and (8) provided the control functions are chosen as

$$u_{1l} = -c_{1l}^{-1}a_{1i}f_{1i} - c_{1l}^{-1}b_{1j}g_{1j} - h_{1l} - c_{1l}^{-1}e_{1m_{(ijlm)}}, \quad (i,j,l,m = 1,2,...,n)$$
$$u_{2l} = -c_{2l}^{-1}a_{2i}f_{2i} - c_{2l}^{-1}b_{2j}g_{2j} - h_{2l} - c_{2l}^{-1}e_{2m_{(ijlm)}}, \quad (i,j,l,m = 1,2,...,n)$$

Corollary 3.4. (i) If $a_{1i} = b_{1j} = c_{1l} = d_{1m} = 0$, and $c_{2m} = 0$, $i,j,l,m = 1,2,...,n$ then the drive systems (2) and (5) achieve multi switching combination anti synchronization with the response system (11) provided the control function is chosen as

$$u_{4m} = -d_{2m}^{-1}a_{2i}f_{2i} - d_{2m}^{-1}b_{2j}g_{2j} - k_{2m} - d_{2m}^{-1}e_{2m_{(ijlm)}}, \quad (i,j,l,m = 1,2,...,n)$$

(ii) If $a_{1i} = b_{1j} = c_{1l} = d_{1m} = 0$, and $d_{2m} = 0$, $i,j,l,m = 1,2,...,n$ then the drive systems (2) and (5) achieve multi switching combination anti synchronization with the response system (8) provided the control function is chosen as

$$u_{2l} = -c_{2l}^{-1}a_{2i}f_{2i} - c_{2l}^{-1}b_{2j}g_{2j} - h_{2l} - c_{2l}^{-1}e_{2m_{(ijlm)}}, \quad (i,j,l,m = 1,2,...,n)$$

(iii) If $a_{2i} = b_{2j} = c_{2l} = d_{2m} = 0$, and $c_{1m} = 0$, $i,j,l,m = 1,2,...,n$ then the drive systems (1) and (4) achieve multi switching combination anti synchronization with the response system (10) provided the control function is chosen as

$$u_{3m} = -d_{1m}^{-1}a_{1i}f_{1i} - d_{1m}^{-1}b_{1j}g_{1j} - k_{1m} - d_{1m}^{-1}e_{1m_{(ijlm)}}, \quad (i,j,l,m = 1,2,...,n)$$

(iv) If $a_{2i} = b_{2j} = c_{2l} = d_{2m} = 0$, and $d_{1m} = 0$, $i,j,l,m = 1,2,...,n$ then the drive systems (1) and (4) achieve multi switching combination anti synchronization with the response system (7) provided the control function is chosen as

$$u_{1l} = -c_{1l}^{-1} a_{1i} f_{1i} - c_{1l}^{-1} b_{1j} g_{1j} - h_{1l} - c_{1l}^{-1} e_{1m_{(ijlm)}}, \quad (i,j,l,m = 1,2,...,n)$$

4. Illustration of the synchronization scheme

In this section we realize the dual combination combination multi switching synchronization among eight chaotic systems and perform numerical simulations to show the validity and effectiveness of the proposed scheme. As an example we consider Lorenz system and Chen system to demonstrate the method. let the first two drive systems be given as

$$\begin{cases} \dot{x}_{11} = 10(x_{12} - x_{11}) \\ \dot{x}_{12} = 28x_{11} - x_{11}x_{13} - x_{12} \\ \dot{x}_{13} = x_{11}x_{12} - \frac{8}{3}x_{13} \end{cases} \tag{23}$$

$$\begin{cases} \dot{x}_{21} = 35(x_{22} - x_{21}) \\ \dot{x}_{22} = -x_{21}x_{23} - 7x_{21} + 28x_{22} \\ \dot{x}_{23} = x_{21}x_{22} - 3x_{23} \end{cases} \tag{24}$$

The next two drive systems are considered as

$$\begin{cases} \dot{y}_{11} = 10(y_{12} - y_{11}) \\ \dot{y}_{12} = 28y_{11} - y_{11}y_{13} - y_{12} \\ \dot{y}_{13} = y_{11}y_{12} - \frac{8}{3}y_{13} \end{cases} \tag{25}$$

$$\begin{cases} \dot{y}_{21} = 35(y_{22} - y_{21}) \\ \dot{y}_{22} = -y_{21}y_{23} - 7y_{21} + 28y_{22} \\ \dot{y}_{23} = y_{21}y_{22} - 3y_{23} \end{cases} \tag{26}$$

The first two response system are described as

$$\begin{cases} \dot{z}_{11} = 10(z_{12} - z_{11}) + u_{11} \\ \dot{z}_{12} = 28z_{11} - z_{11}z_{13} - z_{12} + u_{12} \\ \dot{z}_{13} = z_{11}z_{12} - \dfrac{8}{3}z_{13} + u_{13} \end{cases} \tag{27}$$

$$\begin{cases} \dot{z}_{21} = 35(z_{22} - z_{21}) + u_{21} \\ \dot{z}_{22} = -z_{21}z_{23} - 7z_{21} + 28z_{22} + u_{22} \\ \dot{z}_{23} = z_{21}z_{22} - 3z_{23} + u_{23} \end{cases} \tag{28}$$

and the next two response systems are taken as

$$\begin{cases} \dot{w}_{11} = 10(w_{12} - w_{11}) + u_{31} \\ \dot{w}_{12} = 28w_{11} - w_{11}w_{13} - w_{12} + u_{32} \\ \dot{w}_{13} = w_{11}w_{12} - \dfrac{8}{3}w_{13} + u_{33} \end{cases} \tag{29}$$

$$\begin{cases} \dot{w}_{21} = 35(w_{22} - w_{21}) + u_{41} \\ \dot{w}_{22} = -w_{21}w_{23} - 7w_{21} + 28w_{22} + u_{42} \\ \dot{w}_{23} = w_{21}w_{22} - 3w_{23} + u_{43} \end{cases} \tag{30}$$

By the conditions on indices $i, j, l, m = 1, 2, 3$ stated in Definition 2, several multi switching combination exist for defining the error $e = (e_1, e_2)^T$. We will present results for one randomly selected error space vector combination formed out of several possibilities. Let us define $e_1 =$

$(e11_{(1231)}, e12_{(3122)}, e13_{(2313)})$, and $e_2 = (e21_{(3121)}, e22_{(1312)}, e23_{(2233)})$ where

$$\begin{aligned}
e11_{(1231)} =& a_{11}x_{11} + b_{12}y_{12} - c_{13}z_{13} - d_{11}w_{11} \\
e12_{(3122)} =& a_{13}x_{13} + b_{11}y_{11} - c_{12}z_{12} - d_{12}w_{12} \\
e13_{(2313)} =& a_{12}x_{12} + b_{13}y_{13} - c_{11}z_{11} - d_{13}w_{13} \\
e21_{(3121)} =& a_{23}x_{23} + b_{21}y_{21} - c_{22}z_{22} - d_{21}w_{21} \\
e22_{(1312)} =& a_{21}x_{21} + b_{23}y_{23} - c_{21}z_{21} - d_{22}w_{22} \\
e23_{(2233)} =& a_{22}x_{22} + b_{22}y_{22} - c_{23}z_{23} - d_{23}w_{23}
\end{aligned}$$

under the assumption that $A_1 = diag(a_{11}, a_{12}, a_{13})$, $A_2 = diag(a_{21}, a_{22}, a_{23})$, $B_1 = diag(b_{11}, b_{12}, b_{13})$, $B_2 = diag(b_{21}, b_{22}, b_{23})$, $C_1 = diag(c_{11}, c_{12}, c_{13})$, $C_2 = diag(c_{21}, c_{22}, c_{23})$, $D_1 = diag(d_{11}, d_{12}, d_{13})$, and $D_2 = diag(d_{21}, d_{22}, d_{23})$. Assuming $A_1 = A_2 = B_1 = B_2 = C_1 = C_2 = D_1 = D_2 = I$, the controllers are chosen as

$$(31) \qquad \begin{cases}
U_{11} = -10(x_{12} - x_{11}) - (28y_{11} - y_{11}y_{13} - y_{12}) - (z_{11}z_{12} - \frac{8}{3}z_{13}) \\
\qquad - 10(w_{12} - w_{11}) - e11_{(1231)} \\
U_{12} = -(x_{11}x_{12} - \frac{8}{3}x_{13}) - 10(y_{12} - y_{11}) \\
\qquad - (28z_{11} - z_{11}z_{13} - z_{12}) - (28w_{11} - w_{11}w_{13} - w_{12}) - e12_{(3122)} \\
U_{13} = -(28x_{11} - x_{11}x_{13} - x_{12}) - (y_{11}y_{12} - \frac{8}{3}y_{13}) - 10(z_{12} - z_{11}) \\
\qquad - (w_{11}w_{12} - \frac{8}{3}w_{13}) - e13_{(2313)}
\end{cases}$$

$$(32) \qquad \begin{cases}
U_{21} = -(x_{21}x_{22} - 3x_{23}) - 35(y_{22} - y_{21}) - (-z_{21}z_{23} - 7z_{21} + 28z_{22}) \\
\qquad - 35(w_{22} - w_{21}) - e21_{(3121)} \\
U_{22} = -35(x_{22} - x_{21}) - (y_{21}y_{22} - 3y_{23}) - 35(z_{22} - z_{21}) \\
\qquad - (-w_{21}w_{23} - 7w_{21} + 28w_{22}) - e22_{(1312)} \\
U_{23} = -(-x_{21}x_{23} - 7x_{21} + 28x_{22}) - (-y_{21}y_{23} - 7y_{21} + 28y_{22}) \\
\qquad - (z_{21}z_{22} - 3z_{23}) - (w_{21}w_{22} - 3w_{23}) - e23_{(2233)}
\end{cases}$$

where $U_{11} = u_{13} + u_{31}$, $U_{12} = u_{12} + u_{32}$, $U_{13} = u_{11} + u_{33}$, $U_{21} = u_{22} + u_{41}$, $U_{22} = u_{21} + u_{42}$, and $U_{23} = u_{23} + u_{43}$.

These controllers (31) and (32) are designed in accordance with Theorem 1 in order to realise the desired synchronization. In the numerical simulations process the initial conditions of the drive and response systems are chosen as $(x_{11}, x_{12}, x_{13}) = (1.1, 0, -0.1)$, $(x_{21}, x_{22}, x_{23}) = (1.5, 0.9, 0.1)$, $(y_{11}, y_{12}, y_{13}) = (0, -0.1, 2)$, $(y_{21}, y_{22}, y_{23}) = (-1, 1, 1.5)$, $(z_{11}, z_{12}, z_{13}) = (0.5, 1.2, 5)$, $(z_{21}, z_{22}, z_{23}) = (0.7, -3, 1.1)$, $(w_{11}, w_{12}, w_{13}) = (0, -1, 0)$, and $(w_{21}, w_{22}, w_{23}) = (-2, -0.5, 1)$. Figures $(1)-(6)$ illustrates the time response of synchronized states. We can see that the desired dual combination combination multi switching anti synchronization is achieved with the controllers we designed.

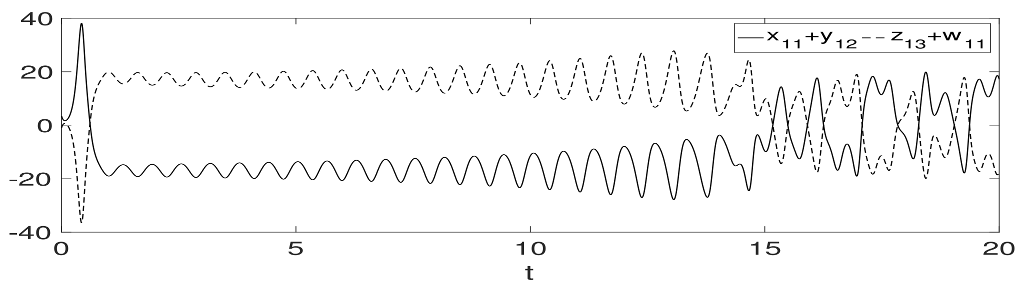

FIGURE 1. Response for states $x_{11} + y_{12}$ and $z_{13} + w_{11}$ for drive systems (23), (25) and response systems (27), (29).

5. Conclusion

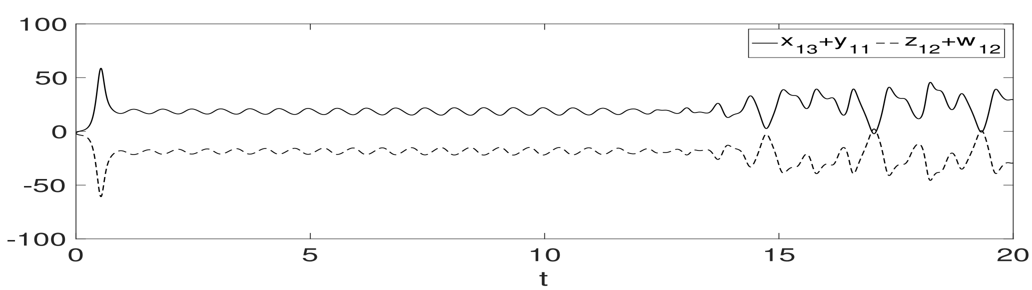

FIGURE 2. Response for states $x_{13} + y_{11}$ and $z_{12} + w_{12}$ for drive systems (23), (25) and response systems (27), (29).

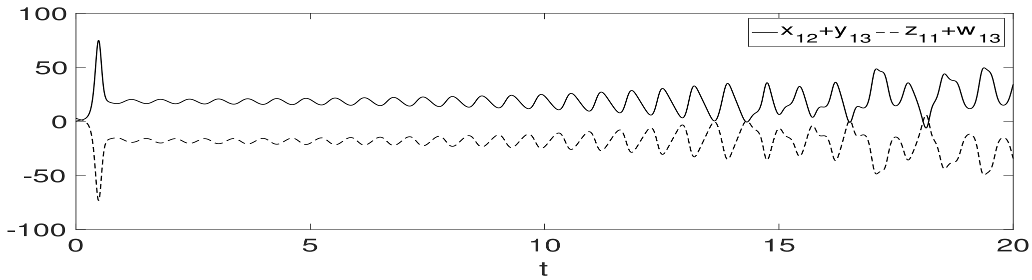

FIGURE 3. Response for states $x_{12}+y_{13}$ and $z_{11}+w_{13}$ for drive systems (23), (25) and response systems (27), (29).

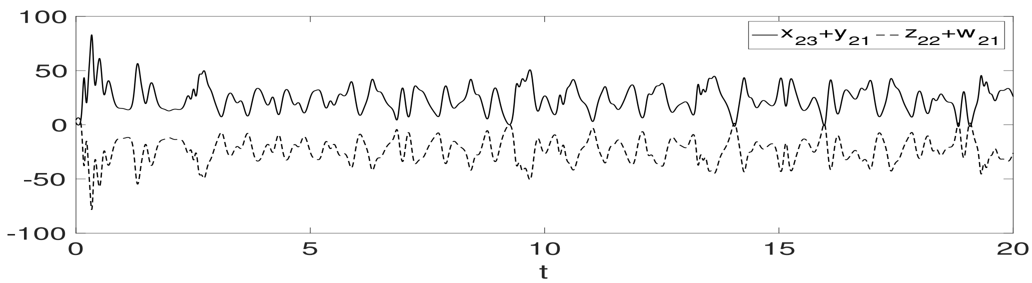

FIGURE 4. Response for states $x_{23}+y_{21}$ and $z_{22}+w_{21}$ for drive systems (24), (26) and response systems (28), (30).

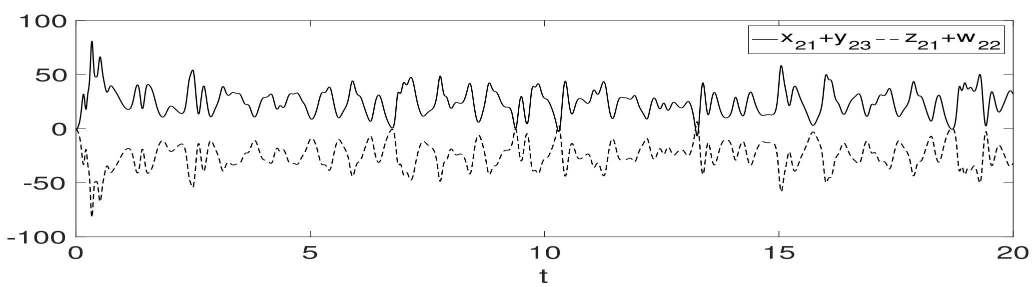

FIGURE 5. Response for states $x_{21}+y_{23}$ and $z_{21}+w_{22}$ for drive systems (24), (26) and response systems (28), (30).

In this paper a novel scheme for synchronization involving eight chaotic systems has been proposed. The proposed scheme dual combination combination multi switching anti synchronization achieves synchronization between four chaotic drive systems and four chaotic response systems in a multi switching manner. The complexity of signal achieved by multiple combination increases the security of transmitted signal, as the dynamic behaviour of resultant signal

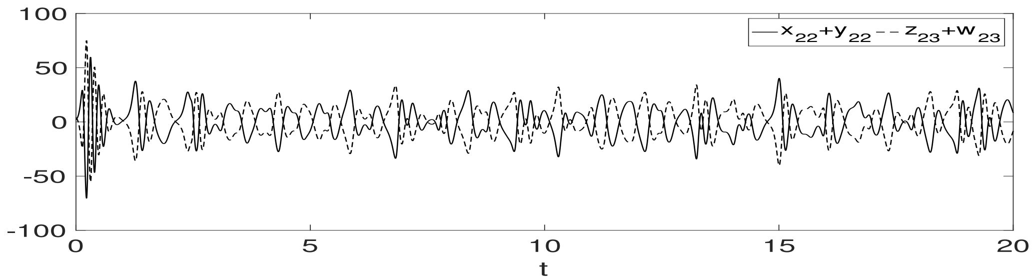

FIGURE 6. Response for states $x_{22} + y_{22}$ and $z_{23} + w_{23}$ for drive systems (24), (26) and response systems (28), (30).

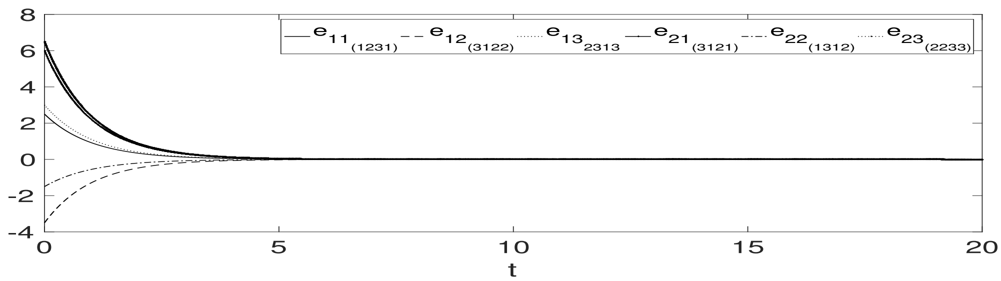

FIGURE 7. Time response of synchronization errors

is so complex that it becomes very difficult, for the intruder, to separate the information signal from the transmitted signal. Thus this scheme may provide improved performance and better resistance in the context of secure communication applications. The concept of multi switching in this scheme further strengthens the anti attack ability of the transmitted signals from drive systems because, for an intruder, determining the correct combination for error space vector is extremely difficult due to large number of possible synchronization directions. Using Lyapunov stability theory, sufficient conditions are obtained for achieving dual combination combination multi switching synchronization. Numerical simulations has been demonstrated using four Lorenz systems and four Chen systems to show the effectiveness and validity of the method.

Conflict of Interests

The authors declare that there is no conflict of interests.

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