OPTIMALITY CONDITIONS OF SECOND-ORDER RADIAL EPIDERIVATIVES

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Abstract. In this paper, we introduce the concepts of second-order radial epiderivative and second-order generalized radial epiderivative for nonconvex set-valued maps. We give existence theorems for the second-order generalized radial epiderivatives. We also establish the second-order optimality conditions by using second-order radial epiderivatives.

Keywords: second-order radial set, second-order radial epiderivative, second-order optimality condition

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1. Introduction

In the last years, the second-order optimality conditions have a great deal of attention in scalar and vector-optimization problems and have been widely investigated [2,3,4,5,8,9,10,11,12,13,14,15,16,17,19,22,24,26]. It can be seen that a second-order contingent set, introduced by Aubin and Frankowska [1], and a second-order asymptotic contingent cone, introduced by Penot [24], play a important role in establishing second-order optimality conditions. Jahn et al. proposed the second-order contingent derivative and the second-order contingent epiderivative in terms of the

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second-order contingent set [15], introduced by Aubin and Frankowska [1]. They obtained the second-order optimality conditions by using these derivatives in set-valued optimization. In [22], Khan and Tammer gave new second-order optimality conditions in set-valued optimization. They presented an extension of the well-known Dubovitski-Milutin approach to set-valued optimization. In [3], Anh and Khanh introduced the higher-order radial sets and corresponding derivatives. They established both necessary and sufficient higher-order conditions for weak efficiency in set-valued vector optimization problem. In [4], Anh and Khanh gave both necessary and sufficient higher-order conditions for various kinds of proper solutions to nonsmooth vector optimization problem in terms of higher-order radial sets and radial derivatives. In [18], İncoğlu introduce the concepts of second-order radial epiderivative and second-order generalized radial epiderivative for nonconvex set-valued maps. They also investigate in [18] some of their properties and give existence theorems for the second-order generalized radial epiderivatives.

Motivated by the work above, we study the second-order radial epiderivatives and the second-order generalized radial epiderivative. We also propose second-order optimality conditions by using second-order radial epiderivatives. This paper is divided into four sections. In Section 2, we recall some basic concepts. In Section 3, we introduce the second-order radial epiderivative and the second-order generalized radial epiderivative and give the existence theorems and some of their basic properties. In Section 4, we establish the second-optimality conditions for weak minimizers.

2. Preliminaries

Throughout this paper, let \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be real normed spaces and let \(Y\) be partially ordered by a closed convex pointed cone \(C \subset Y\). Let \(F : X \to 2^Y\) be a set-valued map, let \((\bar{x}, \bar{y}) \in \text{graph}(F)\), let \((\bar{u}, \bar{v}) \in X \times Y\).

We recall the concept of the radial epiderivative and the generalized radial epiderivative introduced by Kasimbeyli [20], and Kasimbeyli and İncoğlu [21], respectively, together with some standard notions.
Definition 2.1. Let $U$ be a nonempty subset of a real normed space $(Z, \|\|_Z)$, and let $\bar{z} \in \text{cl} (U)$ (closure of $U$) be a given element. The closed radial cone $R(U, \bar{z})$ of $U$ at $\bar{z} \in \text{cl} (U)$ is the set of all $z \in Z$ such that there are $\lambda_n > 0$ and a sequence $(z_n)_{n \in \mathbb{N}} \subset Z$ with $\lim_{n \to \infty} z_n = z$ so that $\bar{z} + \lambda_n z_n \in U$, for all $n \in \mathbb{N}$ [6], [20,21], [25].

It follows from this definitions that $R(U, \bar{z}) = \text{cl} (\text{cone} (U - \bar{z}))$, where cone denotes the conic hull of a set, which is the smallest cone containing $U - \bar{z}$ [6], [7], [20,21].

Definition 2.2. Let $(X, \|\|_X)$ and $(Y, \|\|_Y)$ be real normed spaces, let $F : X \to 2^Y$ be a set-valued map.

(i) The set

$$\text{graph}(F) = \{(x, y) \in X \times Y \mid y \in F(x)\}$$

is called the graph of $F$;

(ii) The set

$$\text{dom}(F) = \{x \in X \mid F(x) \neq \emptyset\}$$

is called the domain of $F$;

(iii) Let $Y$ be partially ordered by a proper, convex, and pointed cone $C \subset Y$. The set

$$\text{epi}(F) = \{(x, y) \in X \times Y \mid y \in F(x) + C\}$$

is called the epigraph of $F$,

(iv) Let $C \subset Y$ a proper, convex and pointed cone. The profile map $P_F : X \to 2^Y$ is defined by

$$P_F(x) = F(x) + C,$$

for every $x \in \text{dom}(F)$.

(v) Let $(\bar{x}, \bar{y}) \in \text{graph}(F)$. A set valued map $D_R F (\bar{x}, \bar{y}) : X \to 2^Y$ whose graph coincides with the contingent cone to graph of $F$ at $(\bar{x}, \bar{y})$, that is

$$\text{graph}(D_R F (\bar{x}, \bar{y})) = R(\text{graph}(F), (\bar{x}, \bar{y})),$$

is called radial derivative of $F$ at $(\bar{x}, \bar{y})$, [6],[25].
Now, we give the definition of the radial epiderivative given by Kasımbeyli without convexity and boundedness [20].

**Definition 2.3.** Let \( Y \) be partially ordered by a convex cone \( C \subset Y \), let \( S \) be a nonempty subset of \( X \) and let \( F : S \to 2^Y \) be a set-valued map. Let a pair \( (\bar{x}, \bar{y}) \in \text{graph}(F) \) be given. A single-valued map \( D_r F (\bar{x}, \bar{y}) : X \to Y \) whose epigraph equals the radial cone to the epigraph of \( F \) at \( (\bar{x}, \bar{y}) \), i.e.

\[
\text{epi} (D_r F (\bar{x}, \bar{y})) = R (\text{epi} (F), (\bar{x}, \bar{y})),
\]

is called radial epiderivative of \( F \) at \( (\bar{x}, \bar{y}) \).

To give the definition of the generalized radial epiderivative, we recall the minimality concept [23].

**Definition 2.4.** Let \((Y, \|\cdot\|_Y)\) be a real normed space partially ordered by a convex cone \( C \subset Y \). Let \( D \) be a subset of \( Y \) and let \( \bar{y} \in D \).

(i) The element \( \bar{y} \) is said to be a minimal element of \( D \), if \( D \cap (\{ \bar{y} \} - C) = \{ \bar{y} \} \).

(ii) Let the ordering cone have a nonempty interior \( \text{int} (C) \). The element \( \bar{y} \) is said to be a weakly minimal element of \( D \), if \( D \cap (\{ \bar{y} \} - \text{int} (C)) = \emptyset \). The set of all minimal, weakly minimal elements of \( D \) with respect to the ordering cone \( C \) is denoted by \( \text{Min} D \), \( W - \text{Min} D \), respectively.

Now, we recall the generalized radial epiderivative for set-valued maps given by Kasımbeyli and İnceoğlu in [21].

**Definition 2.5.** A set valued map \( D_{gr} F (\bar{x}, \bar{y}) : X \to 2^Y \) is called the generalized radial epiderivative of \( F \) at \( (\bar{x}, \bar{y}) \) if

\[
D_{gr} F (\bar{x}, \bar{y}) (x) = \text{Min} (G (x), C),
\]

where \( G : X \to 2^Y \) is the set-valued map given by

\[
G (x) = \{ y \in Y \mid (x, y) \in R (\text{epi} (F), (\bar{x}, \bar{y})) \}, \forall x \in X.
\]

### 3. Second-Order Radial Set and Second-Order Radial Epiderivatives
In this section, we propose the definitions of the second-order radial epiderivatives. By using these definitions, we prove existence theorem and give some of their properties and optimality conditions.

Anh and Khanh defined $m$-th-order radial set and $m$-th-order radial derivative \cite{4}. Based on this, we give the following definitions of second-order radial set and second-order radial derivative.

**Definition 3.1.** Let $(X, \| \cdot \|_X)$ be a real normed space, let $S$ be a nonempty subset of $X$, let $\bar{x} \in \text{cl} (S)$ and let $w \in X$ The second-order radial set of $S$ at $\bar{x}$ with respect to $w$ is

$$R^2 (S, \bar{x}, w) = \{ x \in X \mid \exists t_n > 0, \exists x_n \to x, \forall n, \bar{x} + t_n w + t_n^2 x_n \in S \} .$$

It is also clear that $R^2 (S, \bar{x}, 0_X) = R (S, \bar{x}), 0_X$ the zero element of $X$.

The following definition was presented by Ha in \cite{13}.

**Definition 3.2.** Let $F : X \to 2^Y$ be a set-valued map, let $(\bar{x}, \bar{y}) \in \text{graph} (F)$, let $(\bar{u}, \bar{v}) \in X \times Y$. The second-order radial derivative of $F$ at $(\bar{x}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$ is the set-valued map $D^2_{R}F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \to 2^Y$ whose graph is

$$(1) \quad \text{graph} (D^2_{R}F (\bar{x}, \bar{y}, \bar{u}, \bar{v})) = R^2 (\text{graph} (F), (\bar{x}, \bar{y}), (\bar{u}, \bar{v})).$$

The relation (1) can be expressed equivalently by

$$D^2_{R}F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) (x) = \left\{ y \in Y \mid \exists t_n > 0, \exists x_n \to x, \exists y_n \to y, \forall n, \right.$$ 

$$\left. \bar{y} + t_n \bar{v} + t_n^2 y_n \in F (\bar{x} + t_n \bar{u} + t_n^2 x_n) \right\} .$$

The following definition is a generalization given by Kasimbeyli and Kasimbeyli and İnceoğlu, respectively \cite{20},\cite{21}.

**Definition 3.3.** \cite{18} Let $F : X \to 2^Y$ be a set-valued map, let $(\bar{x}, \bar{y}) \in \text{graph} (F)$, let $(\bar{u}, \bar{v}) \in X \times Y$.

(i) A single-valued map $D^2_r F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \to Y$ whose epigraph equals the second-order radial set to the epigraph of $F$ at $(\bar{x}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$, i.e.,

$$\text{epi} (D^2_r F (\bar{x}, \bar{y}, \bar{u}, \bar{v})) = R^2 (\text{epi} (F), (\bar{x}, \bar{y}), (\bar{u}, \bar{v})) ,$$

is called the second-order radial epiderivative.
(ii) A set-valued map \( D^2_{gr}F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \to 2^Y \) is called the second-order generalized radial epiderivative of \( F \) at \((\bar{x}, \bar{y})\) with respect to \((\bar{u}, \bar{v})\) if

\[
D^2_{gr}F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \text{Min} \left( G^2(x), C \right), \quad x \in \text{dom} \left( G^2(x) \right),
\]

where \( G^2 : X \to 2^Y \) is a set-valued map defined by

\[
G^2(x) = \left\{ y \in Y \mid (x, y) \in R^2(\text{epi}(F), (\bar{x}, \bar{y}), (\bar{u}, \bar{v})) \right\}.
\]

**Example 3.1.** Let \( F : \mathbb{R} \to 2^\mathbb{R} \) be a set-valued map given by

\[
F(x) = \{ y \in \mathbb{R} \mid y \geq x \}, \text{ for all } x \in \mathbb{R}.
\]

Let \((\bar{x}, \bar{y}) = (0, 0)\) and let \((\bar{u}, \bar{v}) = (1, 0)\). Then

\[
R^2(\text{epi}(F), (0, 0), (1, 0)) = \left\{ cz \in \mathbb{R}^2 \mid \exists t_n > 0, \exists (z_n) \to z, \text{ for all } n, t_n (1, 0) + t_n^2 z_n \in \text{epi}F \right\}.
\]

The condition

\[
t_n (1, 0) + t_n^2 z_n \in \text{epi}(F)
\]

is equivalent to

\[
t_n^2 z_{n_2} \geq t_n + t_n^2 z_{n_1};
\]

hence,

\[
z_{n_2} \geq (1 + t_n z_{n_1})^2
\]

Since \( t_n > 0 \) and \( z_{n_2} \to z_2, z_{n_1} \to z_1 \), we obtain that

\[
R^2(\text{epi}(F), (0, 0), (1, 0)) = \mathbb{R} \times [1, 0)
\]

Consequently, we have

\[
G^2(x) = [1, 0),
\]

for every \( x \in \mathbb{R} \). On the other hand,

\[
D^2_F(0, 0, 1, 0)(x) = \{ 1 \}, \text{ for every } x \in \mathbb{R}
\]

and

\[
D^2_{gr}F(0, 0, 1, 0)(x) = \text{Min} \left( G^2(x), \mathbb{R}_+ \right) = \{ 1 \},
\]

for every \( x \in \mathbb{R} \).
**Proposition 3.1.** For every \( x \in \text{dom} \left( D_R^2 F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) \right) \), the following inclusion holds:

\[
D_R^2 F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) (x) + C_Y \subseteq D_R^2 P_F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) (x).
\]

**Corollary 3.1.** For every \( x \in \text{dom} \left( D_R^2 P_F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) \right) \), the following inclusion holds:

\[
D_R^2 P_F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) (x) + C_Y = D_R^2 P_F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) (x).
\]

The following existence theorem for second-order generalized radial epiderivative is proved in [18].

**Theorem 3.1.** Let the convex cone \( C \subset Y \) be regular. For every \( x \in \text{dom} \left( G^2 \right) \), let the set \( D^2_{gr} F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) (x) \) have a \( C \)-lower bound. Then for every \( x \in \text{dom} \left( G^2 \right) \), \( D^2_{gr} F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) (x) \) exists. Moreover, the following equality holds:

\[
epi (D^2_{gr} F (\bar{x}, \bar{y}, \bar{u}, \bar{v})) = R^2 (\text{epi} (F), (\bar{x}, \bar{y}), (\bar{u}, \bar{v})).
\]

**Proposition 3.2.** Let the convex cone \( C \subset Y \) be regular. Let \( F : X \to 2^Y \) be a set-valued map, let \((\bar{x}, \bar{y}) \in \text{graph} (F)\), let \((\bar{u}, \bar{v}) \in X \times Y\). For every \( x \in \text{dom} \left( G^2 (x) \right) \), let the set \( G^2 (x) \) have a \( C \)-lower bound. The following assertion is satisfied:

\[
epi (D^2_{gr} F (\bar{x}, \bar{y}, \bar{u}, \bar{v})) \subset R^2 (\text{dom} (F), \bar{x}, \bar{u}) \times Y.
\]

**Proof.** Let \((\bar{x}, \bar{y}) \in \text{epi} (D^2_{gr} (\bar{x}, \bar{y}, \bar{u}, \bar{v}))\). Then \((\bar{x}, \bar{y}) \in R^2 (\text{epi} (f), (\bar{x}, \bar{y}), (\bar{u}, \bar{v}))\) It follows from the definition of the second-order generalized radial epiderivative that there exist sequences \( t_n > 0 \) and \((x_n, y_n)\) with \((x_n, y_n) \to (x, y)\) such that

\[
(\bar{x}, \bar{y}) + t_n (\bar{u}, \bar{v}) + t_n^2 (x_n, y_n) \in \text{epi} (F), \text{ for all } n \in \mathbb{N},
\]

\[
\bar{y} + t_n \bar{v} + t_n^2 \in F (\bar{x} + t_n \bar{u} + t_n^2 x_n) + C, \text{ for all } n \in \mathbb{N}.
\]

Therefore we have \( \bar{x} + t_n \bar{u} + t_n^2 x_n \in \text{dom} (F) \). This implies that \((x, y) \in R^2 (\text{dom} (F), \bar{x}, \bar{u}) \times Y \).

**Proposition 3.3.** Let \( A \subset X \) be nonempty set and let \( C \subset Y \) be a convex cone with \( \text{int} (C) \neq \emptyset \).

Let \( F : A \to 2^Y \) be a set-valued map, let \( E = \text{dom} \left( D^2_{gr} F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) \right) \). Then

\[
\bigcup_{x \in E} D^2_{gr} F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) \subset R^2 (F (A) + C, \bar{y}, \bar{v})
\]
**Proof.** Let \( y \in D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(E) \) and let \( x \in E \) be the corresponding element such that \( y \in D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \). Then, \((x, y) \in R^2(epi(F), (\bar{x}, \bar{y}), (\bar{u}, \bar{v})) \). There exist \( t_n > 0 \), \((x_n, y_n) \to (x, y) \) such that, for all \( n \in \mathbb{N} \),

\[
\bar{y} + t_n \bar{v} + t_n^2 y_n \in F(\bar{x} + t_n \bar{u} + t_n^2 x_n) + C \subset F(A) + C
\]

Since \( \lambda_n > 0 \) and \( y_n \to y \), we get \( y \in R^2(F(A) + C, \bar{y}, \bar{v}) \). Because \( y \) is chosen arbitrarily, we have \( D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(E) \subset R^2(F(A) + C, \bar{y}, \bar{v}) \).

The following proposition shows that relationship between second-order radial epiderivative and second-order generalized radial epiderivative.

**Proposition 3.4.** [18] Assume that the second-order radial epiderivative \( D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) \) of \( F : X \to 2^Y \) at \((\bar{x}, \bar{y})\) \( \in \text{graph}(F) \) with respect to \((\bar{u}, \bar{v}) \in X \times Y \) exist. Then

\[
D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \text{Min}(D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}), C_Y),
\]

for all \( x \in \text{dom}(D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})) \).

**Proof.** It follows from the Definition 3.3 that \( D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) \)

\[
epi(D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})) = R^2(epi(F), (\bar{x}, \bar{y}), (\bar{u}, \bar{v})) = \text{graph}(D_{gr}^2P_F(\bar{x}, \bar{y}, \bar{u}, \bar{v})).
\]

Hence,

\[
\{D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)\} + C_Y = D_{gr}^2P_F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x),
\]

for every \( x \in \text{dom}(D_{gr}^2P_F(\bar{x}, \bar{y}, \bar{u}, \bar{v})) \). In view of the Definition ?? and the () equality, the second-order generalized radial epiderivative \( D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \to 2^Y \) is given by

\[
D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \text{Min}(D_{gr}^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}), C_Y).
\]

### 4. Optimality Conditions

Now, we obtain the optimality conditions for set-valued maps in terms of second-order radial epiderivatives. Let \( F : S \to 2^Y \) be a set-valued map.
Consider the following set-valued optimization problem:

\[ (P) \begin{cases} \min F(x) \\ \text{s.t. } x \in S \end{cases} \]

**Definition 4.1.** Let the ordering cone \( C \) have a nonempty interior \( \text{int} (C) \). A pair \((\bar{x}, \bar{y}) \in \text{graph}(F)\) is called weak minimizer of (), if \( \bar{y} \) is a weakly minimal element of the set \( F(S) \) where

\[ F(S) = \bigcup_{x \in S} F(x). \]

Here we present a second-order optimality condition by using the second-order radial derivative.

**Theorem 4.1.** Let \((\bar{x}, \bar{y}) \in \text{graph}(F)\) be a weak minimizer of the problem (P) and let \( \bar{u} \in \text{dom}(DP_F(\bar{x}, \bar{y})) \) be arbitrary. Then, for every \( \bar{v} \in D_{RF}F(\bar{x}, \bar{y}) \cap (-\partial C), \)

for every \( x \in \text{dom}(D_2R F(\bar{x}, \bar{y}, \bar{u}, \bar{v})) \),

\[ D^2_{\bar{y}F}(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \notin (-\text{int}(C) - \{\bar{v}\}). \]

**Proof.** Let \((\bar{x}, \bar{y}) \in \text{graph}(F)\) and let \( \bar{y} \in W - \text{Min}(F(S), C) \). Assume to the contrary that there exist an element \( x \in \text{dom}(D_2R F(\bar{x}, \bar{y}, \bar{u}, \bar{v})) \) with

\[ y \in D^2_{RF}(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \cap (-\text{int}(C) - \{\bar{v}\}). \]

By the definition of the second-order radial epiderivative

\[ (x, y) \in R^2(epi(F), (\bar{x}, \bar{y}), (\bar{u}, \bar{v})). \]

Then \( \exists t_n > 0, \exists (x_n, y_n) \subset epi(F), \) with

\[ (x_n, y_n) \to (x, y) \supseteq \forall n, (\bar{x}, \bar{y}) + t_n(\bar{u}, \bar{v}) + t^2_n(x_n, y_n) \in epi(F). \]

By the definition of \( epi(F) \), we get

\[ (2) \quad \bar{y} + t_n\bar{v} + t^2_n y_n \in F(\bar{x} + t_n\bar{u} + t^2_n x_n) + C. \]

Since

\[ y + \bar{v} \in (-\text{int}(C)), y_n \to y, \]
there exist \( n_0 \in \mathbb{N} \) such that
\[
\bar{v} + t_n^2 y_n \in (-\text{int} (C)), \text{ for every } n \geq n_0.
\]

From \( t_n > 0 \), we get
\[
(3) \quad t_n \bar{v} + t_n^2 y_n \in (-\text{int} (C)), \text{ for every } n \geq n_0.
\]

By using the above equality (2), we have
\[
(4) \quad \bar{y} + t_n \bar{v} + t_n^2 y_n \in (\bar{y} - \text{int} (C)), \text{ for every } n \geq n_0.
\]

We set
\[
\alpha_n = \bar{x} + t_n \bar{u} + t_n^2 x_n, \\
\beta_n = \bar{y} + t_n \bar{v} + t_n^2 y_n.
\]

Because of the equalities (5) and (2), we have
\[
\beta_n \in F (\alpha_n) + C.
\]

Therefore, there exists some \( \vartheta_n \in F (\alpha_n) + C \) with \( \beta_n \in \vartheta_n + C \). From here
\[
\vartheta_n \in \beta_n - C.
\]

Because of the inclusion \( \text{int} (C) + C \subset \text{int} (C) \) and the equality \( \beta_n \in (\bar{y} - \text{int} (C)) \), we have
\[
\vartheta_n \in (\bar{y} - \text{int} (C)), \text{ for every } n \geq n_0.
\]

Therefore, we have shown that
\[
F (\alpha_n) \cap (\bar{y} - \text{int} (C)) \neq \emptyset,
\]

which is a contradiction to the assumption that \((\bar{x}, \bar{y})\) is a weak minimizer.

Now we propose some important properties of the second-order radial epiderivative.

**Lemma 4.1.** \( F : S \to 2^Y \) be a set-valued map and \((\bar{x}, \bar{y}) \in \text{graph} (F), \bar{u} \in S \) and \( \bar{v} \in F (\bar{u}) + C \). If the second-order radial epiderivative \( D^2 r F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) (x) \) exists, then
\[
F (x) - \{ \bar{y} \} + C \subset D^2 r F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) (x - \bar{x} - \bar{u}), \text{ for all } x \in S.
\]
Proof. Let \( x \in S, y \in F(x) - \{\bar{y}\} + C \). Then \( y + \bar{y} \in F(x) + C \). By setting \( t_n = 1, x_n = x - \bar{x} - \bar{u}, y_n = y - \bar{v}, \) for all \( n \in \mathbb{N} \), we have \( \exists t_n > 0, \exists (x_n, y_n) \rightarrow (x - \bar{x} - \bar{u}, y - \bar{v}) \ni y + t_n \bar{v} + t_n^2 y_n \in F(\bar{x} + t_n \bar{u} + t_n^2 x_n) + C \), for all \( n \in \mathbb{N} \). Consequently, we get \( y + \bar{y} \in D^2_F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) (x - \bar{x} - \bar{u}) \).

The following sufficient optimality condition for the weak minimizer will be proved by using the Lemma 4.1.

**Theorem 4.2.** Let the set-valued optimization problem (P) be given, let \((\bar{x}, \bar{y}) \in \text{graph}(F)\). If for every \( \bar{u} \in X \) with \( \bar{v} \in D_F (\bar{x}, \bar{y}) (\bar{u}) \cap (-\partial C) \)

\[
D^2_F (\bar{x}, \bar{y}, \bar{u}, \bar{v}) (x - \bar{x} - \bar{u}) \cap (-\text{int}(C)) = \emptyset,
\]

for every \( x \in S \), then \((\bar{x}, \bar{y})\) is a weak minimizer of the problem (P).

**Proof.** By the Lemma 4.1

\[
(F(x) - \{\bar{y}\} + C) \cap (-\text{int}(C)) = \emptyset,
\]

for every \( x \in S \). This implies that

\[
(F(x) - \{\bar{y}\}) \cap (-\text{int}(C) - C) = \emptyset,
\]

for every \( x \in S \). We obtain with the equality \text{int}(C) + C = \text{int}(C)

\[
(F(x) - \{\bar{y}\}) \cap (-\text{int}(C)) = \emptyset,
\]

for every \( x \in S \). Consequently, \( \bar{y} \) is a weakly minimal element of the set \( F(S) \); that is \((\bar{x}, \bar{y})\) is a weak minimizer of the problem (P).

**5. Conclusion**

In this paper two new concept of second-order epiderivative are presented. The relationship between the second-order radial epiderivative and the second-order generalized radial epiderivative are discussed. Some of their properties are investigated also. In set-valued optimization, second-order optimality conditions are obtained by using these epiderivatives.

**Conflict of Interests**

The author declare that there is no conflict of interests.
REFERENCES


