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COMMON FIXED POINT FOR TWO MAPPING IN N-CONE METRIC SPACES

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Abstract. In this paper, we obtain some new coincidence and common fixed point theorems for two nonlinear contractive mappings in the N-cone metric space without the assumption of normal cone. Our main results improve and generalize the corresponding results of Fan et al (*J. Math. Comput. Sci.* 5 (2015), No. 6, 811-821).

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1. Introduction and Preliminaries

In 2007, Huang and Zhang [6] have replaced the real numbers by ordering Banach space and defining cone metric space. They have proved some fixed point theorems of contractive mappings on cone metric spaces. The study of fixed point theorems in such spaces is followed by some other mathematicians; see [2], [3], [7], [8], [10], [11], [12] . In 2010, Aage and Salunke [1] introduced a generalized D-metric space which generalized cone metric space.

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Very recently, Malviya and Fisher [9] introduced the notion of N-cone metric space and proved fixed point theorems for asymptotically regular maps. This new notion generalized the notion of G-cone metric space and generalized D-metric space. In 2015, Fernandez, Modi and Malviya [5] proved unique fixed point theorems for contractive maps in N-cone metric spaces.

In this paper, we obtain some new coincidence and common fixed point theorems for two nonlinear contractive mappings in the N-cone metric space without the assumption of normal cone. Our main results improve and generalize the corresponding results of Fan et al [4]. Also we give examples as an application of the main result.

The following definitions and results will be needed in the sequel([6]).

Let E be a real Banach space. A subset P of E is called a *cone* if and only If

(1) P is closed, non-empty and $P \neq \{0\}$;
(2) $\alpha P + \beta P \subseteq P$ for all nonnegative real numbers α, β;
(3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is said to be *normal* if there exists a constant $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying the above inequality is called the *normal constant* of P. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stand for $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ denotes the interior of P. If $\operatorname{int} P \neq \emptyset$ then P is called a *solid cone*.

Let P be a cone of a real Banach space E and $u, v, w \in E$. Then the following facts are often used([10]).

(p_1) If $u \leq v$, $v \ll w$, then $u \ll w$.
(p_2) If $0 \leq u \ll c$ for each $c \in \operatorname{int} P$, then $u = 0$.
(p_3) If $u \in P$ and $u \leq ku$ for some $0 \leq k < 1$, then $u = 0$.
(p_4) If $a \leq b + c$ for each $c \in \operatorname{int} P$, then $a \leq b$.

Definition 1.1 ([9]) Let X be a nonempty set. An *N-cone metric* on X is a function $N : X^3 \to E$ satisfies the following conditions: for all $x, y, z, a \in X$,

(1) $N(x, y, z) \geq 0$;

(2) $N(x,y,z) = 0$ if and only if $x = y = z$;

(3) $N(x,y,z) \leq N(x,x,a) + N(y,y,a) + N(z,z,a)$.

Then N is called an *N-cone metric* and (X,N) is called an *N-cone metric space*.

Remark 1.2 ([9]) It is easy to see that every generalized D^*-metric space is an N-cone metric space but in general, the converse is not true, see the following example.

Example 1.3. Let $E = \mathbb{R}^3$, $P = \{(x,y,z) \in E : x,y,z \geq 0\}, X = \mathbb{R}$ and $N : X^3 \to E$ is defined by

$$N(x,y,z) = (\alpha(y+z-2x| + |y-z|), \beta(|y+z-2x| + |y-z|), \gamma(|y+z-2x| + |y-z|))$$

where α, β, γ are positive constants. Then (X,N) is an N-cone metric space but not a generalized D^*-metric space, because N is not symmetric.

Lemma 1.4. ([9]) If (X,N) be an N-cone metric space, then for all $x,y \in X$, we have

$$N(x,x,y) = N(y,y,x)$$

Proof. By the definition of N-cone metric, we get

$$N(x,x,y) \leq N(x,x,x) + N(x,x,x) + N(y,y,x) = N(y,y,x)$$

and similarly

$$N(y,y,x) \leq N(y,y,y) + N(y,y,y) + N(x,x,y) = N(x,x,y).$$

Hence we obtain $N(x,x,y) = N(y,y,x)$. This completes the proof.

Definition 1.5. ([9]) Let (X,N) be an N-cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in E$ with $0 \ll c$, there is an $N \in \mathbb{N}$ such that for all $n > N$, $N(x_n,x_n,x) \ll c$, then $\{x_n\}$ is said to be *convergent*, $\{x_n\}$ converges to x and x is the *limit* of $\{x_n\}$. We denote this by $\{x_n\} \to x$ as $n \to \infty$.

(2) If for any $c \in E$ with $0 \ll c$, there is an $N \in \mathbb{N}$ such that for all $n,m > N$, $N(x_n,x_n,x_m) \ll c$, then $\{x_n\}$ is called a *Cauchy sequence* in X.

(3) If every Cauchy sequence in X is convergent in X, then X is called a *complete* N-cone metric space.

The following Lemma is a generalization of Malviya's result [9].

Lemma 1.6. Let (X,N) be an N-cone metric space and P be a solid cone. If $\{x_n\}$ be a sequence converging to x and y in X, then $x = y$.

Proof. Let $c \in E$ be any element with $0 \ll c$. By definition, there is an N such that for all $n > N$, $N(x_n,x_n,x) \ll \frac{c}{3}$ and $N(x_n,x_n,y) \ll \frac{c}{3}$. Thus for all $n > N$,

$$N(x,x,y) \leq 2N(x,x,x_n) + N(y,y,x_n) = 2N(x_n,x_n,x) + N(x_n,x_n,y) \ll c$$

and so $N(x,x,y) = \theta$ by (p_2). Hence $x = y$. This completes the proof.

Lemma 1.7. ([9]) Let (X,N) be an N-cone metric space and P be a solid cone. Let $\{x_n\}$ be a sequence converging to x in X. Then

(1) $\{x_n\}$ is a Cauchy sequence.
(2) Every subsequence of $\{x_n\}$ converges to x in X.

Lemma 1.8. ([9]) Let (X,N) be an N-cone metric space, P be a cone in a real Banach space E and $k_1,k_2,k_3,k_4,k > 0$. If $x_n \to x, y_n \to y, z_n \to z$ and $p_n \to p$ in X and

$$ka \leq k_1 N(x_n,x_n,x) + k_2 N(y_n,y_n,y) + k_3 N(z_n,z_n,z) + k_4 N(p_n,p_n,p),$$

then $a = 0$.

Definition 1.9. Let f and g be self maps on a set X.

(1) If $w = fx = gx$ for some x in X, then x is called a *coincidence point* of f and g, and w is called a point of coincidence of f and g.
(1) f and g are said to be *weakly compatible* if they commute at every coincidence point.

Proposition 1.10. ([1]) Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

2. Main results

In this section, we prove some new coincidence and common fixed point theorems for two nonlinear contractive mappings in the N-cone metric space without the assumption of normal cone.

Lemma 2.1. If (X,N) be an N-cone metric space, then for all $x,y \in X$, we have $N(x,y,y) \leq N(x,x,y) = N(y,y,x)$.

Proof. By the definition of N-cone metric, we get

$$N(x,y,y) \leq N(x,x,y) + N(y,y,y) + N(y,y,y) = N(x,x,y).$$

Hence we obtain $N(x,y,y) \leq N(x,x,y) = N(y,y,x)$ by Lemma 1.4. This completes the proof.

The following theorem is a generalization of the above theorem of Fan et al [4].

Theorem 2.2. Let (X,N) be an N-cone metric space, P be a solid cone and $f,g : X \to X$ be two mappings which satisfy the following conditions:

(1) $f(X) \subset g(X)$;

(2) $f(X)$ or $g(X)$ is complete;

(3) $N(fx,fy,fz) \leq aN(gx,gy,gz) + bN(gx,fx,fx) + cN(gy,fy,fy) + dN(gz,fz,fz)$

for all $x,y,z \in X$ where $a,b,c,d \geq 0$ and $a+b+c+d < 1$. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be any element of X. Then by (1), we see that there exist $x_1 \in X$ such that $fx_0 = gx_1$. In this way, we construct sequences $\{gx_n\}$ with $fx_{n-1} = gx_n$. From Definition 1.1 and Lemma 2.1, we have

$$\begin{aligned} N(gx_{n+1}, gx_{n+1}, gx_n) &= N(fx_n, fx_n, fx_{n-1}) \\ &\leq aN(gx_n, gx_n, gx_{n-1}) + bN(gx_n, fx_n, fx_n) \\ &+ cN(gx_n, fx_n, fx_n) + dN(gx_{n-1}, fx_{n-1}, fx_{n-1}) \\ &= aN(gx_n, gx_n, gx_{n-1}) + bN(gx_n, gx_{n+1}, gx_{n+1}) \end{aligned}$$

$$
\begin{aligned}
&+ \quad cN(gx_n, gx_{n+1}, gx_{n+1}) + dN(gx_{n-1}, gx_n, gx_n) \\
&\leq \quad aN(gx_n, gx_n, gx_{n-1}) + bN(gx_{n+1}, gx_{n+1}, gx_n) \\
&\quad\quad +cN(gx_{n+1}, gx_{n+1}, gx_n) + dN(gx_n, gx_n, gx_{n-1}) \\
&= \quad (a+d)N(gx_n, gx_n, gx_{n-1}) + (b+c)N(gx_{n+1}, gx_{n+1}, gx_n).
\end{aligned}
$$

This implies $N(gx_{n+1}, gx_{n+1}, gx_n) \leq qN(gx_n, gx_n, gx_{n-1})$ where $q = \frac{a+d}{1-(b+c)}$ and $0 \leq q < 1$. By repeated application of above inequality, we have

$$N(gx_{n+1}, gx_{n+1}, gx_n) \leq q^n N(gx_1, gx_1, gx_0).$$

For all $n, m \in \mathbb{N}$ with $n < m$, by Lemma 1.4 and Lemma 2.1, we see that

$$
\begin{aligned}
N(gx_n, gx_n, gx_m) &\leq 2N(gx_n, gx_n, gx_{n+1}) + N(gx_m, gx_m, gx_{n+1}) \\
&= 2N(gx_n, gx_n, gx_{n+1}) + N(gx_{n+1}, gx_{n+1}, gx_m) \\
&\leq 2N(gx_n, gx_n, gx_{n+1}) + 2N(gx_{n+1}, gx_{n+1}, gx_{n+2}) \\
&+ N(gx_m, gx_m, gx_{n+2}) \\
&\leq 2N(gx_n, gx_n, gx_{n+1}) + \cdots + 2N(gx_{m-2}, gx_{m-2}, gx_{m-1}) \\
&+ N(gx_m, gx_m, gx_{m-1}) \\
&\leq 2N(gx_n, gx_n, gx_{n+1}) + \cdots + 2N(gx_{m-2}, gx_{m-2}, gx_{m-1}) \\
&+ 2N(gx_m, gx_m, gx_{m-1}) \\
&\leq 2q^n N(gx_1, gx_1, gx_0) + \cdots + 2q^{m-1} N(gx_1, gx_1, gx_0) \\
&= 2q^n (1 + q + \cdots + q^{m-n-1}) N(gx_1, gx_1, gx_0) \\
&\leq \frac{2q^n}{1-q} N(gx_1, gx_1, gx_0).
\end{aligned}
$$

Let $0 \ll c$ be given. Choose a natural number K_1 such that $\frac{2q^n}{1-q} N(gx_1, gx_1, gx_0) \ll c$ for all $n \geq K_1$. Thus $N(gx_n, gx_n, gx_m) \ll c$ for all $m > n \geq K_1$. Thus $\{gx_n\}$ is a Cauchy sequence.

Case I: If $g(X)$ is complete, then there exists $u \in g(X)$ such that $gx_n \to u$ as $n \to \infty$. So exist $p \in X$ such that $gp = u$.

Case II: If $f(X)$ is complete, then there exists $u \in f(X)$ such that $gx_n = fx_{n-1} \to u$. Since $f(X) \subset g(X)$ we have $u \in g(X)$, and so there exist $p \in X$ such that $gp = u$.

We claim that $fp = u$. Let $0 \ll \alpha$ be given. Since $gx_n \to u$ as $n \to \infty$ and $\{gx_n\}$ is a Cauchy sequence, choose a natural number K_1 such that $N = N(gx_n, gx_n, u) = N(u, u, gx_n) \ll \frac{\alpha}{3(a+2)}, N(gx_n, g_n, gx_{n+1}) \ll \frac{\alpha}{3d}$ for all $n \geq K_2$. Hence, for all $n \geq K_2$, we have

$$\begin{aligned}
N(fp, fp, u) &= N(u, u, fp) \leq 2N(u, u, fx_n) + N(fp, fp, fx_n) \\
&\leq [aN(gp, gp, gx_n) + bN(gp, fp, fp) + cN(gp, fp, fp) \\
&+ dN(gx_n, fx_n, fx_n)] + 2N(u, u, gx_{n+1}) \\
&= aN(u, u, gx_n) + (b+c)N(u, fp, fp) \\
&+ dN(gx_n, gx_{n+1}, gx_{n+1}) + 2N(u, u, gx_{n+1}) \\
&\ll a\frac{\alpha}{3a} + (b+c)N(u, fp, fp) + d\frac{\alpha}{3d} + 2\frac{\alpha}{6} \\
&= (b+c)N(u, fp, fp) + \alpha
\end{aligned}$$

By (p_4) and Lemma 2.1, it shows that

$$N(fp, fp, u) \leq (b+c)N(u, fp, fp) \leq (b+c)N(fp, fp, u).$$

Since $b + c < 1$, by (p_3), $N(fp, fp, u) = 0$ and so $fp = u$. Hence $fp = gp = u$ and u is a point of coincidence of f and g.

Now we show that f and g have a unique point of coincidence. To this end, let us assume that there exists a point q in X such that $fq = gq$.

$$\begin{aligned}
N(fp, fp, fq) &\leq aN(gp, gp, gq) + bN(gp, fp, fp) \\
&+ cN(gp, fp, fp) + dN(gq, fq, fq) \\
&= aN(gp, gp, gq) = aN(fp, fp, fq).
\end{aligned}$$

Since $a < 1$, by (p_3), $N(fp, fp, fq) = 0$ and so $fp = fq$. Hence f and g have a unique point of coincidence. By Proposition 1.10, f and g have a unique common fixed point. This completes the proof.

Corollary 2.3. ([4]) Let (X,N) be an N-cone metric space, P be a normal cone and $f,g : X \to X$ be two mappings which satisfy the following conditions:

(1) $f(X) \subset g(X)$;

(2) $f(X)$ or $g(X)$ is complete;

(3) $N(fx, fy, fz) \leq aN(gx, gy, gz) + bN(gx, fx, fx) + cN(gy, fy, fy) + dN(gz, fz, fz)$

for all $x,y,z \in X$, where $a,b,c,d \geq 0$ and $a+4b+4c+2d < 1$. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Corollary 2.4. Let (X,N) be an N-cone metric space, P be a solid cone and $f,g : X \to X$ be two mappings which satisfy the following conditions:

(1) $f(X) \subset g(X)$;

(2) $f(X)$ or $g(X)$ is complete;

(3) $N(fx, fy, fz) \leq kN(gx, gy, gz)$

for all $x,y,z \in X$ where $0 \leq k < 1$. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. The proof follows from Theorem 2.2 by taking $a = k, b = c = d = 0$.

Corollary 2.5. Let (X,N) be an N-cone metric space, P be a solid cone and $f,g : X \to X$ be two mappings which satisfy the following conditions:

(1) $f(X) \subset g(X)$;

(2) $f(X)$ or $g(X)$ is complete;

(3) $N(fx, fy, fz) \leq aN(gx, fx, fx) + bN(gy, fy, fy) + cN(gz, fz, fz)$

for all $x,y,z \in X$, where $a,b,c \geq 0$ and $a+b+c < 1$. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. The proof follows from Theorem 2.2 by taking $a = o, b = a, c = b, d = c$.

The following corollary is a generalization of Corollary 3.4 of Fan et al [4].

Corollary 2.6. Let (X,N) be an N-cone metric space, P be a solid cone and $f : X \to X$ be a mapping which satisfy the following conditions:

(1) $f(X)$ is complete;

(2) $N(fx,fy,fz) \leq aN(x,y,z)+bN(x,fx,fx)+cN(y,fy,fy)+dN(z,fz,fz)$

for all $x,y,z \in X$ where $a,b,c,d \geq 0$ and $a+b+c+d<1$. Then f has a unique fixed point.

Proof. The proof follows from Theorem 2.2 by taking $g=I$, the identity mapping.

Corollary 2.7. Let (X,N) be an N-cone metric space, P be a solid cone and $f: X \to X$ be a mapping which satisfy the following conditions:

(1) $f(X)$ is complete;

(2) $N(fx,fy,fz) \leq kN(x,y,z)$

for all $x,y,z \in X$ where $0 \leq k < 1$. Then f has a unique fixed point.

Proof. The proof follows from Theorem 2.2 by taking $a=k, b=c=d=0$ and $g=I$.

Corollary 2.8. Let (X,N) be an N-cone metric space, P be a solid cone and $f: X \to X$ be a mapping which satisfy the following conditions:

(1) $f(X)$ is complete;

(2) $N(f^n x, f^n y, f^n z) \leq kN(x,y,z)$

for all $x,y,z \in X$ where $0 \leq k < 1$. Then f has a unique fixed point.

Proof. From Theorem 2.2, T^n has a unique fixed point x^*. But $T^n(Tx^*) = T(T^n x^*) = Tx^*$. So Tx^* is also a fixed point of T^n. Hence $Tx^* = x^*, x^*$ is a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique,

Corollary 2.9. Let (X,N) be an N-cone metric space, P be a solid cone and $f: X \to X$ be a mapping which satisfy the following conditions:

(1) $f(X)$ is complete;

(2) $N(fx,fy,fz) \leq k[N(x,fx,fx)+N(y,fy,fy)]$

for all $x,y \in X$ where $k \in [0,\frac{1}{2})$. Then f has a unique fixed point.

Corollary 2.10 Let (X,N) be an N-cone metric space, P be a solid cone and $f: X \to X$ be a mapping which satisfy the following conditions:

(1) $f(X)$ is complete;

(2) $N(fx,fx,fy) \leq k[N(x,x,y)+N(y,fy,fy)]$

for all $x,y \in X$ where $k \in [0,\frac{1}{2})$. Then f has a unique fixed point.

Example 2.11. (1) Let $E=\mathbb{R}^3$, $P=\{(x,y,z)\in E: x,y,z\geq 0\}, X=[0,1]$ and let $N: X^3\to E$ be an N-cone metric space defined in Example 1.3. Then P is a normal cone. Let f,g be self maps of X defined by $f(x)=\frac{x}{4}, g(x)=\frac{x}{2}$. Then $f(X)\subseteq g(X)$ and $g(X)$ is complete. Taking $a=\frac{1}{2}, b=c=d=\frac{1}{7}$, the inequality (3) of Theorem 2.2 holds for all $x,y,z\in X$. By Theorem 2.2, f and g have a unique point $x=0$ of coincidence in X. Also f and g are weakly compatible and a unique common fixed point $x=0$.

(2) Let $E=C^1_{\mathbb{R}}[0,1]$ with $\|f\|=\|f\|_\infty+\|f'\|_\infty$ and

$$P=\{f\in E: f(t)\geq 0, t\in[0,1]\}.$$

Then E is a Banach space and P is a nonnormal cone. Let $X=\mathbb{R}$ and $N: X^3\to E$ be a map defined by

$$N(x,y,z)(t)=\alpha(|y+z-2x|+|y-z|)e^t$$

for each $t\in[0,1]$ where α is a positive constant. Then (X,N) is a complete N-cone metric space. Let f,g be a self maps of X defined by $f(x)=\frac{x}{2}, g(x)=x$. Then for any $x,y.z\in X$ and $t\in[0,1]$,

$$N(fx,fy,fz)(t)=N(\frac{x}{2},\frac{y}{2},\frac{z}{2})(t)=\frac{\alpha}{2}(|y+z-2x|+|y-z|)e^t$$

and so $N(fx,fy,fz)(t)\leq\frac{1}{2}N(x,y,z)(t)$. Take $a_1=\frac{1}{2}, a_2=a_3=a_4=a_5=0$. Then by Theorem 2.2 or Corollary 2.7, T has a unique fixed point $x=0$.

The following theorem is a generalization of Theorem 3.5 of Fan et al [4].

Theorem 2.12 Let (X,N) be an N-cone metric space, P be a solid cone and $f,g: X\to X$ be two mappings which satisfy the following conditions:

(1) $f(X)\subset g(X)$;

(2) $f(X)$ or $g(X)$ is complete;

(3)

$$\begin{aligned}N(fx,fy,fz) &\leq a[N(gx,fy,fy)+N(gy,fx,fx)]\\ &+ b[N(gy,fz,fz)+N(gz,fy,fy)]\\ &+ c[N(gx,fz,fz)+N(gz,fx,fx)]\end{aligned}$$

for all $x,y,z \in X$, where $a,b,c \geq 0$ and $2a+3b+3c<1$. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be any element of X. Then, by (1), there exist $x_1 \in X$ such that $fx_0 = gx_1$, in this way we construct sequences $\{gx_n\}$ with $fx_{n-1} = gx_n$. By Definition 1.1 and Lemma 1.4, we have

$$\begin{aligned}
N(gx_{n+1},gx_{n+1},gx_n) &= N(fx_n,fx_n,fx_{n-1}) \\
&\leq a[N(gx_n,fx_n,fx_n)+N(gx_n,fx_n,fx_n)] \\
&+ b[N(gx_n,fx_{n-1},fx_{n-1})+N(gx_{n-1},fx_n,fx_n)] \\
&+ c[N(gx_n,fx_{n-1},fx_{n-1})+N(gx_{n-1},fx_n,fx_n)] \\
&= a[N(gx_n,gx_{n+1},gx_{n+1})+N(gx_n,gx_{n+1},gx_{n+1})] \\
&+ b[N(gx_n,gx_n,gx_n)+N(gx_{n-1},gx_{n+1},gx_{n+1})] \\
&+ c[N(gx_n,gx_n,gx_n)+N(gx_{n-1},gx_{n+1},gx_{n+1})] \\
&= 2aN(gx_n,gx_{n+1},gx_{n+1})+(b+c)N(gx_{n-1},gx_{n+1},gx_{n+1}) \\
&\leq 2aN(gx_{n+1},gx_{n+1},gx_n)+(b+c)N(gx_{n-1},gx_{n-1},gx_n) \\
&+ 2(b+c)N(gx_{n+1},gx_{n+1},gx_n).
\end{aligned}$$

So $(1-2a-2b-2c)N(gx_{n+1},gx_{n+1},gx_n) \leq (b+c)N(gx_n,gx_n,gx_{n-1})$ which implies

$$N(gx_{n+1},gx_{n+1},gx_n) \leq qN(gx_n,gx_n,gx_{n-1})$$

where $q = \frac{b+c}{1-(2a+2b+2c)}$ and $0 \leq q < 1$. By repeated application of above inequality, we have

$$N(gx_{n+1},gx_{n+1},gx_n) \leq q^n N(gx_1,gx_1,gx_0).$$

For all $n, m \in \mathbb{N}$ with $n < m$, by Definition 1.1 and Lemma 1.4, we have

$$\begin{aligned}
N(gx_n, gx_n, gx_m) &\leq 2N(gx_n, gx_n, gx_{n+1}) + N(gx_m, gx_m, gx_{n+1}) \\
&= 2N(gx_n, gx_n, gx_{n+1}) + N(gx_{n+1}, gx_{n+1}, gx_m) \\
&\leq 2N(gx_n, gx_n, gx_{n+1}) + 2N(gx_{n+1}, gx_{n+1}, gx_{n+2}) \\
&+ N(gx_m, gx_m, gx_{n+2}) \\
&\leq 2N(gx_n, gx_n, gx_{n+1}) + \cdots + 2N(gx_{m-2}, gx_{m-2}, gx_{m-1}) \\
&+ N(gx_m, gx_m, gx_{m-1}) \\
&\leq 2N(gx_n, gx_n, gx_{n+1}) + \cdots + 2N(gx_{m-2}, gx_{m-2}, gx_{m-1}) \\
&+ 2N(gx_m, gx_m, gx_{m-1}) \\
&\leq 2q^n N(gx_1, gx_1, gx_0) + \cdots + 2q^{m-1} N(gx_1, gx_1, gx_0) \\
&= 2q^n (1 + q + \cdots + q^{m-n-1}) N(gx_1, gx_1, gx_0) \\
&\leq \frac{2q^n}{1-q} N(gx_1, gx_1, gx_0).
\end{aligned}$$

Let $0 \ll c$ be given. Choose a natural number K_1 such that $\frac{2q^n}{1-q} N(gx_1, gx_1, gx_0) \ll c$ for all $n \geq K_1$. Thus $N(gx_n, gx_n, gx_m) \ll c$ for all $m > n \geq K_1$. Thus $\{gx_n\}$ is a Cauchy sequence.

Case I: If $g(X)$ is complete, then there exists $u \in g(X)$ such that $gx_n \to u$ as $n \to \infty$, and so exist $p \in X$ such that $gp = u$.

Case II: If $f(X)$ is complete, then there exists $u \in f(X)$ such that $gx_n = fx_{n-1} \to u$. Since $f(X) \subset g(X)$ we have $u \in g(X)$, and so there exist $p \in X$such that $gp = u$. We claim that $fp = u$.

Let $0 \ll \alpha$ be given. Since $gx_n \to u$ as $n \to \infty$ and $\{gx_n\}$ is a Cauchy sequence, choose a natural number K_1 such that $N(gx_n, gx_n, u) \ll \frac{\alpha}{2(b+c+2)}$ for all $n \geq K_2$. Hence, for all $n \geq K_2$, we have

$$\begin{aligned}
N(fp, fp, u) &= N(u, u, fp) \leq N(fp, fp, fx_n) + 2N(u, u, fx_n) \\
&\leq a[N(gp, fp, fp) + N(gp, fp, fp)] \\
&+ b[N(gp, fx_n, fx_n) + N(gx_n, fp, fp)]
\end{aligned}$$

$$
\begin{aligned}
&+ \quad c[N(gp, fx_n, fx_n) + N(gx_n, fp, fp)] + 2N(u, u, gx_{n+1}) \\
&= \quad 2aN(gp, fp, fp) + (b+c)N(gp, gx_{n+1}, gx_{n+1}) \\
&+ \quad (b+c)N(gx_n, fp, fp) + 2N(u, u, gx_{n+1}) \\
&\leq \quad (2a+2b+2c)N(fp, fp, u) + (b+c+2)N(gx_{n+1}, gx_{n+1}, u) \\
&+ \quad (b+c)N(gx_n, gx_n, u) \\
&\leq \quad (2a+2b+2c)N(fp, fp, u) + \frac{\alpha}{2} + \frac{\alpha}{2}.
\end{aligned}
$$

By (p_4), it shows that

$$N(fp, fp, u) \leq (2a+2b+2c)N(fp, fp, u).$$

Since $2a+2b+2c < 1$, we have $N(fp, fp, u) = 0$ by (p_3) and so $fp = u$. Hence $fp = gp = u$ and u is a point of coincidence of f and g.

Now we show that f and g have a unique point of coincidence. To this end, assume that there exists a point q in X such that $fq = gq$.

$$
\begin{aligned}
N(fp, fp, fq) \quad &\leq \quad a[N(gp, fp, fp) + N(gp, fp, fp)] \\
&+ \quad b[N(gp, fq, fq) + N(gq, fp, fp)] \\
&+ \quad c[N(gp, fq, fq) + N(gq, fp, fp)] \\
&= \quad (b+c)[N(fp, fq, fq) + N(fq, fp, fp)] \\
&\leq \quad (b+c)[N(fp, fp, fq) + N(fp, fp, fq)] \\
&= \quad 2(b+c)N(fp, fp, fq).
\end{aligned}
$$

Since $2(b+c) < 1$, one has $N(fp, fp, fq) = 0$ by (p_3) and so $fp = fq$. Hence f and g have a unique point of coincidence. By Proposition 1.10, f and g have a unique common fixed point. This completes the proof.

Corollary 2.13. ([4]) Let (X, N) be an N-cone metric space, P be a normal cone with normal constant K and $f, g : X \to X$ be two mappings which satisfy the following conditions:

(1) $f(X) \subset g(X)$;

(2) $f(X)$ or $g(X)$ is complete;

(3)

$$\begin{aligned} N(fx,fy,fz) &\leq a[N(gx,fy,fy)+N(gy,fx,fx)] \\ &+ b[N(gy,fz,fz)+N(gz,fy,fy)] \\ &+ c[N(gx,fz,fz)+N(gz,fx,fx)] \end{aligned}$$

for all $x,y,z \in X$, where $a,b,c \geq 0$ and $8a+4b+4c < 1$. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Corollary 2.14. Let (X,N) be an N-cone metric space, P be a solid cone and $f : X \to X$ be a mapping which satisfy the following conditions:

(1) $f(X)$ is complete;

(2)

$$\begin{aligned} N(fx,fy,fz) &\leq a[N(x,fy,fy)+N(y,fx,fx)]+b[N(y,fz,fz) \\ &+ N(z,fy,fy)]+c[N(x,fz,fz)+N(z,fx,fx)] \end{aligned}$$

for all $x,y,z \in X$, where $a,b,c \geq 0$ and $2a+3b+3c < 1$. Then f has a unique fixed point.

Proof. The proof follows from Theorem 2.12 by taking $g = I$, the identity mapping.

Corollary 2.15. Let (X,N) be an N-cone metric space, P be a solid cone and $f : X \to X$ be a mapping which satisfy the following conditions:

(1) $f(X)$ is complete;

(2) $N(fx,fy,fz) \leq k[N(x,fy,fy)+N(y,fx,fx)]$

for all $x,y,z \in X$, where $k \in [0,\frac{1}{2})$. Then f has a unique fixed point.

Proof. The proof follows from Theorem 2.12 by taking $a = k, b = c = 0$ and $g = I$.

Corollary 2.16. Let (X,N) be an N-cone metric space, P be a solid cone and $f : X \to X$ be a mapping which satisfy the following conditions:

(1) $f(X)$ is complete;

(2) $N(fx,fy,fz) \leq k[N(y,fz,fz)+N(z,fy,fy)]$

for all $x,y,z \in X$, where $k \in [0,\frac{1}{3})$. Then f has a unique fixed point.

Proof. The proof follows from Theorem 2.12 by taking $b=k, a=c=0$ and $g=I$.

Corollary 2.17. Let (X,N) be an N-cone metric space, P be a solid cone and $f: X \to X$ be a mapping which satisfy the following conditions:

(1) $f(X)$ is complete;

(2) $N(fx,fy,fz) \leq k[N(x,fz,fz)+N(z,fx,fx)]$

for all $x,y,z \in X$, where $k \in [0,\frac{1}{3})$. Then f has a unique fixed point.

Proof. The proof follows from Theorem 2.12 by taking $b=k, a=c=0$ and $g=I$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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