ITERATIVE APPROXIMATION FOR THE COMMON SOLUTIONS OF
A INFINITE VARIATIONAL INEQUALITY SYSTEM FOR
INVERSE-STRONGLY ACCRETIVE MAPPINGS

HONGPING LUO AND YUANHENG WANG*

Department of Mathematics, Zhejiang Normal University
Zhejiang 321004, P.R. China

Abstract. The aim of this paper is to introduce and study a system of the infinite variational inequalities for inverse-strongly accretive mappings by using relaxed extragradient method. Results proved in this paper may be viewed as an improvement and refinement of the recent results of X.Qin[1] and Aoyama,K[2]

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1. Introduction

Let H be a real Hilbert space with norm \( \| \cdot \| \) and inner product \( \langle \cdot, \cdot \rangle \), C be a nonempty closed convex subset of H and A be a operator from C into H . The classical variational inequality problem is formulated as finding a point \( u \in C \) such that

\[
\langle Au, v - u \rangle \geq 0
\]

*Corresponding author

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for all $v \in C$. Such a point $u \in C$ is called a solution of the problem. Variational inequalities were initially studied by Stampacchia [3, 4] and ever since have been widely studied. The set of solutions of the variational inequality problem is denoted by $\text{VI}(C,A)$. For given $z \in H, u \in C$, we see that the following inequality holds

$$\langle u - z, v - u \rangle \geq 0$$

if and only if $u = P_C z : \|P_C z - z\| = \inf_{v \in C} \|v - z\|$. It is known that projection operator $P_C$ is nonexpansive. It is also know that $P_C$ satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \forall x, y \in H.$$ 

One can see that the variational inequality is equivalent to a fixed point problem. An element $x^* \in C$ is a solution of the variational inequality if and only if $x^* \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where $I$ is the identity mapping and $\lambda > 0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of variational inequalities and related optimization problems.

In this paper, let $C$ be a nonempty closed convex subset of a real Banach space $E$. Let $A, B$ be two inverse-strongly accretive mappings. We consider the following problem of finding $(\tilde{x}, \tilde{y}) \in C \times C$ such that

$$\begin{align*}
\langle \lambda_n A \tilde{y} + \tilde{x} - \tilde{y}, J(x - \tilde{x}) \rangle &\geq 0, \forall x \in C, \\
\langle \mu_n B \tilde{x} + \tilde{y} - \tilde{x}, J(x - \tilde{y}) \rangle &\geq 0, \forall x \in C,
\end{align*}$$

(1)

which is called a general system of infinite variational inequalities, where $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$. In particular, if $A = B, \lambda_n = \mu_n = \lambda$, then problem reduces to finding $(\tilde{x}, \tilde{y}) \in C \times C$ such that

$$\begin{align*}
\langle \lambda A \tilde{y} + \tilde{x} - \tilde{y}, J(x - \tilde{x}) \rangle &\geq 0, \forall x \in C, \\
\langle \lambda A \tilde{x} + \tilde{y} - \tilde{x}, J(x - \tilde{y}) \rangle &\geq 0, \forall x \in C,
\end{align*}$$

(2)

which is defined by Verma [5] and is called the new system of variational inequalities. Further, if we add up the requirement that $\tilde{x} = \tilde{y}$, then problem (1) reduces to the classical variational inequality $\text{VI}(A, C)$. 

Recently, many authors studied the problem of finding a common element of the fixed point set of nonexpansive mappings and the solution set of variational inequalities for \(\alpha\)-inverse-strongly monotone mappings in the framework of Banach space. In 2006, Aoyama, Iiduka and Takahashi \([2]\) obtained a weak Theorem about weak convergence of an iterative sequence for accretive operators in a uniformly convex and 2-uniformly smooth Banach space. In 2009, X.Qin\([1]\), et al. consider the problem of strong convergence of an iterative algorithm for systems of variational inequalities and nonexpansive mapping with applications.

In this paper, motivated by \([1,2,6,7]\), let \(E\) be a uniformly convex and \(q\)-uniformly smooth Banach space, \(C\) be a nonempty closed convex subset of \(E\). We introduce a general iterative algorithm for the system of infinite variational inequality (1) and a sunny nonexpansive mapping.

\[
\begin{align*}
  x_1 &= u \in C \\
  y_n &= Q_C(x_n - \mu_n Bx_n) \\
  x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n (\delta T x_n + (1 - \delta)Q_C(y_n - \lambda_n Ay_n)), \ n \geq 0.
\end{align*}
\]

The problem (1) is proven to be equivalent to a fixed point problem of nonexpansive mapping. By using a relaxed extradient methods, we prove that under some conditions the iterative sequence \(\{x_n\}\) converges strongly to \(\tilde{x} \in C\) and \((\tilde{x}, \tilde{y})\) is a solution of the problem (1), where \(\tilde{y} = Q_C(\tilde{x} - \mu_n B\tilde{x})\). The results here improve and extend the related results of other authors, such as \([1,2,6]\).

2. Preliminaries

Recall that a mapping \(T\) of \(C\) into itself is called nonexpansive, if

\[
\|Tx - Ty\| \leq \|x - y\|.
\]

for all \(x, y \in C\). We denote by \(F(T)\) the set of fixed points of \(T\).

For \(\alpha > 0\), an operator \(A\) of \(C\) into \(E\) is said to be \(\alpha\)-inverse strongly accretive if

\[
\langle Ax - Ay, J(x - y) \rangle \geq \alpha \|Ax - Ay\|^2.
\]
for all $x, y \in C$. It is obviously that

$$\|Ax - Ay\| \leq \frac{1}{\alpha} \|x - y\|.$$  

Let $D$ be a subset of $C$ and $Q$ be a mapping of $C$ into $D$, then $Q$ is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q$ of $C$ into itself is called a retraction if $Q^2 = Q$. If a mapping $Q$ of $C$ into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of $Q$. A subset $D$ of $C$ is called a sunny nonexpansive retract of $C$ if there exists a sunny nonexpansive retraction from $C$ into $D$.

Assume $E$ be a real Banach space, $C$ be a nonempty closed convex subset of $E$. Let $U = \{x \in E : x = 1\}$, A Banach space $E$ is said to be uniformly convex, if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|x - y\| \leq \epsilon$, which implies $\frac{\|x - y\|}{2} \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex.

A Banach space $E$ is said to be smooth if the limit $\lim_{t \to 0} \frac{\|x - ty\| - \|y\|}{t}$ exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. The norm of $E$ is said to be Fréchet differentiable if for each $x \in U$, the limit is attained uniformly for $y \in U$. And we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of $E$ as follows:

$$\rho(t) = \sup \{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| = 1, \|y\| = t\}.$$  

It is known that $E$ is uniformly smooth if and only if $\lim_{t \to 0} \frac{\rho(t)}{t} = 0$. Let $q$ be a fixed real number with $1 < q \leq 2$. Then a Banach space $E$ is said to be $q$-uniformly smooth if there exists a constant $c > 0$ such that $\rho(t) \leq ct^q$ for all $t > 0$. We could obtain the following lemma.

**Lemma 2.1.** \cite{8,9} Let $q$ be a real number with $1 < q \leq 2$ and let $E$ be a Banach space. Then $E$ is $q$-uniformly smooth if and only if there exists a constant $K \geq 1$ such that

$$\frac{1}{2}(\|x + y\|^q + \|x - y\|^q) \leq \|x\|^q + \|Ky\|^q$$

for all $x, y \in E$. 
The best constant $K$ in Lemma 2.1 is called the $q$-uniformly smoothness constant of $E$. Let $q$ be a given real number with $q > 1$. The (generalized) duality mapping $J_q$ from $E$ into $2^{E^*}$ is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^q, ||x^*|| = ||x||^{q-1}\}$$

for all $x \in E$. In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q$

$$J_q(x) = ||x||^{q-2}J(x)$$

**Lemma 2.2.**[10] Let $q$ be a given real number with $1 < q \leq 2$ and let $E$ be a $q$-uniformly smooth Banach space. Then

$$||x + y||^q \leq ||x||^q + q \langle y, J_q(x) \rangle + 2||Ky||^q$$

for all $x, y \in E$, where $J_q$ is the generalized duality mapping of $E$ and $K$ is the $q$-uniformly smoothness constant of $E$.

**Lemma 2.3.**[1] Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T_1$ and $T_2$ be two nonexpansive mappings from $C$ into itself with a common fixed point. Define a mapping $T : C \rightarrow C$ by $Tx = \delta T_1x + (1 - \delta)T_2x$, where $\delta \in (0, 1)$. Then $T$ is nonexpansive and $F(T) = F(T_1) \cap F(T_2)$.

**Lemma 2.4.**[11] In a Banach space $E$, there holds the inequality

$$||x + y||^2 \leq ||x||^2 + 2\langle y, j(x + y) \rangle, \forall x, y \in C, where j(x + y) \in J(x + y).$$

**Lemma 2.5.**[15] Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$. Let $Q_C$ be a sunny nonexpansive retraction from $E$ onto $C$ and let $A$ be an accretive operator of $C$ into $E$. Then, for all $\lambda > 0$,

$$\Omega = F(Q_C(I - \lambda A)).$$

**Lemma 2.6.**[15] Let $E$ be a uniformly convex Banach space, $C$ a nonempty closed convex subset of $E$ and $T : K \rightarrow K$ a nonexpansive mapping. Then $I - T$ is demi-closed at zero.
Lemma 2.7. Let \( \{\alpha_n\}_{n=0}^{\infty} \) be a sequence of nonnegative real numbers satisfying the property
\[
\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\delta_n, \quad n \geq 0,
\]
where \( \{\gamma_n\} \subset (0, 1) \) and \( \{\delta_n\} \) are such that
\[
\begin{align*}
(1) & \lim_{n \to \infty} \gamma_n = 0, \sum \gamma_n = \infty; \\
(2) & \limsup_{n \to \infty} \frac{\delta_n}{\sum \gamma_n} \leq 0 \text{ (or } \sum |\delta_n| < \infty). \\
\end{align*}
\]
then \( \lim_{n \to \infty} \alpha_n = 0. \)

Lemma 2.8. Let \( \{x_n\} \text{ and } \{y_n\} \) be bounded sequences in a Banach space \( X \) and Let \( \{\alpha_n\} \subset [0, 1] \) with \( 0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1, \quad n \geq 0 \), such that
\[
\begin{align*}
(1) & x_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n; \\
(2) & \limsup_{n \to \infty} (||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0.
\end{align*}
\]
then \( \lim_{n \to \infty} ||y_n - x_n|| = 0. \)

Lemma 2.9. For given \((\tilde{x}, \tilde{y}) \in C \times C\), where \( \tilde{y} = Q_C(\tilde{x} - \mu_n B\tilde{x}) \), \( \tilde{x}, \tilde{y} \) is a solution of problem (1), if and only if \( \tilde{x} \) is a common fixed point of the mapping \( S_n : C \to C \) defined by
\[
S_n(x) = Q_C[Q_C(x - \mu_n Bx) - \lambda_n A Q_C(x - \mu_n Bx)], \forall n \in N,
\]
where \( \{\lambda_n\}, \{\mu_n\} \subset (0, 1) \) and \( Q_C \) is a sunny nonexpansive retraction from \( E \) onto \( C \).

Proof.

\begin{align*}
(4) & \quad \begin{cases}
\langle \lambda_n A\tilde{y} + \tilde{x} - \tilde{y}, J(x - \tilde{x}) \rangle \geq 0, \forall x \in C, \\
\langle \mu_n B\tilde{x} + \tilde{y} - \tilde{x}, J(x - \tilde{y}) \rangle \geq 0, \forall x \in C,
\end{cases} \\
\iff & \quad \begin{cases}
\tilde{x} = Q_C(\tilde{y} - \lambda_n A\tilde{y}) \\
\tilde{y} = Q_C(\tilde{x} - \mu_n B\tilde{x})
\end{cases} \\
& \quad \iff \tilde{x} = Q_C(Q_C(\tilde{x} - \mu_n B\tilde{x}) - \lambda_n A Q_C(\tilde{x} - \mu_n B\tilde{x}))
\end{align*}

3. Main results
Theorem 3.1 Let $E$ be a uniformly convex and $q$-uniformly smooth Banach space with the best smooth constant $K$, $C$ a nonempty closed convex subset of $E$. Let $Q_C : E \to C$ be a sunny nonexpansive retraction and $A, B : C \to E$ be $\alpha$-inverse-strongly accretive mapping and $\beta$-inverse-strongly accretive mapping. Let $T : C \to C$ be a nonexpansive mapping with a fixed point and assume that $F = F(T) \cap (\cap_{n=1}^{\infty} F(S_n)) \neq \emptyset$, where $S_n$ is defined as Lemma 2.9. Suppose $\{\lambda_n\} \subset [a, \sqrt[2q]{\frac{q^2-1}{2K^2}}]$, $\{\mu_n\} \subset [a, \sqrt[2q]{\frac{q^2-1}{2K^2}}]$, $a > 0, \delta \in (0, 1)$. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0, 1]$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n \to \infty} \alpha_n = 0$;
(C3) $0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1$;
(C4) $\lim_{n \to \infty} (\lambda_{n+1} - \lambda_n) = 0, \lim_{n \to \infty} (\mu_{n+1} - \mu_n) = 0$.

Then the sequence $\{x_n\}$ defined by (3) convergence strongly to $\tilde{x} = Q_F u$, and $(\tilde{x}, \tilde{y})$ is a solution of the problem (1), where $\tilde{y} = Q_C(\tilde{x} - \mu_n B \tilde{x})$.

Proof. Step 1 We show that $F$ is closed and convex.

Since $A$ is an $\alpha$-inverse-strongly accretive mapping, applying Lemma 2.1, 2.2 and $\{\lambda_n\} \subset [a, \sqrt[2q]{\frac{q^2-1}{2K^2}}]$, we get

$$
\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^q = \|(x - y) - \lambda_n (Ax - Ay)\|^q \\
\leq \|x - y\|^q - q\lambda_n \langle Ax - Ay, J_q(x - y) \rangle + 2\|K\lambda_n (Ax - Ay)\|^q \\
= \|x - y\|^q - \lambda_n q\|x - y\|^{q-2}\langle Ax - Ay, J(x - y) \rangle \\
+ 2K^q\lambda_n^q\|Ax - Ay\|^q \\
\leq \|x - y\|^q - \lambda_n q\alpha^{q-1}\|Ax - Ay\|^{q} + 2K^q\lambda_n^q\|Ax - Ay\|^q \\
= \|x - y\|^q + \lambda_n (2K^q\lambda_n^{q-1} - q\alpha^{q-1})\|Ax - Ay\|^q
$$

which implies that $I - \lambda_n A$ is nonexpansive, so is $I - \mu_n B$. From Lemma 2.9, we obtain that

$$
S_n = Q_C(Q_C(I - \mu_n B) - \lambda_n AQ_C(I - \mu_n B)) \\
= Q_C(I - \lambda_n A)Q_C(I - \mu_n B)
$$

$S_n$ is nonexpansive. Consequently, $F = (\cap_{n=1}^{\infty} F(S_n)) \cap F(T)$ is closed and convex.
Step 2 We observe \( \{x_n\} \) is bounded.

Indeed, taking a fixed point \( \bar{x} \) of \( F \), we have \( \bar{x} = Q_C(Q_C(\bar{x} - \mu_n B \bar{x}) - \lambda_n A Q_C(\bar{x} - \mu_n B \bar{x})) \)

Let \( \bar{y} = Q_C(\bar{x} - \mu_n B \bar{x}) \), then \( \bar{x} = Q_C(\bar{y} - \lambda_n A \bar{y}) \). And let \( l_n = \delta x_n + (1 - \delta)Q_C(y_n - \lambda_n A y_n) \), we get

\[
\|l_n - \bar{x}\| = \delta_n\|Tx_n - \bar{x}\| + (1 - \delta_n)\|Q_C(y_n - \lambda_n A y_n) - \bar{x}\|
\]
\[
\leq \delta_n\|x_n - \bar{x}\| + (1 - \delta_n)\|Q_C(y_n - \lambda_n A y_n) - Q_C(\bar{y} - \lambda_n A \bar{y})\|
\]
\[
\leq \delta_n\|x_n - \bar{x}\| + (1 - \delta_n)\|y_n - \bar{y}\|
\]
\[
\leq \delta_n\|x_n - \bar{x}\| + (1 - \delta_n)\|Q_C(x_n - \mu_n B x_n) - Q_C(\bar{x} - \mu_n B \bar{x})\|
\]
\[
\leq \|x_n - \bar{x}\|
\]

Then we arrive at

\[
\|x_{n+1} - \bar{x}\| = \|\alpha_n u + \beta_n x_n + \gamma_n l_n - \bar{x}\|
\]
\[
= \alpha_n\|u - \bar{x}\| + \beta_n\|x_n - \bar{x}\| + \gamma_n\|l_n - \bar{x}\|
\]
\[
\leq (1 - \alpha_n)\|x_n - \bar{x}\| + \alpha_n\|u - \bar{x}\|
\]
\[
\leq \max\{\|x_n - \bar{x}\|, \|u - \bar{x}\|\}.
\]
\[
\leq \|u - \bar{x}\|
\]

Hence \( \{x_n\} \) is bounded, so are the sets \( \{y_n\} \) and \( \{l_n\} \).

According to step 1 and by (4), we observe that

\[
\|l_{n+1} - l_n\| \leq \delta\|Tx_{n+1} - T x_n\| + (1 - \delta)\|Q_C(y_{n+1} - \lambda_n A y_{n+1}) - Q_C(y_n - \lambda_n A y_n)\|
\]
\[
\leq \delta\|x_{n+1} - x_n\| + (1 - \delta)\|(y_{n+1} - \lambda_n A y_{n+1}) - (y_n - \lambda_n A y_n)\|
\]
\[
\leq \delta\|x_{n+1} - x_n\| + (1 - \delta)\|(y_{n+1} - \lambda_n A y_{n+1}) - (y_n - \lambda_n A y_n)\|
\]
\[
+ (\lambda_n - \lambda_{n+1}) A y_n\|
\]
\[
\leq \delta\|x_{n+1} - x_n\| + (1 - \delta)\|y_{n+1} - y_n\| + |\lambda_n - \lambda_{n+1}|\|A y_n\|
\]
\[
\leq \delta\|x_{n+1} - x_n\| + (1 - \delta)\|Q_C(x_{n+1} - \mu_n B x_{n+1}) - Q_C(x_n - \mu_n B x_n)\|
\]
\[
+ |\lambda_n - \lambda_{n+1}|\|A y_n\|
\]
\[
\leq \|x_{n+1} - x_n\| + |\mu_n - \mu_{n+1}|\|B x_n\| + |\lambda_n - \lambda_{n+1}|\|A y_n\|
\]
Step 3 We prove that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.

Define $x_{n+1} = \beta_n x_n + (1 - \beta_n) h_n$, observe that

$$h_{n+1} - h_n = \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_{n+1} u + \gamma_{n+1} l_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n l_n}{1 - \beta_n} = \frac{\alpha_{n+1} u}{1 - \beta_{n+1}} + \frac{(1 - \alpha_{n+1} - \beta_{n+1}) l_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u}{1 - \beta_n} - \frac{(1 - \alpha_n - \beta_n) l_n}{1 - \beta_n} = \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (u - l_{n+1}) - \frac{\alpha_n}{1 - \beta_n} (u - l_n) + (l_{n+1} - l_n).$$

Applying the conclusion of step 1, we have

$$\|h_{n+1} - h_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|u - l_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|u - l_n\| + \|\mu - \mu_{n+1}\| B x_n + \|\lambda - \lambda_{n+1}\| A y_n.$$

Since $\{y_n\}$ and $\{l_n\}$ are bounded, by (C2), (C3) and (C4), we obtain that $\lim_{n \to \infty} \sup(\|h_{n+1} - h_n\| - \|x_{n+1} - x_n\|) \leq 0$.

Hence by lemma 2.8, we have $\lim_{n \to \infty} \|h_n - x_n\| = 0$.

Consequently $\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n) \|h_n - x_n\| = 0$.

On the other hand, from $x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n l_n$, we have

$$x_{n+1} - x_n = \alpha_n (u - x_n) + \gamma_n (l_n - x_n),$$

then $\lim_{n \to \infty} \|l_n - x_n\| = 0$.

Step 4 We claim that $\limsup_{n \to \infty} \langle u - \tilde{x}, J(x_n - \tilde{x}) \rangle \leq 0$. where $\tilde{x} = Q_F u$ Define a mapping $W_n : C \to C$ by $W_n x = \delta T x + (1 - \delta) Q_C (I - \lambda_n A) Q_C (I - \mu_n B) x$, $\forall x \in C$, which implies that $W_n x_n = l_n$.

We choose a sequence $\{x_n\}$ of $\{x_n\}$ that converges weakly to $x$ such that

$$\limsup_{n \to \infty} \langle u - \tilde{x}, J(x_n - \tilde{x}) \rangle = \limsup_{i \to \infty} \langle u - \tilde{x}, J(x_n - \tilde{x}) \rangle$$

Since $\{\lambda_n\} \subset [a, q^{-1} \sqrt{2K'}]$, $\{\mu_n\} \subset [a, q^{-1} \sqrt{2K'}]$, $a > 0$, it follows that $\{\lambda_n\}$, $\{\mu_n\}$ are bounded. So there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ which converges to $\{\lambda_0\}$.
It follows from lemma 2.6 that, without loss of generality, we assume that \( \{ \lambda_n \} \rightarrow \lambda_0, \{ \mu_n \} \rightarrow \mu_0 \), then

\[
S_0 = Q_C(Q_C(I - \mu_0 B) - \lambda_0 AQ_C(I - \mu_0 B))
\]

\[
= Q_C(I - \lambda_0 A)Q_C(I - \mu_0 B)
\]

\( S_n \) is nonexpansive.

Since \( Q_C \) is nonexpansive, it follows from \( l_n = \delta T x_n + (1 - \delta) Q_C(y_n - \lambda_n A y_n) \), then

\[
\| W_0 x_n - x_n \| \leq \| \delta T x_n + (1 - \delta) Q_C(y_n - \lambda_0 A y_n) - l_n \| + \| l_n - x_n \|
\]

\[
\leq \| \delta T x_n + (1 - \delta) Q_C(y_n - \lambda_0 A y_n) - \delta T x_n - (1 - \delta) Q_C(y_n - \lambda_0 A y_n) - \lambda_0 A y_{n-i} \| + \| l_n - x_n \|
\]

\[
\leq (1 - \delta) \| \lambda_{n-i} - \lambda_0 \| A y_{n-i} \| + \| l_n - x_n \|
\]

It follows from lemma 2.6 that \( x \in F(W_0) \). By using lemma 2.5 and same as [15], we can obtain that \( x \in F(W_0) = Q_{F,u} \).

We have \( \limsup_{n \to \infty} \langle u - \tilde{x}, J(x_{n+1} - \tilde{x}) \rangle \leq 0 = \limsup_{n \to \infty} \langle u - x, J(x_{n+1} - \tilde{x}) \rangle \leq 0 \) holds.

Step 5 We show that \( \lim_{n \to \infty} \| x_n - \tilde{x} \| = 0 \).

\[
\| x_{n+1} - \tilde{x} \|^2 = \alpha_n \langle u - \tilde{x}, J(x_{n+1} - \tilde{x}) \rangle + \beta_n \| x_n - \tilde{x} \| J(x_{n+1} - \tilde{x}) \rangle + \gamma_n \| l_n - \tilde{x} \| J(x_{n+1} - \tilde{x}) \rangle
\]

\[
= \alpha_n \langle u - \tilde{x}, J(x_{n+1} - \tilde{x}) \rangle + \beta_n \| x_n - \tilde{x} \| \| x_{n+1} - \tilde{x} \| + \gamma_n \| l_n - \tilde{x} \| \| x_{n+1} - \tilde{x} \|
\]

\[
\leq (1 - \alpha_n) \| x_n - \tilde{x} \| \| x_{n+1} - \tilde{x} \| + \alpha_n \langle u - \tilde{x}, J(x_{n+1} - \tilde{x}) \rangle
\]

\[
= \frac{1 - \alpha_n}{2} (\| x_n - \tilde{x} \|^2 + \| x_{n+1} - \tilde{x} \|^2) + \alpha_n \langle u - \tilde{x}, J(x_{n+1} - \tilde{x}) \rangle
\]

Then

\[
\| x_{n+1} - \tilde{x} \|^2 \leq \frac{1 - \alpha_n}{1 + \alpha_n} \| x_n - \tilde{x} \|^2 + \frac{2 \alpha_n}{1 + \alpha_n} \langle u - \tilde{x}, J(x_{n+1} - \tilde{x}) \rangle
\]

\[
= (1 - \frac{2 \alpha_n}{1 + \alpha_n}) \| x_n - \tilde{x} \|^2 + \frac{2 \alpha_n}{1 + \alpha_n} \langle u - \tilde{x}, J(x_{n+1} - \tilde{x}) \rangle
\]

Where \( \gamma_n = \frac{2 \alpha_n}{1 + \alpha_n}, \sigma_n = \langle u - \tilde{x}, J(x_{n+1} - \tilde{x}) \rangle \).

Since by (C2), step 3, we have

\[
\lim_{n \to \infty} \gamma_n = 0, \sum_{n=0}^{\infty} \gamma_n = \infty, \limsup_{n \to \infty} \sigma_n \leq 0.
\]
applying lemma 2.7, we deduce that \( \lim_{n \to \infty} ||x_n - \tilde{x}|| = 0 \).

The proof of Theorem 3.1 is completes.

References


