FIXED POINTS OF CYCLIC WEAK CONTRACTIONS IN METRIC SPACES

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Abstract. In this paper the well known notion of a cyclic contraction for a finite family of non-empty subsets of a metric space $X$ and a mapping $T$ of $X$ into $X$ (respectively, into the collection of nonempty subsets of $X$) has been generalized. Subsequently, the above idea is used to obtain some new fixed point theorems for single and multi-valued mappings. The results obtained herein generalize some recent fixed point theorems.

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1. Introduction and Preliminaries

Throughout this paper $N$ denotes the set of natural numbers and $\Phi$ the class of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying:

(a): $\varphi$ is continuous and monotone nondecreasing,

(b): $\varphi(t) = 0 \Leftrightarrow t = 0$.

The function $\varphi \in \Phi$ is also known as altering distance function (see, for instance, [1]).

In [2], Dutta and Chaudury obtained the following generalization of the well known Banach contraction principle.
**Theorem 1.1.** Let \((X, d)\) be a complete metric space and \(T : X \rightarrow X\) a self-mapping satisfying

\[
\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))
\]

for all \(x, y \in X\), where \(\psi, \varphi \in \Phi\). Then \(T\) has a unique fixed point.

The mapping \(T\) satisfying (1.1) is known as \((\psi, \varphi)\)-weakly contraction [3].

Notice that when \(\psi(t) = t\) and \(\varphi(t) = (1 - k)t\), we get the well known Banach contraction principle as a special case of Theorem 1.1.

On the other hand Kirk et al. [4] introduced the following notion of cyclic mappings and obtained a fixed point theorem (see Theorem 1.3 below).

**Definition 1.2.** Let \(A_1, A_2, ..., A_p\) be nonempty subsets of a metric space \((X, d)\). A mapping \(T : \bigcup_{i=1}^{p} A_i \rightarrow \bigcup_{i=1}^{p} A_i\) is called a cyclic mapping (or \(p\)-cyclic mapping) if

\[T(A_i) \subset A_{i+1}, \text{ where } A_{p+1} = A_1.\]

**Theorem 1.3.** Let \(A_1, A_2, ..., A_p\) be nonempty closed subsets of a complete metric space and \(T : \bigcup_{i=1}^{p} A_i \rightarrow \bigcup_{i=1}^{p} A_i\) a cyclic mapping. Assume that there exists \(k \in (0, 1)\) such that

\[d(Tx, Ty) \leq kd(x, y) \quad \forall x \in A_i \text{ and } y \in A_{i+1}.\]

Then \(T\) has a unique fixed point.

For a detailed study of cyclic mappings, we refer to [4 -13] and references thereof.

Recently, Karapinar and Sadarangani [12] (see also [11]) combined the ideas of \((\psi, \varphi)\)-weakly contractions, and cyclic contractions and introduced the notion of cyclic weak \((\psi, \varphi)\)-contraction as follows:

**Definition 1.4.** Let \(A_1, A_2, ..., A_p\) be nonempty subsets of a metric space \((X, d)\) such that \(X = \bigcup_{i=1}^{p} A_i\). A mapping \(T : X \rightarrow X\) is said to be cyclic weak \((\psi, \varphi)\)-contraction if

\[
\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))
\]

for all \(x, y \in X\), where \(\psi, \varphi \in \Phi\). Then \(T\) has a unique fixed point.

For a detailed study of cyclic mappings, we refer to [4 -13] and references thereof.
where $\psi, \varphi \in \Phi$ and $A_{p+1} = A_1$.

**Example 1.5.** [11, Example 4]. Let $X = [-1, 1]$ with the usual metric, i.e., $d(x, y) = |x - y|$. Let $A_1 = [-1, 0] = A_3$ and $A_2 = [0, 1] = A_4$. Then $X = \bigcup_{i=1}^{4} A_i = [-1, 1]$. Define $T : X \to X$ by

$$Tx = \frac{-x}{3} \text{ for all } x \in X.$$  

It is clear that $T$ is a cyclic mapping on $X$. Further, if $\psi, \varphi : [0, \infty) \to [0, \infty)$ are defined by $\psi(t) = t$ and $\varphi(t) = t/2$, then $\psi, \varphi \in \Phi$ and $T$ is a cyclic weak $(\psi, \varphi)$-contraction.

Following theorem is the main result in [12].

**Theorem 1.6.** Let $(X, d)$ be a metric space and $A_1, A_2, ..., A_p$ nonempty closed subsets of $X$ such that $X = \bigcup_{i=1}^{p} A_i$. Let $T : X \to X$ be a cyclic weak $(\psi, \varphi)$-contraction. Then $T$ has a unique fixed point $z \in \bigcap_{i=1}^{p} A_i$.

In this paper we obtain two types of generalizations of the above theorem, One, for single valued mappings, and other for multi-valued mappings in a metric space. Our results extend and generalize certain fixed point theorems of [4], [11], [12] and others.

### 2. Generalized cyclic weak $(\psi, \varphi)$-contraction

First we extend Definition 1.2 as follows.

**Definition 2.1.** Let $A_1, A_2, ..., A_p$ be nonempty subsets of a metric space $(X, d)$. A cyclic mapping $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ will be called a Generalized cyclic weak $(\psi, \varphi)$-contraction if

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(M(x, y))$$  

for all $x \in A_i$ and $y \in A_{i+1}$, where $\psi, \varphi \in \Phi$, $A_{p+1} = A_1$ and

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$  

**Remark 2.2.** When $M(x, y) = d(x, y)$ in Definition 2.1, we recover Definition 1.4. Hence the class of generalized cyclic weak $(\psi, \varphi)$-contraction is larger than cyclic weak $(\psi, \varphi)$-contraction.
Now we present our first result.

**Theorem 2.3.** Let \( A_1, A_2, \ldots, A_p \) be nonempty closed subsets of a complete metric space \((X, d)\) and \( T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i \) be a generalized cyclic weak \((\psi, \varphi)\)-contraction on \( X \). Then \( T \) has a unique fixed point \( z \in \bigcap_{i=1}^p A_i \).

**Proof.** Suppose for some \( i \in \{1, 2, \ldots, p\} \) there exists an \( x \in A_i \) satisfying (2.1). Since for any \( n \in \mathbb{N} \), either \( n \) or \( n + 1 \) is even, we have

\[
\psi(d(T^n x, T^{n+1} x)) \leq \psi(M(T^{n-1} x, T^n x)) - \varphi(M(T^{n-1} x, T^n x)) 
\leq \psi(M(T^{n-1} x, T^n x)).
\]

Since \( \psi \) is nondecreasing, we have

\[
d(T^n x, T^{n+1} x) \leq \max\{d(T^{n-1} x, T^n x), d(T^n x, T^{n+1} x), \frac{d(T^{n-1} x, T^n x) + d(T^n x, T^{n+1} x)}{2}\}
\leq d(T^{n-1} x, T^n x).
\]

for \( n \in \mathbb{N} \). Thus \( \{d(T^n x, T^{n+1} x)\} \) is a decreasing sequence of nonnegative real numbers. If \( \lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0 \) then we are done. Suppose that \( \lim_{n \to \infty} d(T^n x, T^{n+1} x) = r \) for some \( r > 0 \). Making \( n \to \infty \) in (2.2) and using the continuity of \( \psi \) and \( \varphi \), we have

\[
\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r),
\]

which is a contradiction. Hence

\[
\lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0.
\]

We show that \( \{T^n x\} \) is a Cauchy sequence. Suppose \( \{T^n x\} \) is not Cauchy. Then there exists \( \mu > 0 \) and increasing sequences \( \{m_k\} \) and \( \{n_k\} \) of positive integers such that for all \( n \leq m_k < n_k \),

\[
d(T^{m_k} x, T^{n_k} x) \geq \mu \text{ and } d(T^{m_k} x, T^{n_k-1} x) < \mu.
\]

By the triangle inequality,

\[
d(T^{m_k} x, T^{n_k} x) \leq d(T^{m_k} x, T^{m_k-1} x) + d(T^{m_k-1} x, T^{n_k} x).
\]
It follows that \( \lim_{k \to \infty} d(T^{m_k}x, T^{n_k}x) = \mu \). Now by (2.1), we have

\[
\psi(d(T^{m_k+1}x, T^{n_k+1}x)) = \psi(d(TT^{m_k}x, TT^{n_k}x)) \\
\leq \psi(M(T^{m_k}x, T^{n_k}x)) - \varphi(M(T^{m_k}x, T^{n_k}x)) \\
\leq \psi(M(T^{m_k}x, T^{n_k}x)).
\]

Making \( k \to \infty \),

\[
\psi(\mu) \leq \psi(\mu) - \varphi(\mu) \leq \psi(\mu),
\]
a contradiction unless \( \mu = 0 \). Therefore \( \{T^n x\} \) is Cauchy. Since \( X \) is complete there exists a point \( z \in \bigcup_{i=1}^p A_i \) such that \( \{T^n x\} \) converges to \( z \). Now for some \( i \in \{1, 2, ..., p\} \) there exist sequences \( \{T^{2n} x\} \) and \( \{T^{2n-1} x\} \) in \( A_i \) and \( A_{i+1} \) respectively, with \( A_{p+1} = A_1 \), both converging to \( z \).

Using (2.1), we get

\[
\psi(d(T^{2n} x, Tz)) = \psi(d(TT^{2n-1} x, Tz)) \\
\leq \psi(M(T^{2n-1} x, z)) - \varphi(M(T^{2n-1} x, z)) \\
\leq \psi(M(T^{2n-1} x, z)).
\]

Making \( k \to \infty \), we get

\[
\psi(d(z, Tz)) \leq \psi(d(z, z)) = \psi(0) = 0,
\]
and \( \psi(d(z, Tz)) = 0 \). This implies \( d(z, Tz) = 0 \) and \( z = Tz \). Uniqueness of the fixed point follows easily. \( \square \)

**Corollary 2.4. Theorem 1.6.**

*Proof.* It comes from Theorem 2.3, when \( X = \bigcup_{i=1}^p A_i \) and \( M(x, y) = d(x, y) \). \( \square \)

**Corollary 2.5. Theorem 1.3.**

*Proof.* It comes from Theorem 2.3, when \( M(x, y) = d(x, y) \), \( \psi(t) = t \) and \( \varphi(t) = (1 - k)t \) where \( k \in (0, 1) \). \( \square \)
Corollary 2.6. [11, Theorem 6]. Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a complete metric space $(X, d)$ and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ a cyclic mapping such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

for all $x \in A_i$ and $y \in A_{i+1}$, where $\varphi \in \Phi$, $A_{p+1} = A_1$. Then $T$ has a unique fixed point $z \in \bigcap_{i=1}^{p} A_i$.

Proof. It comes from Theorem 2.3, when $M(x, y) = d(x, y)$, $\psi(t) = t$. \hfill \Box

The following example shows the generality of Theorem 2.3 over Theorems 1.3 and 1.6.

Example 2.7. Let $X = \{1, 2, 3, 4, 5\}$ endowed with the metric $d$ defined by

$$d(1, 2) = d(1, 3) = d(3, 5) = \frac{13}{8}, \quad d(1, 4) = \frac{3}{2}, \quad d(3, 4) = 2.$$ $$d(1, 5) = d(2, 4) = \frac{7}{4}, \quad d(2, 3) = d(4, 5) = 1, \quad d(2, 5) = \frac{15}{8}.$$ 

Suppose $A_1 = \{1, 2, 3\}$ and $A_2 = \{1, 4, 5\}$ then $A_1 \cup A_2 = X$. Consider a mapping $T : X \to X$ defined by

$$T1 = 1, \quad T2 = T3 = 4, \quad T4 = 1, \quad T5 = 2.$$ 

We define $\psi(t) = 2t$ and $\varphi(t) = \frac{t}{20}$ for all $t \geq 0$.

Observe that $T(A_1) = \{1, 4\} \subset A_2$ and $T(A_2) = \{1, 2\} \subset A_1$. It can be easily verified that $T$ satisfies all the hypotheses of Theorem 2.3 and $T1 = 1 \in A_1 \cap A_2$. However $T$ does not satisfy Theorems 1.3 and 1.6. For $x = 3, y = 5$ we have

$$d(Tx, Ty) = \frac{7}{4} > \frac{13}{8} - \frac{13}{160} = d(x, y) - \varphi(d(x, y)).$$

4. Multi-valued cyclic weak $(\psi, \varphi)$-contraction

Throughout this section $X$ denotes a metric space $(X, d)$, $CB(X)$ the collection of all nonempty closed and bounded subsets of $X$, $C(X)$ the collection of all nonempty compact subsets of $X$ and $H$ the Hausdorff metric induced by $d$, i.e.,

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\},$$
for all $A, B \subseteq CB(X)$, where $d(x, B) = \inf_{y \in B} d(x, y)$.

First we extend Definitions 1.2 and 1.4 for a multi-valued mapping.

**Definition 4.1.** Let $A_1, A_2, \ldots, A_p$ be nonempty subsets of a metric space $X$ such that $X = \bigcup_{i=1}^{p} A_i$. A mapping $T : X \to CB(X)$ is said to be a cyclic representation of $X$ with respect to $T$ if

$$Tx \subset A_{i+1} \text{ for all } x \in A_i, \text{ where } A_{p+1} = A_1.$$

**Definition 4.2.** Let $A_1, A_2, \ldots, A_p$ be nonempty subsets of a metric space $X$ such that $X = \bigcup_{i=1}^{p} A_i$. A mapping $T : X \to CB(X)$ will be called a multi-valued cyclic weak $(\psi, \varphi)$-contraction if

(i): $X = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of $X$ with respect to $T$;

(ii): $\psi(H(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))$ for all $x \in A_i$ and $y \in A_{i+1}$,

where $\psi, \varphi \in \Phi$ and $A_{p+1} = A_1$.

**Theorem 4.3.** Let $A_1, A_2, \ldots, A_p$ be nonempty closed subsets of a complete metric space $X$ such that $X = \bigcup_{i=1}^{p} A_i$. Let $T : X \to C(X)$ be a multi-valued cyclic weak $(\psi, \varphi)$-contraction on $X$. Then $T$ has a fixed point $z \in \bigcap_{i=1}^{p} A_i$.

**Proof.** We construct a sequence $\{x_n\}$ in $X$ in the following way. Let $x_0 \in A_1$ and $x_1 \in Tx_0 \subset A_2$. If $H(Tx_0, Tx_1) = 0$ then $x_1 \in Tx_1$ i.e., $x_1$ is fixed point of $T$ and we are done. Assume that $H(Tx_0, Tx_1) > 0$. There exits a point $x_2 \in Tx_1 \subset A_3$ such that $d(x_1, x_2) \leq H(Tx_0, Tx_1)$. Such a choice is admissible, since $Tx_1$ is compact (see Nadler Jr. [14, p. 480]). Since $Tx_2$ is compact, we choose a point $x_3 \in A_4$ such that $d(x_2, x_3) \leq H(Tx_1, Tx_2)$. Again, if $H(Tx_1, Tx_2) = 0$ then $x_2 \in Tx_2$ i.e., $x_2$ is fixed point of $T$. For $n > 0$ there exists $i_{n_0} \in \{1, 2, \ldots, p\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Continuing in the same manner for $n \in \mathbb{N}$, we get

$$d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n).$$
Since $T$ is a multi-valued cyclic weak $(\psi, \varphi)$-contraction, we have

\begin{equation}
\psi(d(x_n, x_{n+1})) \leq \psi(H(Tx_{n-1}, Tx_n)) \leq \psi(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n)) \\
\leq \psi(d(x_{n-1}, x_n)).
\end{equation}

(4.1)

Since $\psi$ is nondecreasing, we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

for $n \in \mathbb{N}$. Thus $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. Let $\lim_{n \to \infty} d(x_n, x_{n+1}) = r$ for some $r \geq 0$. Making $n \to \infty$ in (4.1) and using the continuity of $\psi$ and $\varphi$, we have

$$\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r),$$

which is a contradiction unless $r = 0$. Hence

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$

We show that $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ is not Cauchy. Then there exists $\mu > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that for all $n \leq m_k < n_k$,

$$d(x_{m_k}, x_{n_k}) \geq \mu \text{ and } d(x_{m_k}, x_{n_k-1}) < \mu.$$

By the triangle inequality,

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}).$$

It follows that, $\lim_{k \to \infty} d(x_{m_k}, x_{n_k}) = \mu$. Using (ii), we get

$$\psi(d(x_{m_k+1}, x_{n_k+1})) \leq \psi(H(Tx_{m_k}, Tx_{n_k})) \\
\leq \psi(d(x_{m_k}, x_{n_k})) - \varphi(d(x_{m_k}, x_{n_k})) \\
\leq \psi(d(x_{m_k}, x_{n_k})).$$

Making $k \to \infty$,

$$\psi(\mu) \leq \psi(\mu) - \varphi(\mu) \leq \psi(\mu),$$
a contradiction unless $\mu = 0$. Therefore $\{x_n\}$ has a limit in $X$. Call it $z$. By the property that $X = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of $X$ with respect to $T$, the sequence $\{x_n\}$ has infinite number of terms in each $A_i$ for $i \in \{1, 2, ..., p\}$. Suppose $z \in A_i$, $Tz \in A_{i+1}$ and we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ (the existence of this subsequence is guaranteed by the fact that $\{x_n\}$ has infinite number of terms in each $A_i$ for $i \in \{1, 2, ..., p\}$). Again by (ii), we have

$$
\psi(d(x_{n_k+1}, Tz)) \leq \psi(H(Tx_{n_k}, Tz)) \\
\leq \psi(d(x_{n_k}, z)) - \varphi(d(x_{n_k}, z)) \\
\leq \psi(d(x_{n_k}, z)).
$$

Making $k \to \infty$, we get

$$
\psi(d(z, Tz)) \leq \psi(d(z, z)) = \psi(0) = 0,
$$

and $\psi(d(z, Tz)) = 0$. This implies $d(z, Tz) = 0$ and $z \in Tz$. □

**Corollary 4.4.** Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a complete metric space $X$ such that $X = \bigcup_{i=1}^{p} A_i$. Let $T : X \to C(X)$ such that

$$
H(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))
$$

for all $x \in A_i$ and $y \in A_{i+1}$, where $\varphi \in \Phi$ and $A_{p+1} = A_1$. Then $T$ has a fixed point $z \in \bigcap_{i=1}^{p} A_i$.

**Proof.** It comes from Theorem 4.3, when $\psi(t) = t$. □

**Corollary 4.5.** Let $A_1, A_2, ..., A_p$ be nonempty closed subsets of a complete metric space $X$ such that $X = \bigcup_{i=1}^{p} A_i$. Let $T : X \to C(X)$ such that

$$
H(Tx, Ty) \leq kd(x, y)
$$

for all $x \in A_i$ and $y \in A_{i+1}$, where $k \in (0, 1)$ and $A_{p+1} = A_1$. Then $T$ has a fixed point $z \in \bigcap_{i=1}^{p} A_i$.

**Proof.** It comes from Theorem 4.3, when $\psi(t) = t$ and $\varphi(t) = (1-k)t$, where $k \in (0, 1)$. □
References


