A CLASS OF CAYLEY DIGRAPH STRUCTURES INDUCED BY LOOPS

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Abstract. In this paper, we generalize the results in [8] to produce a new classes of Cayley digraph structures induced by loops.

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1. Introduction

A binary relation on a set $V$ is a subset $E$ of $V \times V$. A digraph is a pair $(V, E)$ where $V$ is a non empty set (called vertex set) and $E$ is a binary relation on $V$. The elements of $E$ are called edges. Let $V$ be a non empty set and let $E_1, E_2, \ldots, E_n$ be mutually disjoint binary relations on $V$. Then the $(n + 1)$-tuple $G = (V; E_1, E_2, \ldots, E_n)$ is called a digraph structure[8]. The elements of $V$ are called vertices and the elements of $E_i$ are called $E_i$-edges. The following definition were introduced in [8].

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called (i)$E_1E_2 \cdot \cdot \cdot E_n$-trivial if $E_i = \emptyset$ for all $i$, and $E_i$- trivial if $E_i = \emptyset$ (ii)$E_1E_2 \cdot \cdot \cdot E_n$- reflexive if for all $x \in G$, $(x, x) \in E_i$ for some $i$, and $E_i$- reflexive if for all $x \in V$, $(x, x) \in E_i$(iii) $E_1E_2 \cdot \cdot \cdot E_n$- symmetric if $E_i = E_i^{-1}$ for

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all $i$, and $E_i$-symmetric if $E_i = E_i^{-1}$(iv) $E_1E_2\cdots E_n$-anti symmetric, if $(x, y) \in E_i$ and $(y, x) \in E_i$ implies $x = y$ for all $i$, and $E_i$- anti symmetric if $(x, y) \in E_i$ and $(y, x) \in E_i$ implies $x = y$(v) $E_1E_2\cdots E_n$-transitive if for every $i$ and $j$, $E_i \circ E_j \subseteq E_k$ for some $k$, and $E_i$ transitive if $E_i \circ E_i \subseteq E_i$(vi) an $E_1E_2\cdots E_n$ hasse diagram if for every positive integer $n \geq 2$ and every $v_0, v_1, \ldots, v_n$ of $V$, $(v_i, v_{i+1}) \in \cup E_i$ for all $i = 0, 1, 2, \ldots, n - 1$, implies $(v_0, v_n) \notin E_i$ for all $i$, and $E_i$-hasse diagram if for every positive integer $n \geq 2$ and every $v_0, v_1, \ldots, v_n$ of $V$, $(v_i, v_{i+1}) \in E_i$ for all $i = 0, 1, 2, \ldots, n - 1$, implies $(v_0, v_n) \notin E_i$(vii)$E_1E_2\cdots E_n$-complete if $\cup E_i = V \times V$, and $E_i$ complete if $E_i = V \times V$.

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called (i) an $E_1E_2\cdots E_n$-quasi ordered set if it is both $E_1E_2\cdots E_n$-reflexive and $E_1E_2\cdots E_n$-transitive (ii) an $E_1E_2\cdots E_n$-partially ordered set if it is $E_1E_2\cdots E_n$-anti symmetric and $E_1E_2\cdots E_n$-quasi ordered set. Similarly, we can define $E_i$ quasi ordered set and $E_i$ partially ordered set as in the case of ordinary relations.

An $E_1E_2\cdots E_n$-walk of length $k$ in a digraph structure is an alternating sequence $W = v_0, e_0, v_1, \ldots, e_{k-1}, v_k$, where $e_i = (v_i, v_{i+1}) \in \cup E_i$. An $E_1E_2\cdots E_n$-walk $W$ is called a $E_1E_2\cdots E_n$-path if all the internal vertices are distinct. We use notation $(v_0, v_1, v_2, \ldots, v_n)$ for the $E_1E_2\cdots E_n$-path $W$. As in digraphs, we define $E_i$-walk and $E_i$-path. For example, an $E_i$-path between two vertices $u$ and $v$ consists of only $E_i$-edges.

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called (i) $E_1E_2\cdots E_n$-connected if there exits at least one $E_1E_2\cdots E_n$-path from $v$ to $u$ for all $u, v \in V$, (ii)$E_1E_2\cdots E_n$-quasi connected if for every pair of vertices $x, y$ there is a vertex $z$ such that there is an $E_1E_2\cdots E_n$-path from $z$ to $x$ and an $E_1E_2\cdots E_n$-path from $z$ to $y$, (iii) $E_1E_2\cdots E_n$-locally connected iff for every pair of vertices $u, v \in V$ there is an $E_1E_2\cdots E_n$-path from $v$ to $u$ whenever there is an $E_1E_2\cdots E_n$-path from $u$ to $v$ and (iv) $E_1E_2\cdots E_n$-semi connected for every pair of vertices $u, v$, there is an $E_1E_2\cdots E_n$-path from $u$ to $v$ or an $E_1E_2\cdots E_n$-path from $v$ to $u$.

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called $E_i$-connected if there exits at least one $E_i$ path from $v$ to $u$ for all $u, v \in V$. Similarly we can define $E_i$ quasi connected, $E_i$
-locally connected and $E_i$ - semi connected digraph structures.

The $E_1E_2\cdots E_n$ - distance between two vertices $x$ and $y$ in a digraph structure $G$ is the length of the shortest $E_1E_2\cdots E_n$- path between $x$ and $y$, denoted by $d_{1,2,3,\ldots,n}(x,y)$. Let $G = (V; E_1, E_2, \ldots, E_n)$ be a finite $E_1E_2\cdots E_n$- connected digraph structure. Then the $E_1E_2\cdots E_n$ diameter of $G$ is defined as $d(G) = \max_{x,y\in G}\{d_{1,2,3,\ldots,n}(x,y)\}$. Similarly we can define $E_i$ distance and $E_i$ diameter as in digraphs.

Two digraph structures $(V_1; E_1, E_2, \ldots, E_n)$ and $(V_2; R_1, R_2, \ldots, R_m)$ are said to be isomorphic if (i) $m = n$ and (ii) there exists a bijective function $f: V_1 \to V_2$ such that $(x,y) \in E_i \iff (f(x), f(y)) \in R_i$. This concept of isomorphism is a generalization of isomorphism between two digraphs. An isomorphism of a digraph structure onto itself is called an automorphism. A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is said to be vertex-transitive if, given any two vertices $a$ and $b$ of $V$, there is some digraph automorphism $f: V \to V$ such that $f(a) = b$. Let $(V; E_1, E_2, \ldots, E_n)$ be a digraph structure and let $v \in V$. Then the $E_1E_2\cdots E_n$ out-degree of $u$ is $|\{v \in V : (u, v) \in \cup E_i\}|$ and $E_1E_2\cdots E_n$ in-degree of $u$ is $|\{v \in V : (v, u) \in \cup E_i\}|$. Similarly we can define the $E_i$ out-degree and $E_i$ in-degree as in the case of digraphs.

Let $(V_1; E_1, E_2, \ldots, E_n)$ be a digraph structure. A vertex $v \in G$ is called an $E_1E_2\cdots E_n$ -source if for every vertex $x \in G$, there is an $E_1E_2\cdots E_n$ - path from $v$ to $x$. Similarly a vertex $u \in G$ is called an $E_1E_2\cdots E_n$ - sink if for every vertex $y \in G$ there is an $E_1E_2\cdots E_n$ - path from $y$ to $u$. As in digraphs, we define $E_i$ - source and $E_i$ - sink. Let $(V_1; E_1, E_2, \ldots, E_n)$ be a digraph structure and let $v \in G$. Then the $E_1E_2\cdots E_n$ reachable set $R_{1,2,3,\ldots,n}(u)$ is $\{x \in G : \text{ there is an } E_1E_2\cdots E_n - \text{ path from } u \text{ to } x\}$. Similarly, the $E_1E_2\cdots E_n$ - antecedent set $Q_{1,2,\ldots,n}(u)$ is defined as

$$Q_{1,2,\ldots,n}(u) = \{x \in G : \text{ there is an } E_1E_2\cdots E_n - \text{ path from } x \text{ to } u\}.$$  

As in the case of digraphs, we can define the $E_i$- reachable set and $E_i$- antecedent set of a vertex.

A non empty set $G$, together with a mapping $*: G \times G \longrightarrow G$ is called a groupoid. The mapping $*$ is called a binary operation on the set $G$. If $a, b \in G$, we use the symbol
$ab$ to denote $*(a,b)$. A groupoid $(G,*)$ is called a quasigroup, if for every $a, b \in G$, the equations, $ax = b$ and $ya = b$ are uniquely solvable in $G$ [6]. This implies both left and right cancelation laws. A quasigroup with an identity element is called a loop. Observe that a loop is a weaker algebraic structure than a group.

A subset $A$ of a loop $G$ is said to be a right associative subset of $G$ ($\mathcal{R}$ associative), if for every $x, y \in G$, $(xy)A = x(yA)$. This means, if $x, y \in G$ and $a \in A$, then $(xy)a = x(ya')$ for some $a' \in A$. Observe that the $\mathcal{R}$ associative law not only allows to interchange the positions of parenthesis, the two elements that are on the left should be in $G$ and they will be same on both sides, the rightmost element in the left hand side is in $A$ and is changed to another element $a' \in A$ as the right most element in the right side [12].

Here we have the following result:

**Theorem 1.1.** ([9]) Let $A$ and $B$ be $\mathcal{R}$ associative subsets of a loop $G$. Then $AB$ is also $\mathcal{R}$ associative.

### 3. Cayley digraph structures induced by loops

In [11] the authors introduced a class of Cayley digraph structures induced by groups. In this paper, we introduce a class of Cayley digraph structures induced by loops. These class of Cayley digraphs structures can be viewed as a generalization of those obtained in [11]. Further, many graph properties are studied in terms of algebraic properties.

We start with the following definition:

**Definition 2.1.** Let $G$ be a loop and $S_1, S_2, \ldots, S_n$ be mutually disjoint $\mathcal{R}$ associative subsets of $G$. Then Cayley digraph structure of $G$ with respect to $S_1, S_2, \ldots, S_n$ is defined as the digraph structure $X = (G; E_1, E_2, \ldots, E_n)$, where

$$E_i = \{(x, y) : z \in S_i\}$$

where $z$ denotes the solution of the equation $y = xz$. 

The sets \( S_1, S_2, \ldots, S_n \) are called connection sets of \( X \). The Cayley digraph structure of \( G \) with respect to \( S_1, S_2, \ldots, S_n \) is denoted by \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \).

In this paper we may use the following notations:

1. Let \( S_1, S_2, \ldots, S_n \) be subsets of a loop \( G \), then we may define the product \( S_1, S_2, \ldots, S_n \) as follows:

\[
S_1S_2\ldots S_n = \{(((s_1s_2)s_3)\ldots)s_n : s_i \in S_i, i = 1, 2, \ldots, n\}.
\]

If \( S_1 = S_2 = \cdots = S_n = S \), we denote the above product as \( S^n \).

2. Let \( A_k \) be the union of set of all \( k \) products of the form \( S_{i_1}S_{i_2}\cdots S_{i_k} \) from the set \( \{S_1, S_2, \ldots, S_n\} \). Then \( \bigcup_k A_k \) is denoted by \([S]\).

3. Let \( D \) be a subset of \( G \). We define \( D_\ell = \{z_\ell : z_\ell z = 1 \text{ for some } z \in D\} \), where 1 is the identity element in \( G \).

4. Let \( A \) be a subset of a loop \( G \), then the semi group generated by \( A \) is denoted by \(< A >\).

**Theorem 2.2.** If \( G \) is a loop and let \( S_1, S_2, \ldots, S_n \) are mutually disjoint \( R \)-associative subsets of \( G \), then the Cayley digraph structure \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is vertex transitive.

**Proof.** Let \( a \) and \( b \) be any two arbitrary elements in \( G \). Define a mapping \( \varphi : G \to G \) by

\[
\varphi(x) = (b/a)x \text{ for all } x \in G.
\]

where \((b/a)\) denotes the solution of the equation \( b = za \). This mapping defines a permutation of the vertices of \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \). It is also an automorphism. Let \( x, y \in G \) such that \( y = xz \). Note that

\[
(x, y) \in E_i \iff z \in S_i \text{ for some } i.
\]

The equation \( y = xz \) can be written as

\[
(b/a)y = (b/a)(xz)
\]

\[
= ((b/a)x)z' \text{ for some } z' \in S_i.
\]
The above equation tells us that \(((b/a)x, (b/a)y) \in E_i\). That is, \((\varphi(x), \varphi(y)) \in E_i\). Similarly, assume that \((\varphi(x), \varphi(y)) \in E_i\). Then \((b/a)y = ((b/a)x)z\) for some \(z \in S_i\). This implies that \((b/a)y = (b/a)(xz')\) for some \(z' \in S_i\). By left cancellation law, we obtain \(y = xz'\). This tells us that \((x, y) \in E_i\). Also we note that \(\varphi(a) = (b/a)a = b\). Hence \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is vertex transitive.

**Proposition 2.3** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2 \cdots E_n\)-trivial digraph structure if and only if \(S_i = \emptyset\) for all \(i\).

**Proof.** By definition, \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2 \cdots E_n\)-trivial if and only if \(E_i = \emptyset\) for all \(i\). This implies that \(S_i = \emptyset\) for all \(i\).

**Proposition 2.4** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is an \(E_i\)-trivial digraph structure if and only if \(S_i = \emptyset\).

**Proposition 2.5** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2 \cdots E_n\)-reflexive if and only if \(1 \in S_i\) for some \(i\).

**Proof.** Assume that \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2 \cdots E_n\)-reflexive digraph structure. Then for every \(x \in G\), \((x, x) \in E_i\) for some \(i\). This implies that the equation \(x = xz\) has a unique solution in \(S_i\) for some \(i\). That is, \(1 \in S_i\) for some \(i\).

Conversely, assume that \(1 \in S_i\) for some \(i\). This implies for each \(x \in G\), \((x, x) \in E_i\) for some \(i\). That is, \((x, x) \in \bigcup E_i\) for all \(x \in G\).

**Proposition 2.6** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2 \cdots E_n\)-symmetric if and only if \(S_i = S_i^\ell\) for all \(i\).

**Proof.** First, assume that \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2 \cdots E_n\)-symmetric digraph structure. Let \(a \in S_i\). Then \((1, a) \in E_i\). Since \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is symmetric \((a, 1) \in E_i\). This implies that the equation \(1 = at\) has a solution in \(S_i\). That is \(a \in S_i^\ell\). Hence \(S_i \subseteq S_i^\ell\). Similarly, we can prove that \(S_i^\ell \subseteq S_i\).

Conversely, assume that \(S_i = S_i^\ell\) for all \(i\). Suppose that \((x, y) \in E_i\). Then the equation \(y = xz\) has a solution in \(S_i\). That is \(z \in S_i\). Consider the equation \(x = yt\). This equation
can be written as:

$$xz = (yt)z$$

i.e., $y = y(tz')$ for some $z' \in S_i$

i.e., $y1 = y(tz')$

i.e., $1 = tz'$ (by left cancelation law).

The above equation tells us that $t \in S_i$. Since $S_i = S_{it}$, it follows that $t \in S_i$. Hence the equation $x = yt$ has a solution in $S_i$. That is $(y, x) \in E_i$.

**Proposition 2.7** Cay($G; S_1, S_2, \ldots, S_n$) is an $E_1 E_2 \cdots E_n$-transitive if and only if for every $i, j$, $S_i S_j \subseteq S_k$ for some $k$.

**Proof.** First, assume that Cay($G; S_1, S_2, \ldots, S_n$) is $E_1 E_2 \cdots E_n$-transitive. Let $x \in S_i S_j$. Then $x = z_1 z_2$ for some $z_1 \in S_i$ and $z_2 \in S_j$. This implies that $(1, z_1) \in E_i$ and $(z_1, z_1 z_2) \in E_j$. Since Cay($G; S_1, S_2, \ldots, S_n$) is transitive $(1, z_1 z_2) \in E_k$ for some $k$. That is $z_1 z_2 \in S_k$. Hence $S_i S_j \subseteq S_k$ for some $k$.

Conversely assume that for each $i, j$, $S_i S_j \subseteq S_k$ for some $k$. Let $x, y$ and $z \in G$ such that $y = x t_1$ and $z = y t_2$. If $(x, y) \in E_i$ and $(y, z) \in E_j$, then $t_1 \in S_i$ and $t_2 \in S_j$. Note that the equation $z = y t_2$ can be written as:

$$z = (xt_1) t_2$$

$$= x (t_1 t'_2) \text{ for some } t'_2 \in S_j$$

$$= x t_3 \text{ where } t_3 = t_1 t'_2$$

Note that $t_3 \in S_i S_j$. Since $S_i S_j \subseteq S_k$, $t_3 \in S_k$. That the equation $z = x t$ has a solution $t_3$ in $S_k$. Hence Cay($G; S_1, S_2, \ldots, S_n$) is transitive.

**Proposition 2.8** Cay($G; S_1, S_2, \ldots, S_n$) is $E_1 E_2 \cdots E_n$-complete if and only if $G = \cup S_i$.

**Proof.** Suppose Cay($G; S_1, S_2, \ldots, S_n$) is $E_1 E_2 \cdots E_n$-complete. Then for every $x \in G$, we have $(1, x) \in \cup E_i$. This implies that $x \in S_i$ for some $i$. This implies that $G = \cup S_i$. Conversely, assume that $G = \cup S_i$. Let $x$ and $y$ be two arbitrary elements in $G$ such that
Proposition 2.9 \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is \( E_i \)-complete if and only if \( G = S_i \).

Proposition 2.10 \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is \( E_1 E_2 \cdots E_n \)-connected if and only if \( G = [S] \).

Proof. Suppose \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is \( E_1 E_2 \cdots E_n \)-connected and let \( x \in G \). Let \((1, y_1, y_2, \ldots, y_k, x)\) be a \( E_1 E_2 \cdots E_n \)-path leading from 1 to \( x \). Then we have, \( y_1 = z_1, y_2 = y_1 z_2, \ldots, y_k = y_{k-1} z_k, x = y_k z_{k+1} \) for some \( z_j \in S_{i_j}, j = 1, 2, \ldots, k + 1 \). Note that the equation \( x = y_k z_{k+1} \) can be written as

\[
x = (y_{k-1} z_k) z_{k+1} \\
= ((y_{k-2} z_{k-1}) y_{k-1} z_k) z_{k+1} \\
= (z_1 z_2) \cdots z_{k+1}
\]

The last equation tells us that \( x \in S_{i_1} S_{i_2} \cdots S_{i_{k+1}} \). This implies that \( x \in A \) for some \( A \in [S] \). Since \( x \) is arbitrary, \( G = [S] \).

Conversely, assume that \( G = [S] \). Let \( x \) and \( y \) be any arbitrary elements in \( G \). Let \( y = xz \). Then \( z \in G \) Then \( z \in S_i S_j \cdots S_k \) for some \( i, j, \ldots \) and \( k \). This implies that \( z = s_i s_j \cdots s_k \) for some \( i, j \ldots \) and \( k \). Then clearly, \((1, s_i, s_i s_j, \ldots, s_i s_j \cdots s_k)\) is an \( E_1 E_2 \cdots E_n \)-path from 1 to \( z \). That is, \((x, x s_i, x s_i s_j, \ldots, x s_i s_j \cdots s_k)\) is a \( E_1 E_2 \cdots E_n \)-path from \( x \) to \( y \). Hence \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is connected.

Proposition 2.11 \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is \( E_i \)-connected if and only if \( G = < S_i > \), where \( < S_i > \) is the semi group generated by the set \( S_i \).

Proposition 2.12 \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is \( E_1 E_2 \cdots E_n \)-quasi connected if and only if \( G = [S]_t[S] \).

Proof. First, assume that \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is quasi connected. Let \( x \) be any arbitrary element in \( G \). Then there exits a vertex \( y \in G \) such that there is a path from \( y \) to 1, say, \((y, y_1, y_2, \cdots, y_n, 1)\) and a path from \( y \) to \( x \), say,\((y, x_1, x_2, \ldots, x_m, x)\). Then we have
the following system of equations:

\[ y_1 = yz_1 \text{ for some } z_1 \in S_{i_1} \]
\[ y_2 = y_1z_2 \text{ for some } z_2 \in S_{i_2} \]
\[ y_3 = y_2z_3 \text{ for some } z_3 \in S_{i_3} \]
\[ \vdots \]
\[ 1 = y_nz_{n+1} \text{ for some } z_{n+1} \in S_{i_{n+1}}. \]

(1)

and

\[ x_1 = yt_1 \text{ for some } z_1 \in S_{i_1} \]
\[ x_2 = x_1t_2 \text{ for some } z_2 \in S_{i_2} \]
\[ x_3 = x_2t_3 \text{ for some } z_3 \in S_{i_3} \]
\[ \vdots \]
\[ x = x_m t_{m+1} \text{ for some } z_{m+1} \in S_{i_{m+1}}. \]

Observe that equation (1) can be written as:

\[ 1 = y(w_1w_2 \ldots w_{n+1}) \text{ for some } w_k \in S_{i_k}, k = 1, 2, \ldots, n+1. \]

This implies that

(3) \[ y \in [S]_{\ell} \]

Similarly, equation (2) can be written as:

(4) \[ x \in [S]_{\ell}[S] \]

From equations (4) and (5), we have

(6) \[ x \in [S]_{\ell}[S]. \]

Since \( x \) is arbitrary, \( G = [S]_{\ell}[S]. \)

Conversely, assume that \( G = [S]_{\ell}[S]. \) Let \( x \) and \( y \) be two arbitrary vertices in \( G. \) Let \( y = xz. \) Then \( z \in G. \) This implies that \( z \in [S]_{\ell}[S]. \) Then there exits \( z_1 \in [S]_{\ell} \) and \( z_2 \in [S] \) such that \( z = z_1z_2. \) \( z_1 \in [S]_{\ell} \) implies that there exits \( t_k \in S_{i_k} \) such that \( 1 = z_1(t_1t_2 \ldots t_m). \)
That is, \( 1 = ((z_1 r_1) r_2) \ldots r_m \) for some \( r_m \in S_{i_k}, k = 1, 2, \ldots, m \). This implies that \( (z_1, z_1 r_1, z_1 r_1 r_2, \ldots, 1) \) is a path from \( z_1 \) to 1. That is, 

\[ (yz_1, yz_1 r_1, yz_1 r_1 r_2, \ldots, y) \]

is a path from \( yz_1 \) to \( y \). Similarly, \( z_2 \in [S] \) implies that there exits \( a_k \in S_{i_k} \) such that \( z_2 = a_1 a_2 \ldots a_m \). Observe that \( (z_2, a_1 a_2 a_3, \ldots, 1) \) is a path from \( z_2 \) to 1. That is, \( (z_1 z_2, z_1 a_1 a_2, a_1 a_2 a_3, \ldots, z_1) \) is a path from \( z \) to \( yz_1 \). That is, 

\[ (yz, yz_1 a_1 a_2, ya_1 a_2 a_3, \ldots, yz_1) \]

is a path from \( x \) to \( yz_1 \). This implies that the digraph structure \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is \( E_1 E_2 \cdots E_n \)- quasi connected.

**Proposition 2.13** \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is \( E_i \) quasi connected if and only if \( G = < S_i > \).

**Proposition 2.14** \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is \( E_1 E_2 \cdots E_n \)- locally connected if and only if \( [S] = [S]_\ell \).

**Proof.**

Assume that \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is \( E_1 E_2 \cdots E_n \)- locally connected. Let \( x \in [S] \). Then \( x \in A_m \) for some \( m \). Then \( x = s_i s_j \ldots s_m \). Let \( x_0 = 1, x_1 = s_i, x_2 = s_i s_j, \ldots, x_m = s_i s_j \ldots s_m \). Then

\[ (x_0, x_1, x_2, \ldots, x_m) \]

is a path leading from 1 to \( x \). Since \( \text{Cay}(G; S_1, S_2, \ldots, S_m) \)- is locally connected, there exits a path from \( x \) to 1, say:

\[ (x, y_1, y_2, \ldots, y_m, 1) \]

This implies that

\[ y_1 = xt_1 \text{ for some } t_1 \in S_{i_1} \]

\[ y_2 = y_1 t_2 \text{ for some } t_2 \in S_{i_2} \]

\[ \vdots \]

\[ 1 = y_m t_{m+1} \text{ for some } t_{m+1} \in S_{i_n} \]

This implies that \( 1 = x(z_1 z_2 \cdots z_m) \) for some \( z_k \in S_{i_k}, k = 1, 2, 3, \ldots (m + 1) \). That is \( x \in [S]_\ell \). Hence \( [S] \subseteq [S]_\ell \). Similarly, one can prove that \( [S]_\ell \subseteq [S] \). Hence \( [S] = [S]_\ell \).
Conversely, if \([S] = [S]_\ell\), one can easily verify that \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2\cdots E_n\)-locally connected.

**Proposition 2.15** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_i\)-locally connected if and only if \(<S_i>=<S_i>\ell\).

**Proposition 2.16** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2\cdots E_n\)-semi connected if and only if \(G = [S] \cup [S]_\ell\).

**Proof.** Assume that \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2\cdots E_n\)-semi connected and let \(x \in G\). Then there is a path from 1 to \(x\), say: \((1, x_1, x_2, \ldots, x_k, x)\) or a path from \(x\) to 1, say: \((x, y_1, y_2, \ldots, y_m, 1)\). This implies that \(x \in [S]\) or \(x \in [S]_\ell\). This implies that \(G = [S] \cup [S]_\ell\).

Similarly, if \(G = [S] \cup [S]_\ell\), then one can prove that \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2\cdots E_n\)-semi connected.

**Proposition 2.17** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_i\)-semi connected if and only if \(G = <S_i> \cup <S_i>\ell\).

**Proposition 2.18** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2\cdots E_n\)-quasi ordered set if and only if

(i) \(1 \in S_1 \cup S_2 \cdots \cup S_n\),

(ii) for every \((i, j)\), \(S_i S_j \subseteq S_k\) for some \(k\).

**Proposition 2.19** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is an \(E_i\)-quasi ordered set if and only if \(1 \in S_i\), and \(S_i^2 \subseteq S_i\).

**Proposition 2.20** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2\cdots E_n\)-partially ordered set if and only if

(i) \(1 \in S_1 \cup S_2 \cdots \cup S_n\),

(ii) for every \((i, j)\), \(S_i S_j \subseteq S_k\) for some \(k\),

(iii) \(\cup (S_i \cap S_i^2) = \{1\}\).
Proof. Observe that

\[ x \in \bigcup (S_i \cap S_{i\ell}) \iff x \in (S_i \cap S_{i\ell}) \text{ for some } i \]
\[ \iff x \in S_i \text{ and } x \in S_{i\ell} \]
\[ \iff (1, x) \in E_i \text{ and } (x, 1) \in E_i \]
\[ \iff x = 1 \]

From these equivalences, the result follows.

Proposition 2.21 Cay(G; S_1, S_2, \ldots, S_n) if an E_i- partially ordered set if and only if

(i) \( 1 \in S_i \)
(ii) \( S_i^2 \subseteq S_i \)
(iii) \( S_i \cap S_{i\ell} = \{1\} \)

Proposition 2.22 Let \( A_m (m \geq 2) \) be the set of all \( m \) products of the form \( S_{i1}S_{i2} \cdots S_{im} \). Then Cay(G; S_1, S_2, \ldots, S_n) is an \( E_1E_2 \cdots E_n \)- hasse diagram if and only if \( C \cap S_i = \emptyset \) for all \( i \) and for all \( C \in A_m \).

Proof. Suppose the condition holds. Let \( x_0, x_1, \ldots, x_m \) be \( (m + 1) \) elements in \( G \) such that \( (x_i, x_{i+1}) \in \bigcup E_i \) for \( i = 0, 1, \ldots, m - 1 \). This implies that

\[ x_1 = x_0 t_1 \text{ for some } t_1 \in S_{i_1} \]
\[ x_2 = x_1 t_2 \text{ for some } t_2 \in S_{i_2} \]
\[ x_3 = x_2 t_3 \text{ for some } t_3 \in S_{i_3} \]
\[ \vdots \]
\[ x_m = x_{m-1} t_n \text{ for some } t_n \in S_{i_m} \]
The last equation can be written as:

\[ x_n = ((x_{n-2}t_{m-1}))t_m \]

\[ = ((x_0t_1)t_2) \cdots t_n \]

\[ = x_0(z_1z_2 \cdots z_m) \text{ for some } z_k \in S_{i_k}, k = 1, 2, \ldots, m \]

\[ = x_0t, \text{ where } t = z_1z_2 \cdots z_m \in A_m \]

Since \( C \cap S_i = \emptyset \) for all \( i \) and for all \( C \in A_m, (x_0, x_m) \notin \cup E_i \).

Conversely, assume that \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is an \( E_1E_2 \cdots E_n \)-hasse diagram. We will show that \( C \cap S_i = \emptyset \) for all \( i \) and for all \( C \in A_m \). Let \( S_{i_1}S_{i_2}S_{i_3} \cdots S_{i_m} \) be any element in \( A_m \). Let \( x \in S_{i_1}S_{i_2}S_{i_3} \cdots S_{i_m} \). Then \( x = s_{i_1}s_{i_2}s_{i_3} \cdots s_{i_m} \) for some \( s_{i_k} \in S_{i_k} \). This implies that \( (1, s_{i_1}, s_{i_2}s_{i_3}, \ldots, x) \) is a path from 1 to \( x \). Since \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is an \( E_1E_2 \cdots E_n \)-hasse diagram, \( x \notin S_i \) for any \( i \). That is, \( A_m \cap S_i = \emptyset \) for all \( i \).

**Proposition 2.23** Let \( A_m \ (m \geq 2) \) be the set of all \( m \) products of the form \( S_{i_1}S_{i_2} \cdots S_{i_m} \). Then \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is an \( E_i \)-hasse diagram if and only if \( S_i^m \cap S = \emptyset \), for all \( m \geq 2 \).

**Proposition 2.24** The \( E_1E_2 \cdots E_n \) out-degree of \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is the cardinal number \( |S_1 \cup S_2 \cup \cdots \cup S_n| \).

**Proof.** Since by Theorem 2.2, \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is vertex-transitive it suffices to consider the out-degree of the vertex \( 1 \in G \). Observe that

\[ \rho(1) = \{ u : (1, u) \in \cup E_i \} \]

\[ = \{ u : u \in S_i \text{ for some } i \} \]

\[ = S_1 \cup S_2 \cup \cdots \cup S_n \]

Hence \( |\rho(1)| = |S_1 \cup S_2 \cup \cdots \cup S_n| \).

**Proposition 2.25** The \( E_i \) out-degree of \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is the cardinal number \( |S_i| \).
Proposition 2.26 The \( E_1E_2 \cdots E_n \) in-degree of \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is the cardinal number \( |S_{1\ell} \cup S_{2\ell} \cup \cdots \cup S_{n\ell}| \).

Proof. Since \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is vertex-transitive it suffices to consider the in-degree of the vertex \( 1 \in G \). Observe that

\[
\sigma(1) = \{ u : (u, 1) \in \cup E_i \} = \{ u : (u, 1) \in E_i \} = \{ u : 1 = uz \text{ for some } z \in S_i \} = \{ z_\ell : z_\ell \in S_{i_\ell} \text{ for some } i \} = S_{1\ell} \cup S_{2\ell} \cup \cdots \cup S_{n\ell}.
\]

Hence \( |\sigma(1)| = |S_{1\ell} \cup S_{2\ell} \cup \cdots \cup S_{n\ell}| \).

Proposition 2.27 The \( E_i \) in-degree of \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is the cardinal number \( |S_{i\ell}| \).

Proposition 2.28 For \( k \geq 1 \), let \( A_k \) be the set of all \( k \) products of the form \( S_{i_1}S_{i_2}S_{i_3} \cdots S_{i_k} \). If \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) has finite diameter, then the \( E_1E_2 \cdots E_n \) diameter of the Cayley digraph structure \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is the least positive integer \( m \) such that \( G = A_m \).

Proof. Let \( m \) be the smallest positive integer such that \( G = A_m \). We will show that the diameter of \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is \( m \). Let \( x \) and \( y \) be any two arbitrary elements in \( G \) such that \( y = xz \). Then \( z \in G \). This implies that \( x \in A_m \). But then \( z \) has a representation of the form \( z = s_{i_1}s_{i_2} \cdots s_{i_m} \). This implies that \( (1, s_{i_1}, s_{i_1}s_{i_2}, \ldots, y) \) is a path of \( m \) edges from \( 1 \) to \( z \). That is, \( (x, xs_{i_1}, xs_{i_1}s_{i_2}, \ldots, y) \) is a path of length \( m \) from \( x \) to \( y \). This shows that \( d_{1,2,\ldots,n}(x, y) \leq m \). Since \( x \) and \( y \) are arbitrary, \( \max_{x,y \in G} \{ d_{1,2,\ldots,n}(x, y) \} \leq m \). Therefore the diameter of \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is less than or equal to \( m \). On the other hand let the diameter of \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) be \( k \). Let \( x \in G \) and \( d_{1,2,\ldots,n}(1, x) = k \). Then we have \( x \in B \) for some \( B \in A_k \). That is, \( G = A_k \). Now by the minimality of \( k \), we have \( m \leq k \). Hence \( k = m \).
Proposition 2.29 If $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ has finite diameter, then the $E_i$ diameter of the Cayley digraph structure $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is the least positive integer $m$ such that $G = S_i^m$.

Proposition 2.30 The vertex 1 is an $E_1 E_2 \cdots E_n$-source of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ if and only if $G = [S]$.

Proof. First, assume that 1 is an $E_1 E_2 \cdots E_n$-source of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$. Then for any vertex $x \in G$, there is an $E_1 E_2 \cdots E_n$-path from 1 to $x$. This implies that $G = [S]$. Conversely, if $G = [S]$, one can prove that 1 is an $E_1 E_2 \cdots E_n$-source.

Proposition 2.31 The vertex 1 is an $E_i$ source of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ if and only if $G = <S_i>$.

Proposition 2.32 The vertex 1 is an $E_1 E_2 \cdots E_n$-sink of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ if and only if $G = [S]_{\ell}$.

Proof. First, assume that 1 is an $E_1 E_2 \cdots E_n$-sink of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$. Then for each $x \in G$, there is an $E_1 E_2 \cdots E_n$-path from $x$ to 1. This implies that $x \in [S]_{\ell}$. Hence $G = [S]_{\ell}$. Conversely, if $G = [S]_{\ell}$, one can easily prove that 1 is an $E_1 E_2 \cdots E_n$-sink of the Cayley digraph structure $\text{Cay}(G; S_1, S_2, \ldots, S_n)$.

Proposition 2.33 The vertex 1 is an $E_i$ sink of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ if and only if $G = <S_i>_{\ell}$.

Proposition 2.34 The $E_1 E_2 \cdots E_n$ reachable set $R_{1,2,\ldots,n}(1)$ of the vertex 1 is the set $[S]$.

Proof. By definition, $R(1) = \{x : \text{there exists an } E_1 E_2 \cdots E_n \text{-path from 1 to } x\}$. Observe that

$x \in R_{1,2,\ldots,n}(1) \iff \text{there exists an } E_1 E_2 \cdots E_n \text{-path from 1 to } x, \text{ say } (1, x_1, x_2, \ldots, x_n, x) \\
\iff x \in [S]$. 
Therefore, $R_{1,2,3,...,n}(1) = [S]$.

**Proposition 2.35** The $E_i$ reachable set $R_i(1)$ of the vertex 1 is the set $<S_i>$.

**Proposition 2.36** The $E_1E_2\cdots E_n$ antecedent set $Q_{1,2,...,n}(1)$ of the vertex 1 is the set $[S]_\ell$.

**Proof.** Observe that

$x \in Q_{1,2,...,n}(1) \iff$ there exits an $E_1E_2\cdots E_n$-path from $x$ to 1, say $(x,x_1,x_2,...,x_n,1)$

$\iff x \in [S]_\ell$

Therefore, $Q_{1,2,...,n}(1) = [S]_\ell$.

**Proposition 2.37** The $E_i$ antecedent set $Q_i(1)$ of the vertex 1 is the set $<S_i>_{\ell}$.

**References**


