AN UPPER BOUND ON THE NUMBER OF EDGES OF GRAPHS CONTAINING NO $r$ VERTEX-DISJOINT ODD CYCLES

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Abstract. In [5], we found an upper bound on the number of edges, $\mathcal{E}(G)$, of a graph $G$ containing no $r$ vertex-disjoint cycles of length 3. In this paper we generalize this result to graphs containing no $r$ vertex-disjoint cycles of length $2k+1$. We showed that $\mathcal{E}(G) \leq \left\lceil \frac{(n-r+1)^2}{4} \right\rceil + (r-1)(n-r+1)$ for every $G \in \mathcal{G}(n, V_r, 2k+1)$, the class of all graphs on $n$ vertices containing no $r$ vertex-disjoint cycles of length $2k+1$. Determination of the maximum number of edges in a given graph that contains no specific subgraphs is one of the important problems in graph theory. Solving such problems has attracted the attention of many researchers in graph theory.

Keywords: upper bound; number of edges; graph theory.

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1. Introduction

In this paper, we only consider simple graphs. That is, graphs that has no loops or multiple edges. Let $V(G)$ denote the set of vertices of a graph $G$ and $E(G)$ be the set of edges of $G$. If an edge $e \in E(G)$ is incident with the two vertices $u$ and $v$ in $V(G)$, we write $e = uv = vu$. For
a vertex \( u \in V(G) \) we denote the neighborhood of \( u \) by \( N_G(u) \), which is the set of all vertices \( v \in V(G) \) such that \( uv \in E(G) \). For a vertex \( u \in V(G) \), we define the degree \( d_G(u) \) to be the number of edges incident with \( u \).

For vertex-disjoint subgraphs \( H_1 \) and \( H_2 \) of \( G \), we let \( E(H_1,H_2) \) to be the set of all edges that are incident to a vertex in \( H_1 \) and a vertex in \( H_2 \). That is \( E(H_1,H_2) = \{ uv \in E(G) \mid u \in V(H_1), v \in H_2 \} \). We also define \( \bar{\epsilon}(G) \) to be the number of edges of \( G \). That is, \( \bar{\epsilon}(G) \) equals the \( |E(G)| \) and \( \bar{\epsilon}(H_1,H_2) = |E(H_1,H_2)| \). The cycle on \( n \) vertices is denoted by \( C_n \) and the complete tripartite graph with partitioning sets of order \( m, n \) and \( k \) is denoted by \( K_{m,n,k} \). For given graphs \( G_1 \) and \( G_2 \) we denote the union of \( G_1 \) and \( G_2 \) by \( G_1 \cup G_2 \) such that \( V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \). We also denote the joint of \( G_1 \) and \( G_2 \) by \( G_1 \vee G_2 \) such that \( V(G_1 \vee G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup E(G_1,G_2) \).

The determination of maximum number of edges in a given graph that has no specific subgraphs has attracted the attention of many graph theorists. For example, Höggkvist et al in [6] proved that \( \bar{\epsilon}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1 \) for a non bipartite graph \( G \) with \( n \) vertices that contains no odd cycle \( C_{2k+1} \) for all positive integers \( k \), Jia in [7] proved that \( \bar{\epsilon}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 3 \) for a nonpartite graph \( G \) with \( n \) vertices such that contains no odd cycle for \( n \geq 10 \), and Hailat in [5] proved that \( \bar{\epsilon}(G) \leq \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1) \) for every \( G \in \mathcal{G}(n,V_r,3) \).

In [2], M. Bataineh and M. Jaradat proved that \( \bar{\epsilon}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1 \) for any graph \( G \in \mathcal{J}(n;r,2k+1) \) for large \( n \) and \( r \geq 2, k \geq 1 \), where \( \mathcal{J}(n;r,2k+1) \) is the set of all graphs on \( n \) vertices containing no \( r \) edge-disjoint cycles of length \( 2k+1 \).

In this paper, we generalize the result of [5] to the case where \( G \) is a graph that contains no \( r \) vertex-disjoint cycle of length \( 2k+1 \). This result is parallel to the result of [1] in which the author considered the case of vertex-disjoint cycles instead of edge-disjoint cycles that was addressed in [2].

2. Important Lemmas and Theorems

In this section, we introduce the following results that will be used to prove the main theorem of this paper.
2.1. **Theorem** (Jia [7]). Let \( G \in \mathcal{G}(n, 5) \), \( n \geq 10 \). Then \( \varepsilon'(G) \leq \lfloor \frac{(n-2)^2}{4} \rfloor + 3 \).

2.2. **Theorem** (Batineh [1]). Let \( k \geq 3 \) be a positive integer and \( G \in \mathcal{S}(n; 2k + 1) \). Then for large \( n \), \( \varepsilon'(G) \leq \lfloor \frac{(n-2)^2}{4} \rfloor + 3 \).

Let \( \mathcal{G}(n, r, 2k+1) \) denote the class of graphs on \( n \) vertices containing no \( r \) edge-disjoint cycles of length \( 2k+1 \), and \( \mathcal{G}(n, V_r, 2k+1) \) denote the class of graphs on \( n \) vertices containing no \( r \) vertex-disjoint cycles of length \( 2k+1 \). Note that \( \mathcal{G}(n, V_r, 2k+1) \subseteq \mathcal{G}(n, r, 2k+1) \).

2.3. **Theorem** (Batineh and Jaradat [2]). Let \( G \in \mathcal{G}(n, 2, 3) \). Then for large \( n \), \( \varepsilon'(G) \leq \lfloor \frac{n^2}{4} \rfloor + 1 \). Furthermore, equality holds if and only if \( G \in \Omega(n, 2) = K_{1, \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil} \).

2.4. **Lemma** (Bondy and Murty [3]). Let \( G \) be a graph on \( n \) vertices. If \( \varepsilon'(G) > \frac{n^2}{4} \), then \( G \) contains a cycle of length \( 2k+1 \) for each \( 1 \leq k \leq \lfloor \frac{n+3}{2} \rfloor - \frac{1}{2} \).

2.5. **Theorem** (Batineh and Jaradat [2]). Let \( k \geq 1 \), \( r \geq 2 \) be two integers and \( g \in \mathcal{G}(n; r, 2k+1) \). For large \( n \), \( \varepsilon'(G) \leq \lfloor \frac{n^2}{4} \rfloor + r - 1 \). Furthermore, equality holds if and only if \( G \in \Omega(n, r) = K_{r-1, \lfloor \frac{n+r+1}{2} \rfloor, \lceil \frac{n-r+1}{2} \rceil} \).

Let \( \mathcal{S}(n, V_{2k+1}) \) denote the class of graphs on \( n \) vertices containing no vertex disjoint cycles of length \( 2k+1 \).

2.6. **Theorem** (Batineh [1]). Let \( k \geq 1 \) be an integer and \( G \in \mathcal{S}(n, V_{2k+1}) \). Then for \( n > \max \{ \frac{4k^3 + 15k^2 + 11k - 5}{2}, 4(4k^2 + 8k - 3) + 1 \} \), \( \varepsilon'(G) \leq \lfloor \frac{(n-1)^2}{4} \rfloor + n - 1 \). Furthermore, equality holds if and only if \( G = \Omega(n, 2) \).

2.7. **Theorem** (Hailat [5]). Let \( G \in \mathcal{S}(n, V_r, 3) \). Then for large \( n \), \( \varepsilon'(G) \leq \lfloor \frac{(n-r+1)^2}{4} \rfloor + (r - 1)(n - r + 1) \). Furthermore, equality holds if and only if \( G = \Omega(n, r) \).

### 3. Main Result

In this section, we generalize the result of Theorem 2.7 to the case where \( G \in \mathcal{S}(n, V_r, 2k+1) \). That is to the case where \( G \) is a graph on \( n \) vertices containing no \( r \) vertex-disjoint cycles of length \( 2k+1 \). We prove our main result using induction on \( r \) and we start with \( r = 2 \).
3.1. Theorem. Let $k$ be a positive integer and $G \in \mathcal{S}(n, 2, 2k + 1)$. Then for large $n$, $\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1$. Furthermore, equality holds if and only if $G = \Omega(n, 2)$.

Proof. Since $G \in \mathcal{S}(n, 2, 2k + 1)$, then $G$ has no two vertex-disjoint cycles of length $2k + 1$. Suppose first that $G$ has no cycle of length $2k + 1$. The for $n \geq 4k - 1$, we have $3 \leq 2k + 1 \leq \frac{1}{2} (4k + 2) \leq \left\lfloor \frac{n+3}{3} \right\rfloor$, so that, using Lemma 2.4 (Bondy and Murty [3])

$$\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$$

$$= \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 2(n-1) + 1$$

$$\leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + (n-1) \quad \text{for } n \geq 4k - 1$$

Suppose second that $G$ has a cycle of length $2k + 1$. Then for large $n$, $\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1$ by Theorem 2.6. Note that if $G = \Omega(n, 2) = K_{1, \left\lfloor \frac{n-1}{2} \right\rfloor, \left\lceil \frac{n+1}{2} \right\rceil}$ then

$$\mathcal{E}(G) = \left\lceil \frac{n-1}{2} \right\rceil + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + (n-1)$$

Therefore equality holds if and only if $G = \Omega(n, 2)$. \qed

To prove the main result we need to introduce Turán graphs, since these graphs play a major role in the proof.

3.2. Definition. The complete $s$-partite graph on $n$ vertices with part sizes being $\left\lfloor \frac{n}{s} \right\rfloor$ or $\left\lceil \frac{n}{s} \right\rceil$ is called Turán graph. We denote this graph by $T_{n,s}$.

Note that Turán graph is $K_{s+1}$ free, where $K_{s+1}$ is the complete graph on $(s + 1)$-vertices. In [4], David Conlon introduced the following statement of Turán’s theorem.

3.3. Theorem. (Turán) If $G$ is an $n$-vertex $K_{s+1}$-free graph, then it contains at most $\mathcal{E}(T_{n,s})$ edges.

In addition, Conlon introduced three different proofs of Turán’s Theorem. In this paper we use the result of 2 (Zykov’s Symmetrization). In this proof it was concluded that the set of vertices
of a $K_{s+1}$-free graph $G$ on $n$ vertices with maximum number of edges can be partitioned into $s$ equivalence classes. In these classes, vertices in the same class are non-adjacent and vertices in different classes are adjacent. Since the graph $G$ is $K_{s+1}$-free, it must be a complete $s$-partite graph. Note that $T_{n,s}$ is the unique graph that maximizes the number of edges among such graphs.

3.4. Theorem. Let $G$ be a graph that has $(r - 1)$ vertex-disjoint cycles $C_1, C_2, \ldots, C_{r-1}$, but has no $r$ vertex disjoint cycles of length $2k + 1$ and let $H = G - \bigcup_{i=1}^{r-1} G(C_i)$. Then $\mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i), H) \leq 2(r-1)(n-r+1) - 4k(r-1)^2$ and $\mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i)) \leq (2k+1)(k+1)(r-1)^2$.

Proof. Note that $H$ is $K_{2k+1}$-free graph since, otherwise, $G$ would have $r$ vertex-disjoint cycles of length $2k + 1$, a contradiction to the assumption. Let $H'$ be a graph on the vertices of $H$ with a maximum number of edges. Note that $|V(H)| = |V(H')| = n - (2k+1)(r-1) = (n-r+1) + 2k(r-1)$, $\mathcal{E}(H) \leq \mathcal{E}(H')$, and $\mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i), H) = \mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i), H')$.

Let $n' = n - (2k+1)(r-1) = (n-r+1) - 2k(r-1) = (|V(H')|)$. Since $H'$ is $K_{2k+1}$-free graph then, using proof 2 of Turán's theorem, $H'$ is $T'_{n',2k}$ and the vertices of $H'$ can be partitioned into $2k$ equivalent classes $H'_1, H'_2, \ldots, H'_2k$, where $|V(H'_i)| = \left\lceil \frac{n'}{2k} \right\rceil$ or $\left\lfloor \frac{n'}{2k} \right\rfloor$. Note that vertices of $H'_i$ are non-adjacent for all $i = 1, \ldots, 2k$, but vertices of $H'_i$ are adjacent to all vertices of $H'_j$. In Figure 1, let

$$C_1 = v_{11} \ldots v_{1(2k+1)}v_{11}$$

$$\vdots$$

$$C_{r-1} = v_{(r-1)1} \ldots v_{(r-1)(2k+1)}v_{(r-1)1}$$

Note that $|H'_i| = \left\lceil \frac{n-(2k+1)(r-1)}{2} \right\rceil$ or $\left\lfloor \frac{n-(2k+1)(r-1)}{2} \right\rfloor$, so that

$$\mathcal{E}(v_{ij}, H') \leq \sum_{i=1}^{2k} |H'_i| = n - (2k+1)(r-1)$$

$$= (n - r + 1) - 2k(r-1)$$
In Figure 1, if \( v_{ij} \in V(C_i) \) is adjacent to a vertex \( x \in V(H'_t) \) and to a vertex \( y \in V(H'_j) \) then we can construct a cycle of length \( 2k + 1 \), \( C'_i = v_{ij}x\ldots yv_{ij} \) since each vertex in \( H'_t \) is adjacent to every vertex in \( H'_m \), for \( t \neq m \). Now if we take another vertex \( w_{ij} \in V(C_i) \) and assume that its adjacent to \( x' \in V(H_t) \) and to \( y' \in V(H_l) \) then we can construct another disjoint cycle, \( C''_i \) of length \( 2k + 1 \). If we replace \( C_i \) with \( C'_i \) and \( C''_i \) then we have \( r \) vertex-disjoint cycles in \( G \), a contradiction. This implies that if a vertex in \( V(C_i) \) is adjacent to more that one component of \( V(H') = V(H) \) then the other vertices of \( C_i \) cannot be adjacent to more than one component of \( V(H') \). It follows that

\[
\mathcal{E}(G(C_i), H) = \mathcal{E}(G(C_i), H') \\
\leq (n - r + 1) - 2k(r - 1) + 2k \left( \frac{1}{2k} ((n - r + 1) - 2k(r - 1)) \right) \\
= (n - r + 1) - 2k(r - 1) + (n - r + 1) - 2k(r - 1) \\
= 2(n - r + 1) - 4k(r - 1).
\]
Therefore

\[ \mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i), H) \leq (r-1)(2(n-r+1) - 4k(r-1)) \]

\[ = 2(n-r+1)(r-1) - 4k(r-1)^2. \]

Now, since \(|V(\bigcup_{i=1}^{r-1} G(C_i))| = (2k+1)(r-1)\) then

\[ \mathcal{E}(\bigcup_{i=1}^{r-1} G(C_i)) \leq \frac{(2k+1)(r-1)((2k+1)(r-1) - 1)}{2} \]

\[ \leq \frac{(2k+1)(r-1)((2k+2)(r-1))}{2} \]

\[ = (2k+1)(k+1)(r-1)^2. \]

\[ \square \]

The following lemma is needed for the proof of Theorem 3.6.

3.5. Lemma. Let \(n, r, k\) be three positive integers such that \(r \geq 2\) and \(n \geq 6k(r-1)\). Then

\[(2-k)(r-1)(n-r+1) + (3k^2-k+1)(r-1)^2 < (r-1)(n-r+1).\]

Proof. Suppose not. Then

\[(2-k)(r-1)(n-r+1) + (3k^2-k+1)(r-1)^2 \geq (r-1)(n-r+1),\]

so that

\[(2-k)(n-r+1) + (3k^2-k+1)(r-1) \geq (n-r+1).\]

This implies that

\[n-r+1 \leq \frac{(3k^2-k+1)(r-1)}{k-1},\]

so that

\[n \leq (r-1) \left( \frac{3k^2-k+1}{k-1} + 1 \right) \]

\[= (r-1) \left( \frac{3k^2}{k-1} \right) \]

\[\leq (r-1)(3k^2)\left( \frac{2}{k} \right) = 6k(r-1),\]

a contradiction to the fact that \(n > 6k(r-1)\). Therefore Lemma 3.5 follows. \[\square\]
3.6. **Theorem.** Let \( k \) be a positive integer and \( G \in \mathcal{S}(n,r,2k+1) \). Then for \( n > 6k(r-1) \):

\[
\mathcal{E}(G) \leq \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1).
\]

Furthermore, equality holds if and only if \( G = \Omega(n,r) \).

**Proof.** We prove the theorem using induction on \( r \). For \( r = 2 \) the theorem holds by Theorem 3.1.

Assume that the result is true for \( r-1 \). We need to show that the result is true for \( r \geq 3 \). Let \( G \in \mathcal{S}(n;r,2k+1) \). If \( G \) contains no \( r-1 \) vertex disjoint cycles of length \( 2k+1 \), then by induction

\[
\mathcal{E}(G) \leq \left\lfloor \frac{(n-(r-1)+1)^2}{4} \right\rfloor + ((r-1)-1)(n-(r-1)+1)
= \left\lfloor \frac{(n-r+2)^2}{4} \right\rfloor + (r-2)(n-r+2)
\leq \frac{(n-r+1)^2 + 2(n-r+1) + 1}{4} + 4((r-1)-1)(n-(r-1)+1) + 1
= \frac{(n-r+1)^2}{4} + 2(n-r+1) + 4(r-1)(n-r+1) + 4(r-1) - 4(n-r+1) - 4 + 1
= \frac{(n-r+1)^2}{4} + (r-1)(n-r+1) - \frac{1}{2}(n-r+1)(r-1) - 1 + 1
\leq \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1), \quad \text{for } n \geq 3r-3.
\]

Assume that \( G \) has \( r-1 \) vertex-disjoint cycles each of length \( 2k+1 \) and has no \( r \) vertex-disjoint cycles of length \( 2k+1 \). Let \( C_1, C_2, \ldots, C_{r-1} \) be such cycles in \( G \). Let \( H = G - \bigcup_{i=1}^{r-1} G(C_i) \), so that \( H \) has no cycle of length \( 2k+1 \) since, otherwise, \( G \) will have \( r \) vertex-disjoint cycles of length \( 2k+1 \). Since \( |V(H)| = n' = n - (r-1)(2k+1) \) then, using Lemma 2.5, we have

\[
\mathcal{E}(H) \leq \left\lfloor \frac{n'^2}{4} \right\rfloor = \left\lfloor \frac{(n-r+1)^2 - 2k(r-1))}{4} \right\rfloor
\leq \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor - k(r-1)(n-r+1) + k^2(r-1)^2.
\]

From Theorem 3.4 we have:

\[
\mathcal{E}\left(\bigcup_{i=1}^{r-1} G(C_i), H\right) \leq 2(n-r+1)(r-1) - 4k(r-1)^2
\]
and
\[
E\left(\bigcup_{i=1}^{r-1} G(C_i)\right) \leq (2k+1)(k+1)(r-1)^2.
\]

It follows that:
\[
E(G) = E(H) + E\left(\bigcup_{i=1}^{r-1} G(C_i), H\right) + E\left(\bigcup_{i=1}^{r-1} G(C_i)\right)
\leq \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor - k(r-1)(n-r+1) + k^2(r-1)^2
\]
\[+ 2(n-r+1)(r-1) - 4k(r-1)^2 + (2k+1)(k+1)(r-1)^2
\]
\[= \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (2-k)(r-1)(n-r+1) + (k^2 - 4k + 2k^2 + 3k + 1)(r-1)^2
\]
\[= \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (2-k)(r-1)(n-r+1) + (3k^2 - k + 1)(r-1)^2
\]
\[\leq \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1) \quad \text{(using Lemma 3.5)}
\]

Furthermore, equality holds for \(\Omega(n,r) = K_{r-1,\lfloor\frac{n-r+1}{2}\rfloor,\lceil\frac{n-r+1}{2}\rceil}\) since
\[
E(\Omega(n,r)) = (r-1)\left\lfloor \frac{n-r+1}{2} \right\rfloor + (r-1)\left\lfloor \frac{n-r+1}{2} \right\rfloor + \left\lfloor \frac{n-r+1}{2} \right\rfloor \left\lfloor \frac{n-r+1}{2} \right\rfloor
\]
\[= (r-1)[n-r+1] + \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor
\]
\[= \left\lfloor \frac{(n-r+1)^2}{4} \right\rfloor + (r-1)(n-r+1).
\]

\[\square\]

Conflict of Interests

The authors declare that there is no conflict of interests.

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