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POSITIVE SOLUTIONS FOR NONLINEAR SINGULAR SECOND ORDER NEUMANN BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we study the existence of positive solutions to second order nonlinear differential equations with Neumann boundary conditions, our nonlinearity $f(t,u)$ may be singular at $u=0$ and our proof relies on a nonlinear alternative of Leray-Schauder type, together with a truncation technique. Some examples will be given.

Keywords:Positive solutions; Leray-Schauder alternative; Neumann boundary value problem

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1. Introduction

In this paper, we establish the existence of positive solutions for the second order Neumann boundary value problem

$$\begin{cases} -u'' + a(t)u = f(t,u), & 0 \le t \le 1, \\ u'(0) = 0, \quad u'(1) = 0, \end{cases} \tag{1}$$

where $a : [0,1] \to (0,\infty)$ is continuous and nonlinearity $f : [0,1] \times (0,\infty) \to (0,\infty)$. In particular, the nonlinearity may have a repulsive singularity at $u = 0$, which means that

$$\lim_{u \to 0} f(t,u) = +\infty, \quad \text{uniformly in } t.$$

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Electrostatic or gravitational forces are the most important examples of singular interactions.

Due to a wide range of applications in physics and engineering, second order Neumann boundary value problems have been extensively investigated by numerous researchers in recent years. For a small sample of such work, we refer the reader to [1–6, 9, 10, 12, 13, 16] and the references therein. Here we mention the following results: if $a(t) = M > 0$, in [13], Jiang and Liu obtain the existence of one positive solution of (1) when f is either superlinear or sublinear, and in [12] Sun and Li gave some existence results for at least two positive solutions to (1) under weaker conditions than [13]. In the above two papers, existence results were obtained by using Krasnoselskii's fixed point theorem on compression and expansion of cones [7]. Besides fixed point theorems in cone, the method of upper and lower solutions [10] is also used in the literature [1, 3, 9].

The aim of this paper is to study the existence of solutions of problem (1) by using alternative of Leray-Schauder type, which was used in [11] to deal with periodic singular problems. It is proved that such a problem has at least one positive solutions under reasonable conditions (See Theorem 3.1). The paper is organized as follows. In section 2, some preliminary results will be given, including a famous nonlinear alternative of Leray-Schauder type. In section 3, we will state and prove the main results, some illustrating examples will be given.

2. Preliminaries

Let $u(t)$ and $v(t)$ be the solutions of the following homogeneous equations

$$-u'' + a(t)u = 0, \quad 0 \le t \le 1,$$

satisfying the initial conditions

$$u(0) = 1, u'(0) = 0, v(0) = 0, v'(0) = 1.$$

Lemma 2.1 [8] Suppose that $h : [0,1] \to [0,\infty)$ is continuous, then problem

$$\begin{cases} -u'' + a(t)u = h(t), & 0 \le t \le 1, \\ u'(0) = 0, \quad u'(1) = 0, \end{cases}$$

has a unique solution $u \in \mathbb{C}^2[0,1]$ given by the formula

$$u(t) = \int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s) = \begin{cases} \frac{u(t)v'(1)-v(t)u'(1)}{u'(1)}u(s), & 0 \le s \le t \le 1, \\ \frac{u(s)v'(1)-v(s)u'(1)}{u'(1)}u(t), & 0 \le t \le s \le 1, \end{cases}$$

is the Green's function.

Remark 2.2 [13] If $a(t) = M > 0$, then the Green's function $G(t,s)$ of the boundary value problem (1) has the form

$$G(t,s) = \begin{cases} \frac{\cosh\sqrt{M}(1-t)\cosh\sqrt{M}s}{\sqrt{M}\sinh\sqrt{M}}, & 0 \le s \le t \le 1, \\ \frac{\cosh\sqrt{M}(1-s)\cosh\sqrt{M}t}{\sqrt{M}\sinh\sqrt{M}}, & 0 \le t \le s \le 1, \end{cases}$$

where

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Lemma 2.3 [8] Suppose $a, h : [0,1] \to (0,+\infty)$ are continuous functions. Then the Green's function $G(t,s)$ of problem (1) is positive, i.e., $G(t,s) > 0, t, s \in [0,1]$.

We denote

$$A = \min_{0\le s,t\le 1} G(t,s), \quad B = \max_{0\le s,t\le 1} G(t,s), \quad \sigma = A/B.$$

Thus $B > A > 0$ and $0 < \sigma < 1$. When $a(t) = M > 0$, a direct calculation shows that

$$A = \frac{1}{\sqrt{M}\sinh\sqrt{M}}, \quad B = \frac{\cosh^2\sqrt{M}}{\sqrt{M}\sinh\sqrt{M}}, \quad \sigma = \frac{1}{\cosh^2\sqrt{M}} < 1.$$

In order to prove the main result of this paper, we need the following nonlinear alternative of Leray-Schauder, which can be found in [15] or [17], and has been used by M.Meehan, D.O'Regan in [14, 18].

Theorem 2.4 Assume Ω is a relatively compact subset of a convex set E in a normed space X. Let $T : \overline{\Omega} \to E$ be a compact map with $0 \in \Omega$. Then one of the following two conclusions holds:

(i) T has at least one fixed point in $\overline{\Omega}$.

(ii) There exist $u \in \partial\Omega$ and $0 < \lambda < 1$ such that $u = \lambda T u$.

3. Main results

In this section, we state and prove the main results of this paper. Let $\| \cdot \|$ denote the supremum norm of $\mathbb{C}[0, 1]$.

Theorem 3.1 Suppose that there exists a constant $r > 0$ such that

(H_1) There exists a continuous function $\phi_r \succ 0$ such that $f(t, u) \geq \phi_r(t)$ for all $(t, u) \in [0, 1] \times (0, r]$.
where the notation $\phi_r \succ 0$ means that $\phi_r \geq 0$ for all $t \in [0, 1]$ and $\phi_r > 0$ for t in a subset of positive measure.

(H_2) there exist continuous nonnegative function $p(u), q(u)$ such that

$$f(t, u) \leq p(u) + q(u) \quad \text{for all } (t, u) \in [0, 1] \times (0, \infty),$$

$p(u) > 0$ is nonincreasing and $q(u)/p(u)$ is nondecreasing in $u \in (0, \infty)$.

(H_3) $\frac{r}{p(\sigma r)\left\{1+\frac{q(r)}{p(r)}\right\}} > g^*$, here $g^* = \sup\limits_{0 \leq t \leq 1} \int_0^1 G(t, s) ds$.
where σ and $G(t, s)$ are given as in Section 2.

Then boundary value problem (1) has at least one solution u with $0 < \| u \| < r$.

proof Since (H_3) holds, we can choose $n_0 \in \{1, 2, \cdots\}$ such that

$$g^* p(\sigma r) \left\{1 + \frac{q(r)}{p(r)}\right\} + \frac{1}{n_0} < r.$$

Let $N_0 = \{n_0, n_0 + 1, \cdots\}$. Consider the family of equation

$$\begin{cases} -u'' + a(t)u = \lambda f_n(t, u(t)) + \frac{a(t)}{n}, \\ u'(0) = 0, \quad u'(1) = 0. \end{cases} \tag{2}$$

where $\lambda \in [0,1], n \in N_0$ and

$$f_n(t,u) = \begin{cases} f(t,u), & \text{if} \quad u \geq 1/n, \\ f(t,\frac{1}{n}), & \text{if} \quad u \leq 1/n. \end{cases}$$

Problem (2) is equivalent to the following fixed point problem

$$u(t) = \lambda \int_0^{2\pi} G(t,s) f_n(s,u(s))ds + \frac{1}{n} = \lambda T_n u(t) + \frac{1}{n}, \tag{3}$$

where T_n is defined by

$$(T_n u)(t) = \int_0^1 G(t,s) f_n(s,u(s))ds,$$

and we used the fact

$$\int_0^1 G(t,s)a(s)ds \equiv 1. \quad \text{(see Lemma 2.1 with } h = a)$$

We claim that any fixed point u of (3) for any $\lambda \in [0,1]$ must satisfy $\|u\| \neq r$. Otherwise assume that u is a fixed point of (3) for some $\lambda \in [0,1]$ such that $\|u\| = r$. Note that $f_n(t,x) \geq \phi_r(t)$ for $0 < u \leq r$. It is to see that

$$\begin{aligned} u(t) - \frac{1}{n} &= \lambda \int_0^1 G(t,s) \left[f_n(s,u(s)) \right] ds \\ &\geq \lambda A \int_0^1 \left[f_n(s,u(s)) \right] ds \\ &\geq \lambda \frac{A}{B} \max_{t \in [0,1]} \int_0^1 G(t,s) f_n(s,u(s)) ds \\ &= \sigma \left\| u - \frac{1}{n} \right\|. \end{aligned}$$

Hence for $t \in [0,1]$, we have $u(t) \geq \; 1/n$ and

$$\begin{aligned} u(t) \geq \sigma \left\| u - \frac{1}{n} \right\| + \frac{1}{n} &\geq \sigma \left(\|u\| - \frac{1}{n} \right) + \frac{1}{n} \\ &= \sigma \left(r - \frac{1}{n} \right) + \frac{1}{n} \\ &\geq \sigma r. \end{aligned}$$

Thus we have from condition (H_2), for all $t \in [0,1]$,

$$\begin{aligned} u(t) = &\lambda \int_0^1 G(t,s) f_n(s,u(s))ds + \frac{1}{n} \\ \le &\int_0^1 G(t,s) f(s,u(s))ds + \frac{1}{n} \\ \le &\int_0^1 G(t,s) p(u(s)) \left\{1 + \frac{q(u(s))}{p(u(s))}\right\} ds + \frac{1}{n} \\ \le & p(\sigma r) \left\{1 + \frac{q(r)}{p(r)}\right\} \int_0^1 G(t,s)ds + \frac{1}{n} \\ \le & p(\sigma r) \left\{1 + \frac{q(r)}{p(r)}\right\} g^* + \frac{1}{n_0}. \end{aligned}$$

Therefore,

$$r = \|u\| \le p(\sigma r) \left\{1 + \frac{q(r)}{p(r)}\right\} g^* + \frac{1}{n_0}.$$

This is a contradiction to the choice of n_0 and the claim is proved.

From this claim, the Leray-Schauder alternative principle guarantees that

$$\begin{cases} -u'' + a(t)u = f_n(t,u(t)) + \frac{a(t)}{n}, \\ u'(0) = 0, \quad u'(1) = 0, \end{cases} \tag{4}$$

has a periodic solution u_n with $\|u_n\| < r$. Since $u_n(t) \ge \frac{1}{n} > 0$ for all $t \in [0,1]$ and u_n is actually a positive periodic solution of (4).

Next we claim that these solutions u_n have a uniform positive lower bound, i.e., there exists a constant $\delta > 0$, independent of $n \in N_0$, such that

$$\min_{t\in[0,1]} u_n(t) \ge \delta \tag{5}$$

for all $n \in N_0$. To see this , let $u_r(t)$ be the unique solution to the problem

$$\begin{cases} -u'' + a(t)u = \phi_r(t), \quad 0 \le t \le 1, \\ u'(0) = 0, \quad u'(1) = 0. \end{cases}$$

then $u_r(t) = \int_0^1 G(t,s)\phi_r(t)ds \ge A\|\phi_r\|_1 > 0$, here $\|.\|_1$ denotes the usual L^1- norm over $(0,1)$.

So, we have

$$\begin{aligned} u_n(t) &= \int_0^1 G(t,s) f_n(s,u(s))ds + \frac{1}{n} \\ &= \int_0^1 G(t,s) f(s,u(s))ds + \frac{1}{n} \\ &\geq \int_0^1 G(t,s)\phi_r(t)ds + \frac{1}{n} \\ &\geq A\|\phi_r\|_1 := \delta > 0. \end{aligned}$$

In order to pass the solution u_n of the truncations problems (2) to that of the original problem (1), we need the following fact:

$$\|u_n'\| \leq H \tag{6}$$

for some constant $H > 0$, and for all $n \geq n_0$. To show this, first integrate the first equation in (4) from 0 to 1, we obtain

$$\int_0^1 a(t)u_n(t)dt = \int_0^1 \left[f_n(t,u_n(t)) + \frac{a(t)}{n} \right] dt.$$

Then

$$\begin{aligned} \|u_n'\| &= \max_{0\leq t\leq 1} |u_n'| = \max_{0\leq t\leq 1} \left| \int_0^t u_n''(s)ds \right| \\ &= \max_{0\leq t\leq 1} \left| \int_0^t f_n(s,u_n(s)) + \frac{a(s)}{n} - a(s)u_n(s) \right| ds \\ &\leq \int_0^t \left[f_n(s,u_n(s)) + \frac{a(s)}{n} \right] ds + \int_0^1 a(s)u_n(s)ds \\ &\leq 2\int_0^1 a(s)u_n(s)ds < 2r\|a\|_1 := H. \end{aligned}$$

Now $\|u_n\| < r$ and (6) show that $\{u_n\}_{n\in N_0}$ is a bounded and equi-contioous family on $[0,1]$. The Arzela-Ascoli Theorem guarantees that $\{u_n\}_{n\in N_0}$ has a subsequence $\{u_{n_i}\}_{i\in N}$, converging uniformly on $[0,1]$ to a function $u \in \mathbb{C}[0,1]$. From $\|u_n\| < r$ and (5), u satisfies $\delta \leq u(t) \leq r$ for all t. Moreover u_{n_i} satisfies the integral equation

$$u_{n_i}(t) = \int_0^1 G(t,s) f(s,u_{n_i}(s))ds + \frac{1}{n_i}.$$

Let $i \to \infty$, we arrive at

$$u_n(t) = \int_0^1 G(t,s)f(s,u_n(s))ds,$$

where the uniform continuity of $f(t,u)$ on $[0,1] \times [\delta, r]$ is used. Therefore, u is a positive solution of (1).

Finally it is not difficult to show that $\|u\| < r$, by noting that if $\|u\| = r$, an argument similar to the proof of the first claim will yield a contradiction.

Corollary 3.2 Assume that there exist continuous functions $d, \hat{d}$ and $\lambda > 0$ such that $0 \le \hat{d}(t)u^{-\lambda} \le f(t,u) \le d(t)u^{-\lambda}$ for all $u > 0$ and $t \in [0,1]$. Then problem (1) has at least one positive solution.

proof We will apply Theorem 3.1, (H_1) and (H_2) are satisfied if we take

$$\phi_r(t) = \hat{d}(t)r^{-\lambda}, \quad q(u) = 0, \quad p(u) = d(t)u^{-\lambda}.$$

The existence condition (H_3) become

$$\sigma^\lambda r^{\lambda+1} > \sup_{0 \le t \le 1} \int_0^1 G(t,s)d(s)ds \tag{7}$$

for some $r > 0$. Since $\lambda > 0$ and $u(t) > 0$, we can choose $r > 0$ large enough such that (7) is satisfied.

Corollary 3.3 Let the nonlinearity in (1) be

$$f(t,u) = b(t)u^{-\alpha} + \mu c(t)u^\beta, \quad 0 \le t \le 1,$$

where $\alpha > 0, \beta \ge 0, b(t), c(t) \in \mathbb{C}[0,1]$ are non-negative functions and $b(t) > 0$ for all t, and μ is a positive parameter. Then

(i) if $\beta < 1$, then (1) has at least one positive solution for each $\mu > 0$,

(ii) if $\beta \ge 1$, then (1) has at least one positive solution for each $0 < \mu < \mu_1$, where μ_1 is some positive constant.

proof We will apply Theorem 3.1. To this end, notice assumption (H_1) is fulfilled if $\phi_r(t) = \mu r^{-\alpha} \min_{0 \le t \le 1} b(t)$. If we take

$$p(u) = b_0 u^{-\alpha}, \quad q(u) = \mu c_0 u^\beta,$$

where

$$b_0 = \max_{0\le t\le 1} b(t) > 0, \quad c_0 = \max_{0\le t\le 1} c(t) > 0,$$

then (H_2) is satisfied.

Now the existence condition (H_3) become

$$u < \frac{\sigma^\alpha r^{\alpha+1} - g^* b_0}{g^* c_0 r^{\alpha+\beta}}.$$

for some $r > 0$, so (1) has at least one positive periodic solution for

$$0 < \mu < \mu_1 := \sup_{r>0} \frac{\sigma^\alpha r^{\alpha+1} - g^* b_0}{g^* c_0 r^{\alpha+\beta}}$$

Note that $\mu_1 = \infty$ if $\beta < 1$ and $\mu_1 < \infty$ if $\beta \ge 1$. We have the desired results.

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