ON SOME CLASSES OF CONCIRCULAR CURVATURE TENSOR ON LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract. The present paper deals with the study of different classes of concircular curvature tensor on Lorentzian para-Sasakian manifold admitting a quarter-symmetric metric connection.

Keywords: Lorentzian para-Sasakian manifolds; quarter-symmetric metric connection; concircular curvature tensor; $\eta$--Einstein manifold.

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1. INTRODUCTION

K. Matsumoto [8] introduced the concept of Lorentzian para- Sasakian manifolds in 1989. Later, the same concept was independently introduced by I. Mihai and R. Rosca [10]. The Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [9], U. C. De and A. A. Shaikh [11] and several others such as ([12], [14], [15]). K. Matsumoto and I. Mihai obtained some interesting results for conformally recurrent and conformally symmetric $P$--Sasakian manifold in [1]. In 1924, the notion of semi-symmetric connection on a differentiable manifold was firstly introduced by Friedmann and Schouten [18]. A linear connection $\nabla$
on a differentiable manifold \( M \) is said to be a semi-symmetric connection if the torsion tensor \( T \) of the connection satisfies

\[
T(U,V) = \eta(V)U - \eta(U)V,
\]

where \( \eta \) is a 1-form and \( \xi \) is a vector field defined by \( \eta(U) = g(U,\xi) \), for all vector fields \( U \) on \( \Gamma(TM) \), \( \Gamma(TM) \) is the set of all differentiable vector fields on \( M \). A. Barman ([2], [3]) studied para-Sasakian manifold admitting semi-symmetric metric and non metric connection.

On the other hand, in 1975, Golab [6] introduced and studied quarter-symmetric connection in differentiable manifolds along with affine connections.

A linear connection \( \tilde{\nabla} \) on an \( n \)-dimensional Riemannian manifold \( (M,g) \) is called a quarter-symmetric connection [6] if its torsion tensor \( T \) satisfies

\[
(1.1) \quad T(U,V) = \eta(V)\phi U - \eta(U)\phi V,
\]

where \( \phi \) is a \((1,1)\) tensor field.

The quarter-symmetric connection generalizes the notion of the semi-symmetric connection because if we assume \( \phi U = U \) in the above equation, the quarter-symmetric connection reduces to the semi-symmetric connection [18].

Moreover, if a quarter-symmetric connection \( \tilde{\nabla} \) satisfies the condition

\[
(1.2) \quad (\tilde{\nabla}Ug)(V,W) = 0,
\]

for all \( U,V,W \) on \( \Gamma(TM) \), then \( \tilde{\nabla} \) is said to be a quarter-symmetric metric connection.

Venkatesha and C.S. Bagewadi [19] obtain some interesting results on concircular \( \phi \)-recurrent Lorentzian para-Sasakian manifolds which generalize the concept of locally concircular \( \phi \)-symmetric Lorentzian para-Sasakian manifolds. If curvature tensor \( R \) of Riemannian manifold \( M \) satisfies \( \nabla R = 0 \), then \( M \) is called locally symmetric. Later, many geometers have considered semi-symmetric spaces as a generalization of locally symmetric spaces. A Riemannian manifold \( M \) is said to be semi-symmetric if its curvature tensor \( R \) satisfies \( R(U,V).R = 0 \), where \( R(U,V) \) acts on \( R \) as a derivation and also it is called Ricci-semisymmetric manifold if the relation \( R(U,V).S = 0 \) holds, where \( R(U,V) \) the curvature operator.
A transformation which transforms every geodesic circle of a Riemannian manifold $M$ into a geodesic circle, is known as concircular transformation ([7], [16]), where geodesic circle means a curve in $M$ whose first curvature is constant and second curvature is identically zero. A concircular transformation is always a conformal transformation [7]. Thus the geometry of concircular transformations is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [5]). An invariant of a concircular transformation is the concircular curvature tensor $C$, which is defined by ([16], [17])

\[
C(U, V)W = R(U, V)W - \frac{r}{n(n-1)}[g(V, W)U - g(U, W)V].
\]

Using (1.3), we obtain

\[
g(C(U, V)W, Z) = g(R(U, V)W, Z) - \frac{r}{n(n-1)}[g(V, W)g(U, Z) - g(U, W)g(V, Z)],
\]

where $U, V, W, Z \in \Gamma(TM)$ and $r$ is the scalar curvature on Lorentzian para-Sasakian manifolds. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In this paper, we study a type of quarter-symmetric metric connection on Lorentzian para-Sasakian manifolds. The paper is organized as follows: After introduction section two is equipped with some prerequisites of a Lorentzian para-Sasakian manifold. In section three, curvature tensor and Ricci tensor of Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection are given. Section four is devoted to study $\xi$-concircularly flat Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection. Quasi-concircularly flat and $\phi$-concircularly flat Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection have been studied in section five and six respectively. In next section, we investigate Ricci-semisymmetric manifolds with respect to the quarter-symmetric metric connection of a Lorentzian para-Sasakian manifold.
2. **Preliminaries**

An n-dimensional differentiable manifold \( M \) is said to be a Lorentzian almost para-contact manifold, if it admits an almost para-contact structure \((\phi, \xi, \eta, g)\) consisting of a \((1, 1)\) tensor field \(\phi\), vector field \(\xi\), 1-form \(\eta\) and a Lorentzian metric \(g\) satisfying

\[
\begin{align*}
(2.1) & \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = -1, \quad g(U, \xi) = \eta(U), \\
(2.2) & \quad \phi^2 U = U + \eta(U)\xi, \\
(2.3) & \quad g(\phi U, \phi V) = g(U, V) + \eta(U)\eta(V), \\
(2.4) & \quad (\nabla_U \eta)V = g(U, \phi V) = (\nabla_V \eta)U,
\end{align*}
\]

for any vector fields \(U, V\) on \(M\). Such a manifold \(M\) is termed as Lorentzian para-contact manifold and the structure \((\phi, \xi, \eta, g)\) a Lorentzian para-contact structure [8].

If moreover \((\phi, \xi, \eta, g)\) satisfies the conditions

\[
\begin{align*}
(2.5) & \quad d\eta = 0, \quad \nabla_U \xi = \phi U, \\
(2.6) & \quad (\nabla_U \phi)V = g(U, V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi,
\end{align*}
\]

for \(U, V\) tangent to \(M\), then \(M\) is called a Lorentzian para-Sasakian manifold or briefly LP-Sasakian manifold, where \(\nabla\) denotes the covariant differentiation with respect to Lorentzian metric \(g\).

Moreover, the curvature tensor \(R\), the Ricci tensor \(S\) and the Ricci operator \(Q\) in a Lorentzian para-Sasakian manifold \(M\) with respect to the Levi-Civita connection \(\nabla\) satisfies the following relations [13]

\[
\begin{align*}
(2.7) & \quad \eta(R(U, V)W) = g(W, V)\eta(U) - g(U, W)\eta(V), \\
(2.8) & \quad R(\xi, U)V = g(U, V)\xi - \eta(V)U, \\
(2.9) & \quad R(\xi, U)\xi = -R(U, \xi)\xi = U + \eta(U)\xi,
\end{align*}
\]
\[
R(U,V)\xi = \eta(V)U - \eta(U)V,
\]

(2.10)

\[
S(U,\xi) = (n-1)\eta(U), \quad Q\xi = (n-1)\xi,
\]

(2.11)

\[
S(\phi U,\phi V) = S(U,V) + (n-1)\eta(U)\eta(V),
\]

(2.12)

for all vector fields \(U, V, W \in \Gamma(TM)\).

**Definition 2.1.** A Lorentzian para-Sasakian manifold \(M\) is said to be an \(\eta\)-Einstein manifold \([13]\) if its Ricci tensor \(S\) of the Levi-Civita connection is of the form

\[
S(U,V) = ag(U,V) + b\eta(U)\eta(V) \quad \text{for all } U, V \in \Gamma(TM)
\]

(2.13)

where \(a\) and \(b\) are smooth functions on the manifold \(M\).

3. **Curvature Tensor of Lorentzian Para-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection**

A relation between the quarter-symmetric metric connection \(\tilde{\nabla}\) and the Levi-Civita connection \(\nabla\) in an \(n\)-dimensional Lorentzian para-Sasakian manifold \(M\) is given by \([15]\)

\[
\tilde{\nabla}_U V = \nabla_U V + \eta(V)\phi U - g(\phi U,V)\xi.
\]

(3.1)

The curvature tensor \(\tilde{R}\) of a Lorentzian para-Sasakian manifold \(M\) with respect to the quarter-symmetric metric connection \(\tilde{\nabla}\) is defined by

\[
\tilde{R}(U,V)W = \tilde{\nabla}_U \tilde{\nabla}_V W - \tilde{\nabla}_V \tilde{\nabla}_U W - \tilde{\nabla}_{[U,V]} W.
\]

(3.2)

From the equations (2.1) – (2.6), (3.1) and (3.2), we obtain

\[
\tilde{R}(U,V)W = R(U,V)W + [g(\phi U,W)\phi V - g(\phi V,W)\phi U]
\]

\[
+ [g(V,W)\eta(U) - g(U,W)\eta(V)]\xi
\]

\[
+ \eta(W)[\eta(V)U - \eta(U)V].
\]

(3.3)
where $U, V, W \in \Gamma(TM)$ and $R(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U, V]} W$ is the Riemannian curvature tensor with respect to the Levi-Civita connection $\nabla$.

The Ricci tensor $\bar{S}$ and the Scalar curvature $\bar{r}$ in a Lorentzian para-Sasakian manifold $M$ with respect to the quarter-symmetric metric connection $\bar{\nabla}$ are defined by

$$\tag{3.4} \bar{S}(V, W) = \sum_{i=1}^{n} \varepsilon_i g(\bar{R}(e_i, V)W, e_i),$$

$$\tag{3.5} \bar{r} = \sum_{i=1}^{n} \varepsilon_i \bar{S}(e_i, e_i)$$

where $\{e_1, e_2, ..., e_{n-1}, e_n = \xi\}$ be a local orthonormal basis of vector fields in $M$ and $\varepsilon_i = g(e_i, e_i)$.

Now contracting $U$ in (3.3), we get

$$\tag{3.6} \bar{S}(V, W) = S(V, W) + (n - 1) \eta(V) \eta(W) - (\text{trace} \phi) g(\phi V, W).$$

Again contracting $V$ and $W$ in (3.6), we get

$$\tag{3.7} \bar{r} = r - (n - 1) - (\text{trace} \phi)^2.$$

From equation (3.3) and (3.6), we have

$$\tag{3.8} \bar{R}(U, V) \xi = \bar{R}(\xi, U)V = 0,$$

$$\tag{3.9} \bar{S}(V, \xi) = 0,$$

$$\tag{3.10} \bar{S}(\phi U, \phi V) = \bar{S}(U, V).$$

4. $\xi$-Concircularly Flat Lorentzian Para-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection

**Definition 4.1.** Concircular curvature tensor $\tilde{C}$ of Lorentzian para-Sasakian manifold $M$ with respect to the quarter-symmetric metric connection is defined by

$$\tag{4.1} \tilde{C}(U, V)W = \bar{R}(U, V)W - \frac{\bar{r}}{n(n - 1)} [g(V, W)U - g(U, W)V]$$
for all $U, V, W \in \Gamma(TM)$ where $\bar{R}$ is the curvature tensor and $\bar{r}$ is the scalar curvature of $M$ with respect to the quarter-symmetric metric connection $\bar{\nabla}$.

**Definition 4.2.** A Lorentzian para-Sasakian manifold is said to be $\xi$–concircularly flat [4] with respect to the quarter-symmetric metric connection $\bar{\nabla}$ if

$$\bar{C}(U, V) \xi = 0$$

for all $U, V \in \Gamma(TM)$.

Putting $W = \xi$ in (4.1) and using (3.8) and (4.2), we have

$$\bar{r}[\eta(V)U - \eta(U)V] = 0.$$  

Putting $U = \xi$ in (4.3) and using (2.1), we have

$$\bar{r}[V + \eta(V)\xi] = 0.$$  

Taking inner product of (4.4) with $W$ and replacing $V$ by $QV$, we have

$$\bar{r}[g(QV, W) + \eta(QV)\eta(W)] = 0.$$  

Using $S(V, W) = g(QV, W)$ and equations (2.11) and (3.7) in (4.5), we have

$$[r - (n - 1) - (\text{trace} \phi)^2] [S(V, W) + (n - 1)\eta(V)\eta(W)] = 0.$$  

Equation (4.6) implies that either $r = (n - 1) + (\text{trace} \phi)^2$ or $S(V, W) = -(n - 1)\eta(V)\eta(W)$.

Thus we can state the following:

**Theorem 4.3.** If a Lorentzian para-Sasakian manifold $M$ admitting a quarter-symmetric metric connection is $\xi$–concircularly flat with respect to the quarter-symmetric metric connection, then either scalar curvature of $M$ is $(n - 1) + (\text{trace} \phi)^2$ or the manifold $M$ is a special type of $\eta$–Einstein manifold with respect to the Levi-Civita connection.
5. **Quasi-Concircularly Flat Lorentzian Para-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection**

**Definition 5.1.** A Lorentzian para-Sasakian manifold $M$ is said to be quasi—concircularly flat with respect to the quarter-symmetric metric connection if

\[(5.1) \quad g(\bar{C}(\phi U, V)W, \phi Z) = 0\]

where $U, V, W, Z \in \Gamma(TM)$.

From equation (4.1), we have

\[(5.2) \quad g(\bar{C}(\phi U, V)W, \phi Z) = g(\bar{R}(\phi U, V)W, \phi Z) - \frac{\bar{r}}{n(n-1)} [g(V, W)g(\phi U, \phi Z) - g(\phi U, W)g(V, \phi Z)].\]

Using (5.1) in (5.2), we have

\[(5.3) \quad g(\bar{R}(\phi U, V)W, \phi Z) = \frac{\bar{r}}{n(n-1)} [g(V, W)g(\phi U, \phi Z) - g(\phi U, W)g(V, \phi Z)].\]

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in $M$, then $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis. Putting $U = Z = e_i$ in (5.3) and summing over $i = 1$ to $n-1$, we obtain

\[(5.4) \quad \sum_{i=1}^{n-1} g(\bar{R}(\phi e_i, V)W, \phi e_i) = \frac{\bar{r}}{n(n-1)} \sum_{i=1}^{n-1} [g(V, W)g(\phi e_i, \phi e_i) - g(\phi e_i, W)g(V, \phi e_i)].\]

On LP-Sasakian manifold it can be verify that

\[(5.5) \quad \sum_{i=1}^{n-1} g(\bar{R}(\phi e_i, V)W, \phi e_i) = \bar{S}(V, W),\]

\[(5.6) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1,\]

\[(5.7) \quad \sum_{i=1}^{n-1} g(\phi e_i, W)g(V, \phi e_i) = g(V, W) + \eta(V)\eta(W).\]

So by virtue of (5.5), (5.6) and (5.7), the equation (5.4) takes the form

\[\bar{S}(V, W) = \left[\frac{\bar{r} (n-2)}{n(n-1)}\right] g(V, W) - \left[\frac{\bar{r}}{n(n-1)}\right] \eta(V)\eta(W).\]
or

\[ \bar{S}(V, W) = a g(V, W) + b \eta(V) \eta(W), \]

where \( a = \left[ \frac{\bar{r}(n-2)}{n(n-1)} \right] \) and \( b = -\left[ \frac{\bar{r}}{n(n-1)} \right]. \)

From which it follows that the manifold is an \( \eta \)-Einstein manifold with respect to the quarter-symmetric metric connection.

Hence we can state the following theorem:

**Theorem 5.2.** If a Lorentzian para-Sasakian manifold admitting a quarter-symmetric metric connection is quasi-concircularly flat, then the manifold with respect to the quarter-symmetric metric connection is an \( \eta \)-Einstein manifold.

### 6. \( \phi \)-Concircularly Flat Lorentzian Para-Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection

**Definition 6.1.** A Lorentzian para-Sasakian manifold is said to be \( \phi \)-concircularly flat with respect to the quarter-symmetric metric connection if

\[
(6.1) \quad g(\bar{C}(\phi U, \phi V)\phi W, \phi Z) = 0,
\]

where \( U, V, W, Z \in \Gamma(TM) \).

From equation (4.1), we have

\[
g(\bar{C}(\phi U, \phi V)\phi W, \phi Z) = g(\bar{R}(\phi U, \phi V)\phi W, \phi Z) - \frac{\bar{r}}{n(n-1)}[g(\phi V, \phi W)g(\phi U, \phi Z)
- g(\phi U, \phi W)g(\phi V, \phi Z)].
\]

Using (6.1) in (6.2), we have

\[
(6.3) \quad g(\bar{R}(\phi U, \phi V)\phi W, \phi Z) = \frac{\bar{r}}{n(n-1)}[g(\phi V, \phi W)g(\phi U, \phi Z) - g(\phi U, \phi W)g(\phi V, \phi Z)].
\]

Let \( \{e_1, e_2, \ldots, e_{n-1}, \xi\} \) be a local orthonormal basis of vector fields in \( M \), then

\( \{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\} \) is also a local orthonormal basis. Putting \( U = Z = e_i \) in (6.3) and
summing over \( i = 1 \) to \( n - 1 \), we obtain

\[
(6.4) \quad \sum_{i=1}^{n-1} g(\bar{R}(\phi e_i, \phi V) \phi W, \phi e_i) = \bar{r} \frac{n}{n(n-1)} \sum_{i=1}^{n-1} [g(\phi V, \phi W)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi W)g(\phi V, \phi e_i)],
\]

So by virtue of \((3.10), (5.5), (5.6)\) and \((5.7)\), the equation \((6.4)\) takes the form

\[
\bar{S}(V, W) = \left[ \bar{r} \frac{(n-2)}{n(n-1)} \right] g(V, W) - \left[ \bar{r} \frac{(n-2)}{n(n-1)} \right] \eta(V) \eta(W).
\]

or

\[
\bar{S}(V, W) = a g(V, W) + b \eta(V) \eta(W),
\]

where \( a = \left[ \bar{r} \frac{(n-2)}{n(n-1)} \right] \) and \( b = -\left[ \bar{r} \frac{(n-2)}{n(n-1)} \right] \).

From which it follows that the manifold is an \( \eta \)–Einstein manifold with respect to the quarter-symmetric metric connection.

Hence we can state following theorem:

**Theorem 6.2.** A \( \phi \)-concircularly flat Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection is an \( \eta \)–Einstein manifold with respect to the quarter-symmetric metric connection.

**7. Lorentzian Para-Sasakian Manifold Satisfying \( \bar{C} \cdot \bar{S} = 0 \) with Respect to the Quarter-Symmetric Metric Connection**

We consider Lorentzian para-Sasakian manifolds with respect to a quarter-symmetric metric connection \( \bar{\nabla} \) satisfying the curvature condition \( \bar{C} \cdot \bar{S} = 0 \). Then

\[
(\bar{C}(U, V) \cdot \bar{S})(W, Z) = 0.
\]

So,

\[
(7.1) \quad \bar{S}(\bar{C}(U, V) W, Z) + \bar{S}(W, \bar{C}(U, V) Z) = 0.
\]
Putting $U = \xi$ in (7.1), we get

(7.2) $\bar{S}(\bar{C}(\xi, V) W, Z) + \bar{S}(W, \bar{C}(\xi, V) Z) = 0.$

Now from (3.8) and (4.1), we have

(7.3) $\bar{C}(\xi, V) W = -\frac{\bar{r}}{n(n-1)}[g(V, W)\xi - \eta(W)V].$

Using (7.3) in (7.2) and putting $W = \xi$ and using (3.9), we obtain

$\bar{r}\bar{S}(V, Z) = 0.$

This implies that $\bar{r} = 0.$

Hence we can state following:

**Theorem 7.1.** If Lorentzian para-Sasakian manifolds satisfying $\bar{C} \cdot \bar{S} = 0$ with respect to the quarter-symmetric metric connection, then the manifold is scalar flat with respect to the quarter-symmetric metric connection.

**Example 1.** Example of a LP-Sasakian manifold with respect to Quarter-symmetric metric connection.

Taking a 3–dimensional manifold $M = \{(x, y, v) \in R^3\}$, where $(x, y, v)$ are standard coordinates of $R^3$. Let $e_1, e_2, e_3$ are vector fields on $M$, given by

$e_1 = -e^v \frac{\partial}{\partial x}, \quad e_2 = -e^{v-x} \frac{\partial}{\partial y}, \quad e_3 = -\frac{\partial}{\partial v} = \xi,$

Clearly, $\{e_1, e_2, e_3\}$ is linearly independent set of vectors on $M$. So it forms a basis of $\Gamma(TM)$.

The Lorentzian metric $g$ is defined by

$g(e_i, e_j) = \begin{cases} 0, & \text{for } i \neq j \text{ and } 1 \leq i, j \leq 3 \\ g(e_1, e_1) = g(e_2, e_2) = 1, & g(e_3, e_3) = -1. \end{cases}$

Let $\eta$ be a 1–form on $M$ defined as $\eta(U) = g(U, e_3) = g(U, \xi)$, for all $U \in \Gamma(TM)$, and let $\phi$ be a $(1, 1)$ tensor field on $M$ defined as
\[ \phi(e_1) = -e_1, \quad \phi(e_2) = -e_2, \quad \phi(e_3) = 0. \]

By applying linearity of \( \phi \) and \( g \), we have

\[ \eta(e_3) = -1, \quad \phi^2(U) = U + \eta(U)\xi, \]

and

\[ g(\phi U, \phi V) = g(U, V) + \eta(U)\eta(V) \quad \text{for all} \quad U, V \in (TM). \]

Let \( \nabla \) be a Levi-Civita connection with respect to the Riemannian metric \( g \), we have

\[ [e_1, e_2] = -e^v e_2, \quad [e_2, e_3] = -e_2, \quad [e_1, e_3] = -e_1, \]

The Riemannian connection \( \nabla \) of the metric \( g \) is given by

\[
2g(\nabla U V, W) = U g(V, W) + V g(W, U) - W g(U, V) \\
- g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]),
\]

which is known as Koszul’s formula, we can easily calculate

\[
\begin{align*}
\nabla_{e_1} e_1 &= e_3, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= -e_2, \\
\nabla_{e_2} e_1 &= -e^v e_2, & \nabla_{e_2} e_2 &= -e_3 - e^v e_1, & \nabla_{e_2} e_3 &= -e_2, \\
\nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= 0,
\end{align*}
\]

From the above it follows that the manifold satisfies \( \nabla U \xi = \phi U \), for \( \xi = e_3 \) and \( (\nabla U \phi)V = g(U, V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi \). Hence the manifold is \( LP \)-Sasakian manifold.

Using (3.1), we have
\[
\begin{align*}
\nabla_{e_1} e_1 &= 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \\
\nabla_{e_2} e_1 &= -e^v e_2, \quad \nabla_{e_2} e_2 = -e^v e_1, \quad \nabla_{e_2} e_3 = 0, \\
\nabla_{e_3} e_1 &= 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0,
\end{align*}
\]

Using (1.1), the torsion tensor \( T \), with respect to quarter symmetric metric connection \( \nabla \) as follows:

\[
\begin{align*}
T(e_i, e_i) &= 0, \quad \forall i = 1, 2, 3, \\
T(e_1, e_2) &= 0, \quad T(e_1, e_3) = e_3, \quad T(e_2, e_3) = e_2.
\end{align*}
\]

Also,

\[
(\nabla_{e_1} g)(e_2, e_3) = 0, \quad (\nabla_{e_2} g)(e_3, e_1) = 0, \quad (\nabla_{e_3} g)(e_1, e_2) = 0,
\]

Thus \( M \) is a Lorentzian para-Sasakian manifold admitting quarter-symmetric metric connection \( \nabla \).

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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