CONCIRCULAR CURVATURE TENSOR OF KENMOTSU MANIFOLDS ADMITTING GENERALIZED TANAKA-WEBSTER CONNECTION

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Abstract. The objective of the present paper is to study concircular curvature tensor of Kenmotsu manifold with respect to generalized Tanaka-Webster connection, whose concircular curvature tensor satisfies certain conditions and it is shown that if the curvature tensor of a Kenmotsu manifold admitting generalized Tanaka-Webster connection ∇* vanishes, then the Kenmotsu manifold is locally isometric to the hyperbolic space \( H_{2n+1}(-1) \). Further we have studied \( \xi \)-concircularly flat, \( \phi \)-concircularly flat, pseudo-concircularly flat, \( C^\alpha \cdot \phi = 0 \), \( C^\alpha \cdot S^\ast = 0 \) and we have shown that \( R^\ast \cdot C^\ast = R^\ast \cdot R^\ast \). Finally, an example of a 5-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection is given to verify our result.

Keywords: Kenmotsu manifolds; generalized Tanaka-Webster connection; concircular curvature tensor; \( \xi \)-concircularly flat; \( \phi \)-concircularly flat; pseudo-concircularly flat.

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1. INTRODUCTION

The Tanaka-Webster connection is canonical affine connection defined on a non-degenerate pseudo-Hermition CR-manifold [18, 21]. The generalized Tanaka-Webster connection for contact metric manifolds by the canonical connection was first studied by Tanno [19]. This connection coincides with the Tanaka-Webster connection if the associated CR-structure is integrable.

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For a real hypersurface in a Kahler manifold with almost contact structure \((\phi, \xi, \eta, g)\), Cho [4, 5] adapted Tanno’s generalized Tanaka-Webster connection for a non-zero real number \(k\).

Using the generalized Tanaka-Webster connection, some geometers have studied some characterizations of real hypersurfaces in complex space forms [17]. Kenmotsu manifolds introduced by Kenmotsu in 1971[10]. Kenmotsu manifolds have been studied by various others such as Ozgur [14], yildz et al [25], Hui et al [8, 9], Nagaraja et al [11, 12, 13] and many others [2, 22]. Recently many authors[15, 7, 16] have been studied generalized Tanaka-Webster connection in Kenmotsu manifolds.

The present paper is organized as follows: After a brief review of Kenmotsu manifolds in section 2, we study concircular curvature tensor of Kenmotsu manifold with generalized Tanaka-Webster connection and prove that if the curvature tensor of a Kenmotsu manifold admitting generalized Tanaka-Webster connection \(\nabla^*\) vanishes, then the Kenmotsu manifold is locally isometric to the hyperbolic space \(H^{2n+1}(-1)\). Next, we study \(\xi\)-concircularly flat, \(\phi\)-concircularly flat, pseudo-concircularly flat, \(C^*.\phi = 0\) and \(C^*.S^* = 0\) with respect to generalized Tanaka-Webster connection. Then we have proved \(R^*.C^* = R^*.R^*\). Finally, in the last section we give an example of a 5-dimensional Kenmotsu manifold admitting generalized Tanaka-Webster connection to verify our results.

2. Preliminaries

A \((2n+1)\)-dimensional smooth manifold \(M\) is said to be an almost contact metric manifold if it admits an almost contact metric structure \((\phi, \xi, \eta, g)\) consisting of a tensor field \(\phi\) of type \((1, 1)\), a vector field \(\xi\), a 1-form \(\eta\) and a Riemannian metric \(g\) compatible with \((\phi, \xi, \eta)\) satisfying

\[\phi^2 X = -X + \eta(X)\xi, \phi\xi = 0, g(X, \xi) = \eta(X), \eta(\xi) = 1, \eta \circ \phi = 0\]

and

\[g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).\]

An almost contact metric manifold is said to be a Kenmotsu manifold [3] if

\[\nabla_X \phi Y = -g(X, \phi Y)\xi - \eta(Y)\phi X,\]
where $\nabla$ denotes the Riemannian connection of $g$.

In a Kenmotsu manifold the following relations hold [6].

\begin{align}
\nabla_X \xi &= X - \eta(X)\xi, \\
(\nabla_X \eta)Y &= g(\nabla_X \xi, Y), \\
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \\
\eta(R(X, Y)Z) &= g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \\
S(X, \xi) &= -2n\eta(X), \\
Q\xi &= -2n\xi, \\
S(\phi X, \phi Y) &= S(X, Y) + 2n\eta(X)\eta(Y),
\end{align}

for any vector fields $X, Y, Z$ on $M$, where $R$ denote the curvature tensor of type $(1, 3)$ on $M$.

## 3. Main results

Throughout this paper we associate $*$ with the quantities with respect to generalized Tanaka-Webster connection. The generalized Tanaka-Webster connection $\nabla^*$ associated to the Levi-Civita connection $\nabla$ is given by [20, 7]

\begin{align}
\nabla^*_X Y &= \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi - \eta(X)\phi Y,
\end{align}

for any vector fields $X, Y$ on $M$.

Using (4) and (5), the above equation yields,

\begin{align}
\nabla^*_X Y &= \nabla_X Y + g(X, Y)\xi - \eta(Y)X - \eta(X)\phi Y.
\end{align}

By taking $Y = \xi$ in (12) and using (4) we obtain

\begin{align}
\nabla^*_X \xi &= 0.
\end{align}
We now calculate the Riemann curvature tensor $R^*$ using (12) as follows:

\begin{equation}
R^*(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y.
\end{equation}

Using (6) and taking $Z = \xi$ in (14) we get

\begin{equation}
R^*(X, Y)\xi = 0.
\end{equation}

On contracting (14), we obtain the Ricci tensor $S^*$ of a Kenmotsu manifold with respect to the generalized Tanaka-Webster connection $\nabla^*$ as

\begin{equation}
S^*(Y, Z) = S(Y, Z) + 2ng(Y, Z).
\end{equation}

This gives

\begin{equation}
Q^*Y = QY + 2nY.
\end{equation}

Contracting with respect to $Y$ and $Z$ in (16), we get

\begin{equation}
r^* = r + 2n(2n + 1),
\end{equation}

where $r^*$ and $r$ are the scalar curvatures with respect to the generalized Tanaka-Webster connection $\nabla^*$ and the Levi-Civita connection $\nabla$ respectively.

**Definition 3.1.** A Kenmotsu manifold with respect to the Levi-Civita connection is of constant curvature if its curvature tensor $R$ is of the form

\[ g(R(X, Y)Z, U) = k\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}, \]

where $k$ is a constant.

If $R^* = 0$, then the equation (14) becomes

\begin{equation}
R(X, Y, Z, U) = -\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}.
\end{equation}

From which it follows that the Kenmotsu manifold with respect to the Levi-Civita connection is of constant curvature $-1$.

This leads to the following:

**Theorem 3.1.** If curvature tensor of a Kenmotsu manifold with respect to generalized Tanaka-Webster connection $\nabla^*$ is vanishes, then the Kenmotsu manifold is locally isometric to the
hyperbolic space $H^{2n+1}(-1)$.

**Definition 3.2.** [1] For each plane $p$ in the tangent space $T_x(M)$, the sectional curvature $K(p)$ is defined by $K(p) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}$, where $\{X,Y\}$ is orthonormal basis for $p$. Clearly $K(p)$ is independent of the choice of the orthonormal basis $\{X,Y\}$.

Taking $Z = X$, $U = Y$ in (19), we get

$$R(X,Y,X,Y) = \{g(X,X)g(Y,Y) - g(X,Y)g(X,Y)\}.$$  

Then from the above equation we conclude that

$$K(p) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} = -1.$$  

Thus we can state the following theorem:

**Theorem 3.2.** If in a Kenmotsu manifold, the curvature tensor of a generalized Tanaka-Webster connection $\nabla^*$ vanishes, then the sectional curvature of the plane determined by two vectors $X, Y \in \xi^\perp$ is $-1$.

Now, an interesting invariant of a concircular transformation is the concircular curvature tensor. The concircular curvature tensor [23] $C^*$ with respect to the generalized Tanaka-Webster connection $\nabla^*$ is defined by

$$C^*(X,Y)Z = R^*(X,Y)Z - \frac{r^*}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\},$$  

for all vector fields $X$, $Y$, $Z$ on $M$.

By interchanging $X$ and $Y$ in (22), we have

$$C^*(Y,X)Z = R^*(Y,X)Z - \frac{r^*}{2n(2n+1)}\{g(X,Z)Y - g(Y,Z)X\}.$$  

On adding (22) and (23) and using the fact that $R(X,Y)Z + R(Y,X)Z = 0$, we get

$$C^*(X,Y)Z + C^*(Y,X)Z = 0.$$  

From (14), (22) and first Bianchi identity $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ with respect to $\nabla$, we obtain

$$C^*(X,Y)Z + C^*(Y,Z)X + C^*(Z,X)Y = 0.$$
Hence, from (24) and (25), shows that concircular curvature tensor with respect to generalized Tanaka-Webster connection in a Kenmotsu manifold is skew-symmetric and cyclic.

Next, we assume that the manifold $M$ with respect to the generalized Tanaka-Webster connection is concircularly flat, that is, $C^*(X,Y)Z = 0$. Then from (22), it follows that

$$R^*(X,Y)Z = \frac{r^*}{2n(n+1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{26}$$

Taking inner product of the above equation with $\xi$, we have

$$g(R^*(X,Y)Z, \xi) = \frac{r^*}{2n(n+1)}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}. \tag{27}$$

Using (1), (7), (14) and (18) in (27), we get

$$\frac{r + 2n(n+1)}{2n(n+1)}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} = 0. \tag{28}$$

Replacing $X$ by $QX$ in (28), we obtain

$$\frac{r + 2n(n+1)}{2n(n+1)}\{g(Y,Z)\eta(QX) - g(QX,Z)\eta(Y)\} = 0. \tag{29}$$

Using (8) in (29), we get

$$\frac{r + 2n(n+1)}{2n(n+1)}\{-2ng(Y,Z)\eta(X) - S(X,Z)\eta(Y)\} = 0. \tag{30}$$

Taking $Y = \xi$ in (30), yields

$$\frac{r + 2n(n+1)}{2n(n+1)}\{-2n\eta(X)\eta(Z) - S(X,Z)\} = 0. \tag{31}$$

This implies either the scalar curvature of $M$ is $-2n(2n+1)$ or

$$S(X,Z) = -2n\eta(X)\eta(Z). \tag{32}$$

Hence we can state the following theorem:

**Theorem 3.3.** For a concircularly flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection, either the scalar curvature is $-2n(2n+1)$ or the manifold is a special type of $\eta$-Einstein manifold.
Definition 3.3. A Kenmotsu manifold with respect to the generalized Tanaka-Webster connection $\nabla^*$ is said to be $\xi$-concircularly flat if $C^*(X,Y)\xi = 0$.

In view of (14) and (18) in (22), we get

$$C^*(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y - \frac{r + 2n(2n + 1)}{2n(2n + 1)} \{g(Y,Z)X - g(X,Z)Y\}. \tag{33}$$

By taking $Z = \xi$ in (33) and then using (1) and (6), we find

$$C^*(X,Y)\xi = \frac{r + 2n(2n + 1)}{2n(2n + 1)} R(X,Y)\xi. \tag{34}$$

Thus from (14), (18), (33) and (34), we have the following theorem:

**Theorem 3.4.** Let $M$ be a Kenmotsu manifold with generalized Tanaka-Webster connection. In $M$, the following three conditions are equivalent:

i) $M$ is $\xi$-concircularly flat.

ii) $r = -2n(2n + 1)$.

iii) $r^* = 0$.

Now, we assume that the manifold $M$ with respect to the generalized Tanaka-Webster connection is $\xi$-concircularly flat, that is, $C^*(X,Y)\xi = 0$. Then from (22), it follows that

$$R^*(X,Y)\xi = \frac{r^*}{2n(2n + 1)} \{\eta(Y)X - \eta(X)Y\}. \tag{35}$$

In view of (15) and (18), we have

$$\frac{r + 2n(2n + 1)}{2n(2n + 1)} \{\eta(Y)X - \eta(X)Y\} = 0. \tag{36}$$

Taking $Y = \xi$ in (36) and using (1), we get

$$\frac{r + 2n(2n + 1)}{2n(2n + 1)} \{X - \eta(X)\xi\} = 0. \tag{37}$$

Taking inner product of the above equation with $U$, we have

$$\frac{r + 2n(2n + 1)}{2n(2n + 1)} \{g(X,U) - \eta(X)\eta(U)\} = 0. \tag{38}$$

Now, replacing $X$ by $QX$ in (38), we obtain

$$\frac{r + 2n(2n + 1)}{2n(2n + 1)} \{g(QX,U) - \eta(QX)\eta(U)\} = 0. \tag{39}$$
Using (9) in (39), we get

\[ r + 2n(2n+1) \left\{ S(X, U) + 2n\eta(X)\eta(U) \right\} = 0. \]

This implies either the scalar curvature of \( M \) is \(-2n(2n+1)\) or

\[ S(X, U) = -2n\eta(X)\eta(U). \]

Hence we can state the following theorem:

**Theorem 3.5.** For a \( \xi \)-concircularly flat Kenmotsu manifold with respect to the generalized Tanaka-Webster connection, either the scalar curvature is \(-2n(2n+1)\) or the manifold is a special type of \( \eta \)-Einstein manifold.

**Definition 3.4.** A Kenmotsu manifold is said to be \( \phi \)-concircularly flat with respect to the generalized Tanaka-Webster connection \( \nabla^* \) if

\[ g(R^*(\phi X, \phi Y)\phi Z, \phi W) = 0, \]

for any vector fields \( X, Y, Z \) on \( M \).

Using (22) in (42), we have

\[ g(R^*(\phi X, \phi Y)\phi Z, \phi W) = \frac{r^*}{2n(2n+1)} \left\{ g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W) \right\}. \]

Let \( \{e_1, e_2, e_3, \ldots, e_{2n+1}\} \) be a local orthonormal basis of vector fields in \( M \). Then \( \{\phi e_1, \phi e_2, \phi e_3, \ldots, \phi e_{2n+1}\} \) is also a local orthonormal basis. If we put \( X = W = e_i \) in (43) and summing up with respect to \( i, 1 \leq i \leq 2n + 1 \), we obtain

\[ \sum_{i=1}^{2n} g(R^*(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{r^*}{2n(2n+1)} \sum_{i=1}^{2n} \left\{ g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \right\}. \]

From (44), it follows that

\[ S^*(\phi Y, \phi Z) = \frac{r^*(2n-1)}{2n(2n+1)} g(\phi Y, \phi Z). \]

Using (1), (16) and (18) in (45), we get

\[ S(\phi Y, \phi Z) + 2ng(\phi Y, \phi Z) = \frac{(r + 2n(2n+1))(2n-1)}{2n(2n+1)} g(\phi Y, \phi Z). \]
By using (2) and (10) in (46), we obtain

\[ S(Y, Z) + 2n \eta(Y) \eta(Z) + \{ 2n - \frac{(r + 2n(2n + 1))(2n - 1)}{2n(2n + 1)} \} g(\phi Y, \phi Z) = 0. \]  

(47)

Hence contracting (47), we get

\[ r = -2n. \]  

(48)

By substituting equation (48) in (22), we get

\[ C^*(X, Y)Z = R(X, Y)Z + \frac{1}{2n + 1} \{ g(Y, Z)X - g(X, Z)Y \}. \]  

(49)

This leads to the following:

**Theorem 3.6.** Let the Kenmotsu manifold \( M \) with generalized Tanaka-Webster connection be \( \phi \)-concircularly flat. Then \( M \) is of constant sectional curvature \( -\frac{1}{2n + 1} \) if and only if concircular curvature tensor \( C^* \) vanishes.

**Definition 3.5.** A Kenmotsu manifold is said to be pseudo-concircularly flat with respect to the generalized Tanaka-Webster connection \( \nabla^* \) if it satisfies

\[ g(C^*(\phi X, Y)Z, \phi W) = 0, \]  

(50)

for any vector fields \( X, Y, Z \) on \( M \).

In view of (22) and (50), we have

\[ g(R^*(\phi X, Y)Z - \frac{r^*}{2n(2n + 1)} \{ g(Y, Z)\phi X - g(\phi X, Z)Y \}, \phi W) = 0. \]  

(51)

Making use of (14) and (18) in (51), we get

\[ g(R(\phi X, Y)Z, \phi W) - \frac{r}{2n(2n + 1)} \{ g(Y, Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W) \} = 0. \]  

(52)

Let \( \{ e_1, e_2, e_3, \ldots, e_{2n+1} \} \) be a local orthonormal basis of vector fields in \( M \). Then by putting \( Y = Z = e_i \) in (52) and summing up with respect to \( i, 1 \leq i \leq 2n + 1 \), we obtain

\[ S(\phi X, \phi W) = \frac{r}{2n + 1} g(\phi X, \phi W). \]  

(53)

On using (1) and (10) in (53), we get

\[ S(X, W) = \frac{r}{2n + 1} g(X, W) - \{ 2n + \frac{r}{2n + 1} \} \eta(X) \eta(W). \]  

(54)
Again taking $X = W = e_i$ in (54) and summing up with respect to $i, 1 \leq i \leq 2n + 1$, we obtain

\[(55) \quad r = -2n(2n + 1).\]

By virtue of (54) and (55), we get

\[(56) \quad S(X, W) = -2ng(X, W).\]

Thus $M$ is an Einstein manifold.

Again by substituting (55) in (33), we obtain

\[(57) \quad C^*(X, Y)Z = R(X, Y)Z + \{g(Y, Z)X - g(X, Z)Y\}.\]

Thus, from the above discussions we state the following:

**Theorem 3.7.** Let the Kenmotsu manifold $M$ with generalized Tanaka-Webster connection be pseudo-concircularly flat if and only if $S(Y, Z) = -2ng(Y, Z)$.

Further if $C^* = 0$, then $M$ is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.

**Definition 3.6.** A Kenmotsu manifold is said to be $\phi$-concircularly semisymmetric with respect to generalized Tanaka-Webster connection $\nabla^*$ if $C^*(X, Y).\phi = 0$ holds on $M$.

Now, we consider $\phi$-concircularly semisymmetric Kenmotsu manifold with respect to generalized Tanaka-Webster connection. Then

\[(58) \quad (C^*(X, Y).\phi)Z = C^*(X, Y)\phi Z - \phi C^*(X, Y)Z = 0.\]

for all $X, Y, Z$ on $M$.

Taking $Z = \xi$ in (58), we get

\[(59) \quad \phi(C^*(X, Y)\xi) = 0.\]

Using (34) and (6) in (59), we get

\[(60) \quad \frac{r + 2n(2n + 1)}{2n(2n + 1)}\{\eta(X)\phi Y - \eta(Y)\phi X\} = 0.\]

Replace $Y$ by $\xi$ and $X$ by $\phi X$ in (60) and using (1), we get

\[(61) \quad \frac{r + 2n(2n + 1)}{2n(2n + 1)}\{X - \eta(X)\xi\} = 0.\]
Taking inner product of the above equation with $U$, we have
\begin{equation}
(62) \quad \frac{r + 2n(2n+1)}{2n(2n+1)} \{g(X, U) - \eta(X)\eta(U)\} = 0.
\end{equation}

Now, replacing $X$ by $QX$ in (62), we obtain
\begin{equation}
(63) \quad \frac{r + 2n(2n+1)}{2n(2n+1)} \{g(QX, U) - \eta(QX)\eta(U)\} = 0.
\end{equation}

Using (9) in (63), we get
\begin{equation}
(64) \quad \frac{r + 2n(2n+1)}{2n(2n+1)} \{S(X, U) + 2n\eta(X)\eta(U)\} = 0.
\end{equation}

This implies either the scalar curvature of $M$ is $-2n(2n+1)$ or
\begin{equation}
(65) \quad S(X, U) = -2n\eta(X)\eta(U).
\end{equation}

Hence we can state the following:

**Theorem 3.8.** For a $\phi$-concircularly semisymmetric Kenmotsu manifold with respect to the generalized Tanaka-Webster connection, either the scalar curvature is $-2n(2n+1)$ or the manifold is a special type of $\eta$-Einstein manifold.

Now, we consider
\begin{equation}
\end{equation}

By making use of (22) and (16) in (66), we obtain
\begin{equation}
(67) \quad C^*.S^* = S(R(X, Y)Z, U) - \frac{r}{2n(2n+1)} \{g(Y, Z)X - g(X, Z)Y\}, U) + S(Z, R(X, Y)U - \frac{r}{2n(2n+1)} \{g(Y, U)X - g(X, U)Y\}).
\end{equation}

Suppose $C^*.S^* = 0$. Then we have
\begin{equation}
(68) \quad S^*(C^*(X, Y)Z, U) + S^*(Z, C^*(X, Y)U) = 0.
\end{equation}

Taking $U = \xi$ in (68) and using (16), it follows that
\begin{equation}
(69) \quad S^*(Z, C^*(X, Y)\xi) = 0.
\end{equation}

Making use of (1), (6) and (33) in (69), we get
\begin{equation}
(70) \quad \frac{r + 2n(2n+1)}{2n(2n+1)} S^*(Z, \eta(X)Y - \eta(Y)X) = 0.
\end{equation}
Replacing $X$ by $\xi$ in (70) and using (1) and (16), we get

$$\frac{r + 2n(2n + 1)}{2n(2n + 1)} \{ S(Z, Y) + 2ng(Z, Y) \} = 0. \tag{71}$$

Contracting (71) with respect to $Y$ and $Z$, we get

$$r = -2n(2n + 1). \tag{72}$$

From (67) and (72), we obtain

$$S(Y, Z) = -2ng(Y, Z). \tag{73}$$

Thus $M$ is an Einstein manifold.

Again by substituting (72) in (33), we obtain

$$C^*(X, Y)Z = R(X, Y)Z + \{ g(Y, Z)X - g(X, Z)Y \}. \tag{74}$$

Thus, from the above discussions we state the following:

**Theorem 3.9.** Let $M$ be a Kenmotsu manifold with generalized Tanaka-Webster connection, then $C^* . S^* = 0$ if and only if $S(Y, Z) = -2ng(Y, Z)$.

Further if $C^* = 0$ then $M$ is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.

Further, we have

$$\begin{align*}
\end{align*} \tag{75}$$

With the use of (22), (75) becomes

$$\begin{align*}
&- R^*(U, V)R^*(X, Y)W + \frac{r^*}{2n(2n + 1)} \{ g(R^*(X, Y)V, W)U + g(V, R^*(X, Y)W)U \\
&- g(R^*(X, Y)U, W)V - g(U, R^*(X, Y)V)W \}.
\end{align*} \tag{76}$$

By the symmetric properties of the curvature tensor $R^*$ [7, 16], we get

$$\begin{align*}
\end{align*} \tag{77}$$
Finally, we get


Thus we state the following:

**Theorem 3.10.** Let \(M\) be a Kenmotsu manifold with generalized Tanaka-Webster connection. Then \(R^*.C^* = R^*.R^*\).

4. **Example of a 5-dimensional Kenmotsu manifold with respect to the generalized Tanaka-Webster connection**

We consider the five-dimensional manifold \(M = \{(x,y,z,u,v) \in \mathbb{R}^5\}\), where \((x,y,z,u,v)\) are the standard coordinates in \(\mathbb{R}^5\). The vector fields

\[
E_1 = e^{-v} \frac{\partial}{\partial x}, \quad E_2 = e^{-v} \frac{\partial}{\partial y}, \quad E_3 = e^{-v} \frac{\partial}{\partial z}, \quad E_4 = e^{-v} \frac{\partial}{\partial u}, \quad E_5 = \frac{\partial}{\partial v}
\]

are linearly independent at each point of \(M\). Let \(g\) be the Riemannian metric defined by

\[
g_{ij} = \begin{cases} 
1 & \text{for } i = j, \\
0 & \text{for } i \neq j.
\end{cases}
\]

Let \(\eta\) be the 1-form defined by \(\eta(Z) = g(Z, E_3)\) for any \(Z \in \chi(M)\). Let \(\phi\) be the \((1,1)\) tensor field defined by \(\phi E_1 = E_3, \phi E_2 = E_4, \phi E_3 = -E_1, \phi E_4 = -E_2, \phi E_5 = 0\). Then using the linearity of \(\phi\) and \(g\) we have

\[
\eta(E_5) = 1, \quad \phi^2(Z) = -Z + \eta(Z)E_5, \quad g(\phi Z, \phi U) = g(Z,U) - \eta(Z)\eta(U),
\]

for any \(Z, U \in \chi(M)\). Thus for \(E_5 = \xi, (\phi, \xi, \eta, g)\) defines an almost contact metric structure on \(M\).

Let \(\nabla\) be the Levi-Civita connection with respect to the metric \(g\). Then we have

\[
[E_1, E_2] = [E_1, E_3] = [E_1, E_4] = [E_2, E_3] = 0, \quad [E_1, E_5] = E_1,
\]

\[
\]
The Riemannian connection $\nabla$ of the metric $g$ is given by the Koszul’s formula

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y))$$

$$- g([X, Y], Z) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

By Koszul’s formula, we get

$$\nabla_{E_1} E_1 = -E_5, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = 0, \quad \nabla_{E_1} E_4 = 0, \quad \nabla_{E_1} E_5 = E_1,$nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = -E_5, \quad \nabla_{E_2} E_3 = 0, \quad \nabla_{E_2} E_4 = 0, \quad \nabla_{E_2} E_5 = E_2,$nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = -E_5, \quad \nabla_{E_3} E_4 = 0, \quad \nabla_{E_3} E_5 = E_3,$nabla_{E_4} E_1 = 0, \quad \nabla_{E_4} E_2 = 0, \quad \nabla_{E_4} E_3 = 0, \quad \nabla_{E_4} E_4 = -E_5, \quad \nabla_{E_4} E_5 = E_4,$nabla_{E_5} E_1 = 0, \quad \nabla_{E_5} E_2 = 0, \quad \nabla_{E_5} E_3 = 0, \quad \nabla_{E_5} E_4 = 0, \quad \nabla_{E_5} E_5 = 0.$

Further we obtain the following:

$$\nabla^*_E E_j = 0, \quad i, j = 1, 2, 3, 4, 5$$

and hence

$$(\nabla^*_E \phi) E_j = 0, \quad i, j = 1, 2, 3, 4, 5.$$}

From the above expressions it follows that the manifold satisfies (2), (3) and (4) for $\xi = E_5$. Hence the manifold is a Kenmotsu manifold. With the help of the above results we can verify the following results.

$$R(E_1, E_2) E_2 = R(E_1, E_3) E_3 = R(E_1, E_4) E_4 = R(E_1, E_5) E_5 = -E_1,$nabla_{E_2} E_1 = E_2, \quad \nabla_{E_3} E_1 = R(E_1, E_3) E_4 = R(E_1, E_5) E_5 = -E_2, \quad \nabla_{E_3} E_2 = -E_2, \quad \nabla_{E_3} E_4 = E_3,$nabla_{E_4} E_1 = R(E_1, E_4) E_2 = R(E_1, E_5) E_3 = R(E_1, E_5) E_4 = -E_3,$nabla_{E_5} E_1 = R(E_1, E_5) E_2 = R(E_1, E_5) E_3 = R(E_1, E_5) E_4 = R(E_1, E_5) E_5 = -E_4,$nabla_{E_1} E_2 = R^*(E_i, E_j) E_k = 0, \quad i, j, k = 1, 2, 3, 4, 5.
From the above expressions of the curvature tensor of the Kenmotsu manifold it can be easily seen that the manifold has a constant sectional curvature $-1$.

Making use of the above results we obtain the Ricci tensors as follows:

$$S(E_1, E_1) = g(R(E_1, E_2)E_2, E_1) + g(R(E_1, E_3)E_3, E_1) + g(R(E_1, E_4)E_4, E_1)$$
$$+ g(R(E_1, E_5)E_5, E_1) = -4.$$

Similarly, we have

$$S(E_2, E_2) = S(E_3, E_3) = S(E_4, E_4) = S(E_5, E_5) = -4$$

and

$$S^*(E_1, E_1) = S^*(E_2, E_2) = S^*(E_3, E_3) = S^*(E_4, E_4) = S^*(E_5, E_5) = 0.$$

Therefore, it can be easily verified that the manifold is an Einstein manifold with respect to Levi-Civita connection.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**References**


