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FIXED POINT THEOREMS FOR WEAK ψ-QUASI CONTRACTIONS ON A GENERALIZED METRIC SPACE WITH PARTIAL ORDER

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Abstract. In this paper, we obtain conditions for a weak ψ-quasi contraction on a generalized metric space with a partial order to have a fixed point. These results generalize some of the previously known results (Jleli and Samet [11], Sastry et al. [8]).

Keywords: generalized metric space; weak ψ-quasi contraction; fixed point; partial order.

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1. Introduction

The concept of metric was introduced by Frechet [2] as an extension of the distance on the real line. In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the Banach contraction principle was given shape in the context of metric spaces.Later several generalization of Banach contraction principle were obtained.Some of the generalizations of Banach contraction principle were also extended to the generalized version of metric spaces. In 1993, Czerwik [15] introduced the concept of a b-metric space. Since then

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fixed point results in b-metric spaces were obtained by several authors. In 2000, Hitzler and Seda [14] introduced the notation of dislocated metric spaces. In dislocated metric spaces the self distance of a point may be non-zero and this concept has played a very important role in topology and logical programming. For more works we refer [3, 4, 10, 12, 13, 16]. Combining several generalizations of metric spaces in 2015, Jleli and Samet [11] obtained a generalization of the notion of a metric space which they called a generalized metric space.They also stated and proved fixed point theorems for some contractions defined on these spaces.

Recently some researchers have focused on the existence of fixed points in metric spaces endowed with partial orders . Fixed point theorems in partially ordered metric spaces were firstly obtained in 2004 by Ran and Reurings [1], and then by Nieto and Rodriguez-Lopez [6] .Jleli and Samet [11] obtained fixed point results for self maps on a generalized metric space with partial order.

In this paper we prove the existence of fixed points for weak ψ-quasi contractions on a generalized metric space with partial order and obtain results of Jleli and Samet [11] and Sastry et al. [8] as corollaries.

2. PRELIMINARIES

In this section we give the definition of generalized metric space and obtain certain properties of generalized metric which we use in the later development.

Notation: (Jleli and Samet [11]) Let X be a non-empty set and $D : X \times X \to [0,+\infty]$ be a given mapping. For every $x \in X$, let us define the set $\mathscr{C}(D,X,x) = \{\{x_n\} \subset X : \lim_{n\to\infty} D(x_n,x) = 0\}$.

Definition 2.1. (Jleli and Samet [11]) Let X be a non-empty set and $D : X \times X \to [0,+\infty]$ be a function which satisfies the following conditions:

(2.1.1) $D(x,y) = 0$ implies $x = y$

(2.1.2) $D(x,y) = D(y,x)$ for all $x,y \in X$

(2.1.3) there exists $\lambda > 0$ such that if $x,y \in X$ and $\{x_n\} \in \mathscr{C}(D,X,x)$, then $D(x,y) \leq \lambda \limsup_{n\to\infty} D(x_n,y)$.

Then D is called a generalized metric and the pair (X,D) is called a generalized metric space with coefficient λ. In general we drop λ. It may be noted that in a generalized metric space, the distance between two points may be infinite. $D(x,y)$ is called the generalized distance between x and y.

Remark *2.2.* (Jleli and Samet [11]) Obviously , if the set (D,X,x) is empty for every $x \in X$ then (X,D) is a generalized metric space if and only if (2.1.1) and (2.1.2) are satisfied.

Definition 2.3. (Jleli and Samet [11]) Let (X,D) be a generalized metric space . Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that sequence $\{x_n\}$ is D-convergent to x if $\{x_n\} \in \mathscr{C}(D,X,x)$.

Proposition 2.4. (Jleli and Samet [11]) Let (X,D) be a generalized metric space . Let $\{x_n\}$ be a sequence in X and $x,y \in X$. If $\{x_n\}$ is D-convergent to x and $\{x_n\}$ is D-convergent to y then $x = y$.

Definition 2.5. (Jleli and Samet [11]) Let (X,D) be a generalized metric space . Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that sequence $\{x_n\}$ is a D- Cauchy sequence if $\lim_{m,n\to\infty} D(x_n,x_{n+m}) = 0$.

Definition 2.6. (Jleli and Samet [11]) Let (X,D) be a generalized metric space . It is said to be D-complete if every D-Cauchy sequence in X is convergent to some element in X.

Here in after we use converges in place of D-converges when there is no confusion.

Definition 2.7. Let $f : X \to X$ be a self map and $x \in X$. write $f^1(x) = f(x)$ and $f^{n+1}(x) = f(f^n(x))$ for $n = 1,2,3....$ For convenience we write $x = f^0(x)$, $x_1 = f^1(x)$ and $x_{n+1} = f(x_n)$ for $n = 1,2,3....$ Then $\{x_n\}$ is called the sequence of iterates of f at x.

We use the following two results in section 3.

Theorem 2.8. (Sastry et al. [7]) Let (X,D) be a generalized metric space . Suppose $\{x_n\} \subset X, x \in X$ and $x_n \to x$. Then $D(x,x) = 0$

Theorem 2.9. (Sastry et al. [7]) Let (X,D) be a generalized metric space and $x \in X$. Suppose $\mathscr{C}(D,X,x) \neq \phi$ Then $D(x,x) = 0$.

3. MAIN RESULTS

We start with the following notation which we use in the subsequent development.

Suppose $\lambda > 1$. We write

$\Psi_\lambda = \{\psi : [0,\infty] \to [0,\infty] | \psi$ is non-decreasing, $\psi(t) = 0 \Longleftrightarrow t = 0$ and

$$\psi(t) < \frac{t}{\lambda} \text{ for } t > 0\}. \text{ and}$$

$\Psi_1 = \{\psi : [0,\infty] \to [0,\infty] | \psi$is non-decreasing, right continuous,$\psi(t) = 0 \Longleftrightarrow t = 0$ and

$$\psi(t) < t \text{ for } t > 0\}.$$

Lemma 3.1. (Sastry et al. [9]) If $\psi \in \Psi_\lambda$ then $\lim\limits_{n\to\infty} \psi^n(t) = 0$.

Definition 3.2. Let (X,D) be a generalized metric space and $\preceq$ be a partial order on X .Then we say that $(X,D,\preceq)$ is a generalized metric with partial order. If $x,y \in X$ and either $x \preceq y$ or $y \preceq x$ then we say that x and y are comparable.

If $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all n then we say that $\{x_n\}$ is an increasing sequence.

If $x_{n+1} \preceq x_n$ for all n, we say that $\{x_n\}$ is a decreasing sequence.

Suppose $(X,D,\preceq)$ is a generalized metric space with partial order and $f : X \to X$. We say that f is an increasing function if $x \preceq y$ implies $f(x) \preceq f(y)$, we say that f is decreasing if $x \succeq y$ implies $f(x) \preceq f(y)$.

Definition 3.3. Let $(X,D,\preceq)$ be a generalized metric space with partial order, $\psi \in \Psi_\lambda$ and $f : X \to X$ be a mapping.Write

$$M(x,y) = \max\{D(x,y), D(x,fx), D(y,fy), D(x,fy), D(y,fx)\}.$$

We say that f is weak ψ- quasi contraction if

$$D(fx,fy) \le \psi(M(x,y)), \tag{1}$$

whenever x and y are comparable.

Definition 3.4. (Jleli and Samet [11]) Suppose $(X,D,\preceq)$ is a generalized metric space with partial order . We say that

(3.4.1) X is D-regular (increasing) if $\{x_n\}$ is an increasing sequence in X, $\{x_n\}$ is D- convergent to x implies $x_n \preceq x$ for all n and $x_n \preceq y$ for all n implies $x \preceq y$.

(3.4.2) X is D-regular (decreasing) if $\{x_n\}$ is a decreasing sequence in X, $\{x_n\}$ is D- convergent to x implies $x_n \succeq x$ for all n and $x_n \succeq y$ for all n implies $x \succeq y$.

Definition 3.5. (Jleli and Samet [11]) We say that f is weak continuous if the following condition holds:if $\{x_n\} \subset X$ is D-convergent to $x \in X$, then there exists subseqence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{f(x_{n_k})\}$ is D-convergent to $f(x)$ (as $k \to \infty$).

Now we state and prove our main result.

Theorem 3.6. Let $(X, D, \preceq)$ be a D-complete generalized metric space with partial order. Assume the following conditions holds in X.

(3.6.1) $(X, D, \preceq)$ is D-regular(increasing);

(3.6.2) $f : X \to X$ is an increasing function;

(3.6.3) f is a weak ψ -quasi contraction;

(3.6.4) There exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

$$sup_n D(x_0, f^n(x_0)) < \alpha < \infty \text{ and} \tag{2}$$

$$D(f^n(x_0), f^{n+1}(x_0)) \leq \psi^n(\alpha) \text{ for every } n \tag{3}$$

Then $\{f^n(x_0)\}$ is Cauchy and hence converges, say, to $w \in X$. If $\limsup_n D(f^n(x_0), fw) < \infty$ then w is a fixed point of f.

Moreover, if w' is another fixed point of f comparable with w such that $D(w, w') < \infty$ and $D(w', w') < \infty$ then $w = w'$.

Proof. If $x_0 = fx_0$ then f has a fixed point . Suppose that $x_0 \prec fx_0$ then we construct a sequence $\{f^n x_0\} \in X$ as follows $f^{n+1}x_0 = f(f^n x_0)$ for $n = 0, 1, 2, \ldots$.

Since f is a nondecreasing map and $x_0 \prec fx_0$ then we have $fx_0 \preceq f^2 x_0$. In general, by induction we can show $f^n x_0 \preceq f^{n+1}x_0$ for every n. Thus $\{f^n x_0\}$ is an increasing sequence.

Fist we show that

$$D(f^n(x_0), f^n(x_0)) \leq \psi^n(\alpha) \text{ for every } n \tag{4}$$

The result is true for $n = 0$, by (2). Now assume the truth for n i.e.,

$$D(f^n(x_0), f^n(x_0)) \leq \psi^n(\alpha). \tag{5}$$

We show that $D(f^{n+1}(x_0), f^{n+1}(x_0)) \leq \psi^{n+1}(\alpha)$.

Now $D(f^{n+1}(x_0), f^{n+1}(x_0))$= $D(f(f^n(x_0)), f(f^n(x_0))) \leq \psi(M(f^n(x_0), f^n(x_0))$

where $M(f^n(x_0), f^n(x_0) = \max\{D(f^n(x_0), f^n(x_0)), D(f^n(x_0), f^{n+1}(x_0)), D(f^n(x_0), f^{n+1}(x_0)),$
$D(f^n(x_0), f^{n+1}(x_0)), D(f^n(x_0), f^{n+1}(x_0))\}$
$\leq \max\{\psi^n(\alpha), \psi^n(\alpha), \psi^n(\alpha), \psi^n(\alpha), \psi^n(\alpha)\}$
$= \psi^n(\alpha)$, by (3) and (5).

Therefore $D(f^{n+1}(x_0), f^{n+1}(x_0)) \leq \psi(\psi^n(\alpha)) = \psi^{n+1}(\alpha)$.

Therefore (4) holds for every n.

Now we show that $D(f^n(x_0), f^{n+m}(x_0)) \leq \psi^n(\alpha)$ for $n = 0, 1, 2, ..., m = 0, 1, 2, ...$

If $n = 0$ then $D(x_0, f^m(x_0)) < \alpha = \psi^0(\alpha)$, by (2).

Assume that this is true for n i.e.,

$$D(f^n(x_0), f^{n+m}(x_0)) \leq \psi^n(\alpha) \text{ for } m = 0, 1, 2, ... \tag{6}$$

We show that $D(f^{n+1}(x_0), f^{n+1+m}(x_0)) \leq \psi^{n+1}(\alpha)$ for $m = 0, 1, 2, ...$

The result is true if. $m = 0$, by (4). Now assume the truth for m i.e.,

$$D(f^{n+1}(x_0), f^{n+1+m}(x_0)) \leq \psi^{n+1}(\alpha). \tag{7}$$

We show that $D(f^{n+1}(x_0), f^{n+1+m+1}(x_0)) \leq \psi^{n+1}(\alpha)$.

We have $f^n(x_0)$ and $f^{n+m}(x_0)$ are comparable (since $f^n(x_0)$ is an increasing sequence).

Consider $D(f^{n+1}(x_0), f^{n+1+m+1}(x_0)) = D(f(f^n(x_0)), f(f^{n+1+m}(x_0)))$
$\leq \psi(M(f^n(x_0), f^{n+1+m}(x_0))$

where

$M(f^n(x_0), f^{n+1+m}(x_0) = \max\{D(f^n(x_0), f^{n+1+m}(x_0), D(f^n(x_0), f^{n+1}(x_0),$
$D(f^{n+1+m}(x_0), f^{n+1+m+1}(x_0), D(f^n(x_0), f^{n+1+m+1}(x_0),$
$D(f^{n+1+m}(x_0), f^{n+1+}(x_0)\}$
$\leq \max\{\psi^n(\alpha), \psi^n(\alpha), \psi^{n+1+m}(\alpha), \psi^n(\alpha), \psi^{n+1}(\alpha)\}$

$= \psi^n(\alpha)$, by (3), (6) and (7).

Therefore $D(f^{n+1}(x_0), f^{n+1+m+1}(x_0)) \leq \psi(\psi^n(\alpha)) = \psi^{n+1}(\alpha)$.

This is true for $m+1$. Therefore this is true for every n.

We have $D(f^n(x_0), f^{n+m}(x_0)) \leq \psi^n(\alpha)$ for $n = 0,1,2,...,m = 0,1,2,.....$

On letting $n \to \infty$, we have,

$\lim_{n,m\to\infty} D(f^n(x_0), f^{n+m}(x_0)) \leq \lim_{n\to\infty} \psi^n(\alpha) \to 0$ *as* $n \to \infty$, by Lemma 3.1.

Therefore $\{f^n(x_0)\}$ is Cauchy and hence converges to a limit say $w \in X$.

Since $(X, D, \preceq)$ is D-regular(increasing), it follows that $f^n x_0 \preceq w$ for every n.

Now $D(f^{n+1}(x_0), fw) = D(f(f^n(x_0), fw) \leq \psi(M(f^n(x_0), w))$,

where

$$M(f^n(x_0), w) = \max\{D(f^n(x_0), w), D(f^n(x_0), f^{n+1}(x_0)), D(w, fw), D(f^n(x_0), fw), D(w, f^{n+1}(x_0))\}$$
$$\leq \max\{\varepsilon, \varepsilon, \lambda \overline{\lim} D(f^n(x_0), fw), \overline{\lim} D(f^n(x_0), fw)\} \text{ for large } n.$$

Write $\mu = \overline{\lim} D(f^n(x_0), fw)$. On letting $n \to \infty$ we get,

$\mu \leq \psi(\max\{\varepsilon, \lambda\mu, \mu\})$ for large n. Therefore $\mu \leq \psi(\varepsilon) < \varepsilon$ for small ε.

Therefore $\mu = 0$ i.e., $\overline{\lim} D(f^n(x_0), fw) = 0$

(since ε is arbitrary and $\lim sup_n D(f^n(x_0), fw) < \infty$ by hypothesis).

Therefore $f^n(x_0) \to f(w)$ as $n \to \infty$. Hence $f(w) = w$. Therefore w is a fixed point of f.

Suppose w' is also a fixed point of f comparable with w such that $D(w, w') < \infty$ and $D(w', w') < \infty$.

Now $D(w, w') = D(fw, fw') \leq \psi(M(w, w'))$,

where

$M(w, w') = \max\{D(w, w'), D(w, w), D(w', w'), D(w, w'), D(w, w')\} = D(w, w')$.

Therefore $D(w, w') \leq \psi(D(w, w')) < D(w, w')$ so that $D(w, w') = 0$.

Hence uniqueness follows.

The following Theorem can be proved following the lines of proof of the above Theorem.

Theorem 3.7. Let $(X, D, \preceq)$ be a D-complete generalized metric space with partial order. Assume the following conditions holds in X.

(3.7.1) $(X, D, \preceq)$ is D-regular(decreasing);

(3.7.2) $f : X \to X$ is a decreasing function;

(3.7.3) f is a weak ψ -quasi contraction;

(3.7.4) There exists $x_0 \in X$ such that $x_0 \succeq fx_0$, for every n

$$sup_n D(x_0, f^n(x_0)) < \alpha < \infty \text{ and} \tag{8}$$

$$\text{for every } n\ D(f^n(x_0), f^{n+1}(x_0)) \leq \psi^n(\alpha). \tag{9}$$

Then $\{f^n(x_0)\}$ is Cauchy and hence converges, say, to $w \in X$. If $\limsup_n D(f^n(x_0, fw)) < \infty$ then w is a fixed point of f. Moreover if w' is another fixed point of f comparable with w such that $D(w, w') < \infty$ and $D(w', w') < \infty$ then $w = w'$.

□

The following example is in support of our main result.

Example 3.8. Let $X = \{1, \frac{1}{2}, \frac{1}{3}, ...\} \cup \{0\}$. Define $x \preceq y$ if $x \geq y$ and $D : X \times X \to [0, \infty]$ be given by $D(1, x) = D(x, 1) = \infty$ if $x = 0$ or $\frac{1}{n}$ for some n, $D(x, y) = |x - y|$ otherwise.Then (X, D) is a generalized metric space with $\lambda = 1$ and also $(X, D, \preceq)$ is D-complete generalized metric space with partial order.

Define $f(x) = \begin{cases} 1 & if\ x = 1 \\ \frac{x}{2} & \text{otherwise.} \end{cases}$ and $\psi(t) = kt$ for $t \geq 0$ and $k < 1$.

Then 0 and 1 are fixed points, $D(0, 1) = \infty$ and all the hypothesis of Theorem 3.6 is satisfied.

The following theorems are corollaries of our main result.

Theorem 3.9. (Jleli and Samet [11]) Suppose that the following conditions hold:

(i) (X, D) is complete;

(ii) $(X, \preceq)$ is D- regular;

(iii) f is a weak k-contraction for some $k \in (0, 1)$;

(iv) there exists $x_0 \in X$ such that $\delta(D, f, x_0) < \infty$ and $x_0 \preceq f(x_0)$;

(v) f is $\preceq$- monotone.

Then $\{f^n(x_0)\}$ converges to some $w \in X$ such that w is a fixed point of f . Moreover, if $D(w, w) < \infty$ then $D(w, w) = 0$.

Theorem 3.10. (Sastry et al. [8]) Suppose $(X,D,\preceq)$ is D- complete generalized metric space with a partial order. Suppose the following conditions hold in X:

(i) $(X,D,\preceq)$ is D-regular(increasing).

(ii) $f : X \to X$ is an increasing function;

(iii) f is a weak k -quasi contraction for some $k \in (0,1)$;

(iv) There exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

$$sup_n D(x_0, f^n(x_0)) < \alpha < \infty \text{ and} \tag{10}$$

$$\text{for every } n\ D(f^n(x_0), f^{n+1}(x_0)) \leq k^n(\alpha). \tag{11}$$

Then $\{f^n(x_0)\}$ is Cauchy and hence converges say to $w \in X$. If $\limsup_n D(f^n(x_0), fw) < \infty$ and $k\lambda < 1$then w is a fixed point of f. Moreover if w' is another fixed point of f comparable with w such that $D(w,w') < \infty$ and $D(w',w') < \infty$ then $w = w'$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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