THE MONOCHROMATIC CONNECTIVITY OF 3-CHROMATIC GRAPHS

YIRONG YANG\textsuperscript{1,*}, HUAPING WANG\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, Zhejiang Normal University, Jinhua 321004, P.R. China
\textsuperscript{2}Department of Mathematics, Jiangxi Normal University, Nanchang 330022, P.R. China

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Abstract. In this paper, we solve completely the monochromatic connectivity of 3-chromatic graphs.

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1. INTRODUCTION

An edge-coloring of a connected graph is a \textit{monochromatically connecting} coloring (MC-coloring, for short) if there is a monochromatic path joining any two vertices. The \textit{monochromatic connection number} of a graph $G$, denoted by $mc(G)$, is defined to be the maximum number of colors used in an MC-coloring of a graph $G$. As proved in \cite{3}, an important property of an extremal MC-coloring (a coloring that use $mc(G)$ colors) is that each color forms a tree.

For a color $c$, let $T_c$ be the tree whose edges colored $c$. The color $c$ is nontrivial if $T_c$ has at least two edges. Otherwise $c$ is trivial. A nontrivial color tree with $m$ edges is said waste $m-1$ colors. For any two nontrivial colors $b$ and $c$, the corresponding trees $T_b$ and $T_c$ intersect in at most one vertex \cite{3}. Such an extremal coloring is called simple. Every connected graph has

*Corresponding author

E-mail address: 15558699651@163.com

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a simple extremal MC-coloring[3]. These concepts were introduced by Caro and Yuster in [3] and they gave some upper and lower bounds for $mc(G)$ characterized by other graph parameters. A straightforward lower bounds for $mc(G)$ is $m - n + 2$ (throughout this paper, $n$ and $m$ denote the number of vertices and edges respectively), which can be verified by coloring the edges of a spanning tree with one color, and coloring the remaining edges by new distinct colors.

Now we present some definitions and notations necessary. For a graph $G$, we use $V(G)$, $E(G)$, $|V(G)|$, $|E(G)|$ to denote the vertex set, edge set, number of vertices, number of edges of $G$, respectively. Given a graph $G$ and $D \subseteq V(G)$, let $|D|$ be the number of vertices in $D$ and $G[D]$ be the subgraph of $G$ induced by $D$. If $\chi(G) = k$, then $G$ is $k$-chromatic.

2. Preliminaries

Let $V_i, i = 1, 2, 3$, be the vertex parts of the graph $K_{n_1,n_2,n_3}$. Let $E_{i,j}$ be the set of edges between $V_i$ and $V_j$, $i, j \in \{1, 2, 3\}, i \neq j$. Let $E_0$ be a subset of $E(K_{n_1,n_2,n_3})$.

Lemma 2.1. [3] If $G$ is $K_3$-free, then $mc(G) = m - n + 2$.

Lemma 2.2. [3] Any graph $G$ satisfies $mc(G) \leq m - n + \chi(G)$.

The join of two disjoint graphs $G$ and $H$, denoted by $G + H$, is defined to be the graph $\overline{G + H}$.

Lemma 2.3. [4] Let $G$ be the join of two disconnected graphs $G_1$ and $G_2$. Then $mc(G) = |E(G)| - |V(G)| + 2$.

Moreover we have the following properties of the simple extremal MC-coloring.

Lemma 2.4. If $G$ is a connected spanning subgraph of some graph $H$, then $mc(G) \leq mc(H) - (|E(H)| - |E(G)|)$.

Proof. It is clear that $G$ has a simple extremal MC-coloring. Let $f$ be an MC-coloring of $G$ realizing $mc(G)$. Let the remaining $|E(H)| - |E(G)|$ edges of $H$ receive trivial colors. Then we get an MC-coloring, denoted by $f'$ of $H$. Clearly, $f'$ is simple and it use $mc(G) + (|E(H)| - |E(G)|)$ colors. Then $mc(H) \geq mc(G) + (|E(H)| - |E(G)|)$, i.e., $mc(G) \leq mc(H) - (|E(H)| - |E(G)|)$, and we are done. The proof is completed.

Lemma 2.5. If $G$ is a connected spanning subgraph of some graph $H$ and let $mc(G) = m(G) - n(G) + k_1$, $mc(H) = m(H) - n(H) + k_2$, then $k_1 \leq k_2$. 
**Proof.** By Lemma 2.4, it implies that \( mc(G) \leq mc(H) - (|E(H)| - |E(G)|) \). Since \( G \) is a spanning subgraph of graph \( H, n(G) = n(H) \). And we have \( m(G) - n(G) + k_1 \leq m(H) - n(H) + k_2 - (|E(H)| - |E(G)|) \), i.e., \( m(G) - n(G) + k_1 \leq m(G) - n(G) + k_2 \). Hence we get that \( k_1 \leq k_2 \), and we are done. The proof is completed.

3. **Main results**

**Lemma 3.1.** Let \( V_i, i = 1, 2, 3 \) be the vertex parts of the graph \( K_{n_1,n_2,n_3} \). Let \( G = K_{n_1,n_2,n_3} - \{uv,xy\}, u \in V_1, v,x \in V_2, y \in V_3 \). Then \( mc(G) = m - n + 2 \)

**Proof.** The lower bound \( mc(G) \geq m - n + 2 \) is obvious and we only need to show \( mc(G) \leq m - n + 2 \).

Let \( f \) be a simple extremal MC-coloring of \( G \). Suppose that \( f \) consists of \( k \) nontrivial color trees, denoted by \( T_1, \ldots, T_k \), where \( t_i = |V(T_i)| \). As \( T_i \) has \( t_i - 1 \) edges, it wastes \( t_i - 2 \) colors. Hence it suffices to prove that \( \sum_{i=1}^{k} (t_i - 2) \geq n - 2 \).

**Case 1.** Every vertex appears in at least two distinct nontrivial color trees.

In this case we have \( \sum_{i=1}^{k} t_i \geq 2n \). So if \( k \leq n/2 + 1 \), we have \( \sum_{i=1}^{k} (t_i - 2) \geq 2n - 2k \geq n - 2 \), and we are done. So let \( k > n/2 + 1 \). Now we claim that we still have \( \sum_{i=1}^{k} (t_i - 2) \geq n - 2 \) when \( k > n/2 + 1 \). Since \( T_i \) can monochromatically connect at most \( \binom{n_i - 1}{2} \) pairs of non-neighbors in \( G \), we have \( \sum_{i=1}^{k} \binom{n_i - 1}{2} \geq |E(G)| = \sum_{i=1}^{3} \binom{n_i}{2} + 2 \).

Assume that \( \sum_{i=1}^{k} (t_i - 2) < n - 2 \). Since \( T_j \) is nontrivial, \( t_j - 1 \geq 2 \). By the straightforward convexity, the expression \( \sum_{i=1}^{k} \binom{n_i - 1}{2} \), subject to \( t_i - 1 \geq 2 \), is maximized when \( k - 1 \) of the \( n_i \)'s equal 3 and one of the \( n_i \)'s, say \( k \), is as large as it can be, namely \( t_k - 1 \) is the largest integer smaller than \( n - 2 + k - 2(k - 1) = n - k \). Hence, \( t_k - 1 = n - k - 1 \). We have \( \sum_{i=1}^{k} \binom{n_i - 1}{2} \leq k - 1 + \binom{n - k - 1}{2} \).

Note that \( g(k) = k - 1 + \binom{n - k - 1}{2} \) is a decreasing function of \( k \) for \( n/2 + 1 < k \leq n - 3 \) and then \( g(k) < g(n/2 + 1) \). Note that \( \sum_{i=1}^{3} \binom{n_i}{2} + 2 - g(n/2 + 1) > 0 \). This implies that \( g(k) < g(n/2 + 1) < |E(G)| \), i.e., \( \sum_{i=1}^{k} \binom{n_i - 1}{2} < |E(G)| = \sum_{i=1}^{3} \binom{n_i}{2} + 2 \), a contradiction. Hence \( \sum_{i=1}^{k} (t_i - 2) \geq n - 2 \) and we are done.

**Case 2.** There are vertices that appear in unique nontrivial color trees.
Denote by $S$ the vertices that appear in the unique nontrivial color trees. Note that $u,v$ or $x,y$ are monochromatically connected by a nontrivial color tree. So let $T_u,T_x$ monochromatically connect $u,v$ and $x,y$, respectively.

**Subcase 2.1** \( S \cap V_1, S \cap V_2, S \cap V_3 \neq \emptyset \).

Notice that vertices of the same part are not adjacent in $G$ and any two of each part are monochromatically connected in a nontrivial color tree. So all the vertices of the same part must lie in a nontrivial color tree. So we can assume that $V_i \subseteq T_i, i = 1, 2, 3$, and we have that $V_i \cap V(T_i) \cap S \neq \emptyset$.

**Sub-subcase 2.1.1** $T_1 \neq T_2 \neq T_3 \neq T_1$.

Suppose that $T_u, T_x \notin \{T_1, T_2, T_3\}$. Since $V_i \subseteq V(T_i)$, we have that $t_i \geq n_i + 1$, i.e., $t_i - 2 \geq n_i - 1$, $i = 1, 2, 3$. That is to say that $T_i$ waste at least $n_i - 1$ edges, $i = 1, 2, 3$. Also, both $T_u$ and $T_x$ waste at least one edge. So the total waste of the coloring $f$ is at least $n - 2$ and we are done.

Suppose that $T_u \in \{T_1, T_2, T_3\}$ or $T_x \in \{T_1, T_2, T_3\}$. Without loss of generality, let $T_u = T_1$. Since $uv \notin E(G)$ and vertices of $V_1$ are not adjacent, then $T_1$ contains at least another vertex besides $v$ and vertices of $V_1$. It implies that $t_1 \geq n_1 + 2$, i.e., $t_1 - 2 \geq n_1$. Similarly, $V_i \subseteq V(T_i), i = 2, 3$, and we have that $t_i \geq n_i + 1$, i.e., $t_i - 2 \geq n_i - 1$ for $i = 2, 3$. So the total waste of $T_1, T_2, T_3$ is at least $n - 2$ and we are done.

**Sub-subcase 2.1.2** There are two trees in $\{T_1, T_2, T_3\}$ which are same.

Let $T_1 = T_2 \neq T_3$, now we have $V_1 \cup V_2 \subseteq V(T_1)$. Suppose that $y \in V(T_1)$. Then the waste of $T_1$ is at least $n_1 + n_2 - 1$. Clearly, $t_3 \geq n_3 + 1$, i.e., $t_3 - 2 \geq n_3 - 1$. Hence the total waste of the coloring $f$ is at least $n - 2$ and we are done. Suppose that $y \in V_3 - V(T_1)$. Then the waste of $T_1$ is at least $n_1 + n_2 - 2$ and $T_x \neq T_1$. This implies that $T_x = T_3$. Then $t_3 \geq n_3 + 2$, i.e., $t_x - 2 \geq n_3$. Hence the total waste of the coloring $f$ is at least $n - 2$ and we are done. By the symmetry, if $T_2 = T_3 \neq T_1$, then the total waste of the coloring $f$ is at least $n - 2$ and we are done.

Let $T_1 = T_3 \neq T_2$, now we have $V_1 \cup V_3 \subseteq V(T_1)$. Suppose that $v \notin V(T_1)$ or $x \notin V(T_1)$. Without loss of generality, let $v \notin V(T_1)$, then $T_u = T_2$. It implies that $t_2 \geq n_2 + 2$, i.e., $t_2 - 2 \geq n_2$. Clearly, $t_1 \geq n_1 + n_3$, i.e., $t_1 - 2 \geq n_1 + n_3 - 2$. Hence the total waste of the coloring $f$ is at least $n - 2$ and we are done. Suppose that $v,x \in V(T_1)$. Since $f$ is simple and $x,y \in V(T_2)$, we have that
v = x. Then \( t_1 \geq n_1 + n_3 + 1 \), i.e., \( t_1 - 2 \geq n_1 + n_3 - 1 \). Clearly, \( t_2 \geq n_2 + 1 \), i.e., \( t_2 - 2 \geq n_2 - 1 \). Hence the total waste of the coloring \( f \) is at least \( n - 2 \) and we are done.

**Sub-subcase 2.1.3** \( T_1 = T_2 = T_3 \).

Since \( S \cap V_1 \cap V_2 \cap V_3 \neq \emptyset \), the tree \( T_1 \) is a spanning tree of \( G \). So the waste of \( T_1 \) is \( n - 2 \) and so we are done.

**Subcase 2.2** The set \( S \) is exactly joint with two partite sets of \( G \).

Here we only present the proof details of the case \( S \cap V_1 \neq \emptyset, S \cap V_2 \neq \emptyset \). The other two cases can be proved similarly. Clearly, we can assume that \( V_i \subseteq V(T_i), i = 1, 2 \).

Assume that \( T_1 = T_2 \). Then we have that \( V_1 \cup V_2 \subseteq V(T_1) \). Suppose that \( y \in V(T_1) \). Since \( T_1 \) is not a spanning tree of \( G \), there is a vertex \( v_3 \in V_3 - V(T_1) \). Clearly, \( v_3y \notin E(G) \). Let \( T_{v_3} \) be the nontrivial color tree monochromatically connecting \( v_3, y \). Since \( V_3 \) is an independent set in \( G \), we have that \( |V(T_{v_3}) \cap (V_1 \cup V_2)| \geq 1 \). This implies that \( |V(T_{v_3}) \cap V(T_1)| \geq 2 \), a contradiction. Suppose that \( y \notin V(T_1) \). Since \( xy \notin E(G) \) and \( V_3 \) is an independent set in \( G \), this means that \( |V(T_3) \cap (V_1 \cup V_2)| \geq 2 \), i.e., \( |V(T_3) \cap V(T_1)| \geq 2 \), a contradiction. So \( T_1 \neq T_2 \). Now we claim that \( \sum_{i=1}^{k} (t_i - 2) \geq n - 2 \). Since we have that \( S \cap V_3 = \emptyset \), each vertex of \( V_3 \) appears in at least two nontrivial color trees. In order to monochromatically connect the \( (|V_3|^2) \) distinct pairs of vertices of \( V_3 \), we need a set of nontrivial color trees, say \( T_s, \ldots, T_q \), and each \( T_i, i = s, \ldots, q \) contains at least two vertices of \( V_3 \).

Suppose that \( |V(T_1) \cap V_3| \geq 2 \) and \( |V(T_2) \cap V_3| \geq 2 \), and let \( w_1, w_2 \in V(T_1) \cap V_3, z_1, z_2 \in V(T_2) \cap V_3 \). Notice that \( |V(T_1) \cap V(T_2)| \leq 1 \). Let \( w_1 \in V(T_1) \cap V_3 - V(T_2) \) and \( z_1 \in V(T_2) \cap V_3 - V(T_1) \). Since \( w_1z_1 \notin E(G) \), we have \( w_1, z_1 \) lie in a nontrivial color tree and let \( T_s \) be such nontrivial color tree in \( f \). Since \( V_3 \) is an independent set, we have that \( |V(T_s) \cap (V_1 \cup V_2)| \geq 1 \). This implies that \( V(T_s) \cap V_1 \neq \emptyset \) or \( V(T_s) \cap V_2 \neq \emptyset \). Along with \( w_1 \in V(T_s) \cap V(T_1) \) and \( z_1 \in V(T_s) \cap V(T_2) \), we have that \( |V(T_1) \cap V(T_s)| \geq 2 \) or \( |V(T_2) \cap V(T_s)| \geq 2 \), a contradiction.

Suppose that \( |V(T_1) \cap V_3| < 2 \) and \( |V(T_2) \cap V_3| < 2 \), then \( T_1, T_2 \notin \{ T_s, \ldots, T_q \} \). It is clear that \( t_1 \geq n_1 + 1 \), i.e., \( t_1 - 2 \geq n_1 - 1 \) for \( i = 1, 2 \). Notice that \( t_i \geq 3 \), i.e., \( t_i - 2 \geq 1 \), for \( i = s, \ldots, q \). If \( q - s + 1 \geq n_3 \), then we have \( \sum_{i=s}^{q} (t_i - 2) \geq q - s + 1 \geq n_3 \). Hence we get that \( \sum_{i=1}^{k} (t_i - 2) = \left( \sum_{i=s}^{q} (t_i - 2) \right) + n_1 + n_2 - 2 \geq n - 2 \) and we are done.
So let $q - s + 1 < n_3$. Since $V_3 \subset \cup_{i=s}^{q} V(T_i)$ and each vertex of $V_3$ appears in at least two distinct nontrivial color trees, every vertex of $V_3$ is covered by at least two edges of $T_s, \ldots, T_q$ and each such edge in $G$ exactly covers one vertex of $V_3$. So, the total number of edges of $T_s, \ldots, T_q$ is at least $2n_3$ and we have \( \sum_{i=s}^{q} (t_i - 1) \geq 2n_3 \), i.e., \( \sum_{i=s}^{q} (t_i - 2) = \sum_{i=s}^{q} (t_i - 1) - (q - s + 1) > n_3 \). Hence \( \sum_{i=1}^{k} (t_i - 2) = (\sum_{i=s}^{q} (t_i - 2)) + n_1 + n_2 - 2 > n - 2 \) and we are done.

Suppose that $|V(T_1) \cap V_3| < 2$ or $|V(T_2) \cap V_3| < 2$. Without loss of generality, let $|V(T_1) \cap V_3| \geq 2$ and $|V(T_2) \cap V_3| < 2$. Then we have that $T_1 \in \{T_s, \ldots, T_q\}$ and $T_2 \notin \{T_s, \ldots, T_q\}$. It is clear that $t_1 \geq n_1 + 2$, i.e., $t_1 - 2 \geq n_1$ and that $t_2 \geq n_2 + 1$, i.e., $t_2 - 2 \geq n_2 - 1$. Notice that $t_i \geq 3$, i.e., $t_i - 2 \geq 1$, for $i = s, \ldots, q$ and $t_1 - 2 \geq n_1$. If $q - s + 1 \geq n_3$, then we have $\sum_{i=s}^{q} (t_i - 2) \geq n_1 + q - s \geq n_1 + n_3 - 1$. Hence $\sum_{i=1}^{k} (t_i - 2) \geq n - 2$ and we are done.

So let $q - s + 1 < n_3$. Notice that each $\{T_s, \ldots, T_q\} \setminus \{T_1\}$ contains at least a vertex out of $V_3$. So the sum of the orders of $\{T_s, \ldots, T_q\}$ is at least $2n_3 + n_1 + q - s$. This implies that $\sum_{i=s}^{q} (t_i - 1) \geq 2n_3 + n_1 - 1$, i.e., $\sum_{i=s}^{q} (t_i - 2) = \sum_{i=s}^{q} (t_i - 1) - (q - s + 1) > n_3 + n_1 - 1$. Hence $\sum_{i=1}^{k} (t_i - 2) = (\sum_{i=s}^{q} (t_i - 2)) + n_2 - 1 > n - 2$ and we are done.

**Sub-case 2.3** The set $S$ is exactly joint with one partite set of $G$.

Without loss of generality, let $S \cap V_1 \neq \emptyset, S \cap V_2 = \emptyset, S \cap V_3 = \emptyset$, then $V_1 \subseteq V(T_1)$ and each vertex of $V_2 \cup V_3$ appears in at least two distinct nontrivial color trees. Let $T_2, \ldots, T_k$ be the nontrivial color trees which monochromatically connect all vertices of $V_2 \cup V_3$. Then each $T_i$ contains at least two vertices of $V_2 \cup V_3$ for $2 \leq i \leq k$.

Suppose that $|V(T_1) \cap (V_2 \cup V_3)| < 2$. Then $T_1 \notin \{T_2, \ldots, T_k\}$. It is clearly that every vertex of $V_2 \cup V_3$ appears in at least two distinct nontrivial color trees. By the same way as case 1, we can deduce that $\sum_{i=2}^{k} (t_i - 2) \geq n_2 + n_3 - 1$. Since $V_1 \subseteq V(T_1)$, we have that $t_1 \geq n_1 + 1$, i.e., $t_1 - 2 \geq n_1 - 1$. Hence $\sum_{i=1}^{k} (t_i - 2) \geq n - 2$ and we are done.

Suppose that $|V(T_1) \cap (V_2 \cup V_3)| \geq 2$. Then $T_1 \in \{T_2, \ldots, T_k\}$. Now we still claim that $\sum_{i=1}^{k} (t_i - 2) \geq n - 2$. Recall that we have $\sum_{i=2}^{k} (t_i - 2) \geq n_2 + n_3 - 1$ for $T_1 \notin \{T_2, \ldots, T_k\}$. But now $T_1 \in \{T_2, \ldots, T_k\}$ and $V_1 \subset V(T_1)$, then $T_1$ will have other $n_1 - 1$ edges of $E(G)$ such that all vertices
of $V_1$ are monochromatically connected. That is to say that $\sum_{i=1}^{k} (t_i - 2) \geq n - 2 = \sum_{i=2}^{k} (t_i - 2) \geq n_2 + n_3 - 1 + n_1 - 1 = n - 2$ for this case, and we are done.

The proof is completed.

**Theorem 3.2.** Let $G$ be a connected 3-chromatic spanning subgraph of $K_{n_1,n_2,n_3}$ with partite sets $V_i, |V_i| = n_i, i = 1, 2, 3$. If $G = K_{n_1,n_2,n_3} - E_0$ with $E_0 \cap E_{i,j} \neq \emptyset$ and $E_0 \cap E_{j,k} \neq \emptyset$, $\{i, j, k\} = \{1, 2, 3\}$, then $mc(G) = m - n + 2$.

**Proof.** The lower bound $mc(G) \geq m - n + 2$ is obvious and we only need to show $mc(G) \leq m - n + 2$. It is clearly that $G$ is a connected spanning subgraph of $K_{n_1,n_2,n_3} - \{uv, xy\}$ for some $uv \in E_{i,j}$ and $xy \in E_{j,k}$. By Lemmas 2.5-3.1, we have that $mc(G) \leq m - n + 2$ and we are done.

The proof is completed.

**Theorem 3.3.** Let $G$ be a connected 3-chromatic spanning subgraph of $K_{n_1,n_2,n_3}$ with partite sets $V_i, |V_i| = n_i, i = 1, 2, 3$. If $G = K_{n_1,n_2,n_3} - E_0, E_0 \subset E_{i,j}, \{i, j\} \subset \{1, 2, 3\}$ such that $G[V_i, V_j]$ is disconnected, then $mc(G) = m - n + 2$.

**Proof.** Without loss of generality, we assume that $i = 1, j = 2$. Then $E_0 \subset E_{1,2}$ and $G[V_1, V_2]$ is disconnected. Let $G_1 = G[V_1, V_2]$ and $G_2 = G[V_3]$. So $G = G_1 + G_2$. Notice that both $G_1$ and $G_2$ are disconnected. Hence, from Lemma 2.3 we have that $mc(G) = m(G) - n(G) + 2$, and we are done.

The proof is completed.

**Theorem 3.4.** Let $G$ be a connected 3-chromatic spanning subgraph of $K_{n_1,n_2,n_3}$ with partite sets $V_i, |V_i| = n_i, i = 1, 2, 3$. Let $G = K_{n_1,n_2,n_3} - E_0$. Then $mc(G) = m - n + 3$ if and only if $E_0 \subset E_{i,j}$ and $G[V_i, V_j]$ is still connected for some $i, j \in [3]$.

**Proof.** Now we show the necessity of this proof. Let $mc(G) = m - n + 3$. We show that $E_0 \subset E_{i,j}$ and $G[V_i, V_j]$ is still connected for some $i, j \in [3]$. Suppose that $E_0$ is not a subset of $E_{i,j}$ for any $i, j \in [3]$. This implies that $E_0 \cap E_{i,j} \neq \emptyset, E_0 \cap E_{j,k} \neq \emptyset, \{i, j, k\} = \{1, 2, 3\}$. Then it follows from Theorem 3.2 that $mc(G) = m - n + 2$, a contradiction. So $E_0 \subset E_{i,j}$ for some $i, j \in [3]$. Suppose that $G[V_i, V_j]$ is disconnected. Then it follows from Theorem 3.3 that $mc(G) = m - n + 2$, a contradiction and we are done.
The sufficiency of this proof can be proved by coloring the spanning tree of $G[V_i, V_j]$ with a color $c_1$ and one vertex from $V_i \cup V_j$ is adjacent to all vertices of $V_k$ by a color $c_2$, where $k \neq i, j$ and $k \in [3]$. The remaining edges of $G$ receive trivial colors. Then we get an simple extremal MC-coloring, say $f$ of $G$. Clearly, $f$ contains $m(G) - n(G) + 3$ colors and we are done.

The proof is completed.

Conflict of Interests

The authors declare that there is no conflict of interests.

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