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EXISTENCE RESULTS FOR GENERALIZED MIXED VECTOR VARIATIONAL-LIKE INEQUALITY PROBLEMS WITH EXPONENTIAL TYPE INVEXITIES

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Abstract. In this paper, we study a new kind of existence of solution for set valued exponential type mixed vector variational-like inequality problem in Euclidean space and proposed α_g-relaxed exponentially (γ,η)-monotone mapping. Moreover, we established an example in order to illustrate the main problem. We proved the existence results by KKM-technique with α_g-relaxed exponentially (γ,η)-monotone mapping. Further, we give some consequences of the main result. The results presented in this paper unifies and extends some known results in this area.

Keywords: Euclidean space; α_g-relaxed exponentially (γ,η)-monotone mapping; variational-like inequality problem; set valued mapping.

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1. Introduction

The theory of vector variational inequality has been introduced by Giannessi [7] in 1980 for finite dimensional space. Later, it has been studied by Chen *et al.* [4] in abstract spaces and

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obtained existence theorems. Wu and Huang [16] defined the concepts of relaxed $\eta - \alpha$ pseudomonotone mappings to study vector variational-like inequality problem in Banach spaces. The generalized variational-like inequalities with generalized α-monotone multifunctions studied by Ceng *et al.* [3] [see for instance, [6, 12, 14]]. In 2004, Antczak [1] introduced the class of exponential (p,r)-invex functions for differentiable case [see for more details [8, 13]]. The exponential and logarithmic functions are very important in mathematical modeling of various real-life problems, for example, in mathematical modeling of growth and decline of populations, digital circuit optimization in the field of electrical engineering. Very recently, Jayswal *et al.* [9, 10] introduced exponential type vector variational-like inequality problems with exponential invexities.

Motivated by the work of Antczak [1], Jayswal *et al.* [9, 10], Ho *et al.* [8] and by the ongoing research in this direction, we introduced a generalized mixed exponential type vector variational-like inequality problem (in short, GMEVVLIP) in Euclidean space and defined a new kind of α_g-relaxed exponential (γ,η)-monotone mappings. We proved the existence results of GMEVVLIP by KKM-technique and Nadler results. The results presented in this paper extend and generalize many previously known results in this research area.

2. PRELIMINARIES

Now, we recall some useful concepts and results which are necessary for proving our main result. Throughout the paper unless otherwise stated, we consider E_1 and E_2 as Euclidean spaces of dimensions m and n, K and C be nonempty subsets of E_1 and E_2 respectively.

Let K be a nonempty subset of E_1. Then, K is said to be

(i) cone if $\lambda K \subset K$, $\forall\, \lambda \geq 0$;

(ii) convex cone if $K + K \subset K$;

(iii) pointed cone if K is cone and $K \bigcap \{-K\} = \{0\}$;

(iv) proper cone if $K \neq E_2$.

Let $K : C \to 2^{E_2}$ be a closed pointed convex cone valued mapping with $\text{int}K(u) \neq \emptyset$ with apex at origin, where $\text{int}K(u)$ be a set of interior points of $K(u)$. Then, $K(u)$ induces a partial ordering in E_2 as:

(i) $v \leq_{K(u)} w \Leftrightarrow w - v \in K(u)$;

(ii) $v \nleq_{K(u)} w \Leftrightarrow w - v \notin K(u)$;

(iii) $v \leq_{\text{int}K(u)} w \Leftrightarrow w - v \in \text{int}K(u)$;

(iv) $v \nleq_{\text{int}K(u)} w \Leftrightarrow w - v \notin \text{int}K(u)$.

Let (E_2, K) be an ordered space with the ordering of E_2 defined by a set $K(u)$ and ordering relation $'' \leq_{K(u)} ''$ is a partial order. Then

(i) $v \nleq_{K(u)} w \Leftrightarrow v + s \nleq w + s$, for any $u, v, w, s \in E_2$;

(ii) $v \nleq_{K(u)} w \Leftrightarrow \lambda v \nleq \lambda w$, for any $\lambda \geq 0$.

Let $C \subseteq E_1$ be a nonempty closed convex subset of an Euclidean space $E_1 = R^m$ and (E_2, K) be an ordered space induces by the closed convex pointed cone $K(u)$ whose apex at origin with $\text{int}K(u) \neq \emptyset$.

Lemma 1. [3] *Let (E_2, K) be an ordered space induced by the pointed closed convex cone K with $\text{int}K(u) \neq \emptyset$. Then, for any $u, v, w \in E_2$, the following relation hold:*

(i) $w \nleq_{\text{int}K} x \geq_K v \Rightarrow w \nleq_{\text{int}K} v$;

(ii) $w \ngeq_{\text{int}K} x \leq_K v \Rightarrow w \ngeq_{\text{int}K} v$.

Definition 1. *A mapping $F : E_1 \to E_2$ is a $K(u)-$ convex on E_1 if*

$$F(\lambda u + (1-\lambda)v) \leq_{K(u)} \lambda F(u) + (1-\lambda)F(v),\ \forall u, v \in E_1,\ \lambda \in [0,1],$$

that is,

$$\lambda F(u) + (1-\lambda)F(v) - F(\lambda u + (1-\lambda)v) \in K(u).$$

Remark 1. (i) *If $K(u) = K$, for all $u \in E_1$, where K is convex in E_2 then Definition 1 reduces to the vector convexity of F that is*

$$F(\lambda u + (1-\lambda)v) \leq_K \lambda F(u) + (1-\lambda)F(v),\ \forall u, v \in E_1,\ \lambda \in [0,1].$$

(ii) *If $E_2 = R$ and $K = [0, +\infty)$ in (i) then Definition 1 reduces to the convex function that is*

$$\lambda F(u) + (1-\lambda)F(v) - F(\lambda u + (1-\lambda)v) \geq 0,\ \forall u, v \in E_1,\ \lambda \in [0,1].$$

Definition 2. *A mapping $F : C \to E_2$ is said to be completely continuous if for any sequence $\{u_n\} \in C$, $u_n \rightharpoonup u_0$ weakly, then $F(u_n) \to F(u_0)$.*

Definition 3. *Let E_1 and E_2 be two topological vector spaces, $A : E_1 \to 2^{E_2}$ be a set valued mapping and $A^{-1}(v) = \{u \in E_1 : v \in A(u)\}$. Then,*

(i) *A is said to be upper semicontinuous if for each $u \in E_1$ and each open set V in E_2 with $A(u) \subset V$, then there exists an open neighborhood U of u in E_1 such that $A(u_0) \subset V$, for each $u_0 \in U$.*

(ii) *A is said to be closed if for any set $\{u_\alpha\} \to u$ in E_1 and any net $\{v_\alpha\}$ in E_2 such that $v_\alpha \to v$ and $v_\alpha \in A(u_\alpha)$, for any α, we have $v \in A(u)$.*

(iii) *A is said to have a closed graph if the graph of A, $\mathrm{Graph}(A) = \{(u,v) \in E_1 \times E_2, v \in A(u)\}$ is closed in $E_1 \times E_2$.*

Definition 4. *Let $F : C \to 2^{E_1}$ be a set valued mapping. Then F is said to be a KKM-mapping if for any $\{v_1, v_2, ..., v_n\}$ of C, we have $\mathrm{co}\{v_1, v_2, ..., v_n\} \subset \bigcup_{i=1}^{n} F(v_i)$, where $\mathrm{co}\{v_1, v_2, ..., v_n\}$ denotes the convex hull of $v_1, v_2, ..., v_n$.*

Lemma 2. [5] *Let C be a nonempty subset of a Hausdorff topological vector space E_1 and let $F : C \to 2^{E_1}$ be a KKM-mapping. If $F(v)$ is a closed in E_1 for all $v \in C$ and compact for some $v \in C$, then $\bigcap_{v \in C} F(v) \neq \emptyset$.*

Lemma 3. [11] *Let E be a normed vector space and H be a Hausdorff metric on the collection $CB(E)$ of all closed and bounded subsets of E, induced by a metric d in terms of $d(u,v) = \|u - v\|$, which is defined by*

$$H(X,Y) = \max\{\sup_{u \in X} \inf_{v \in Y} \|u - v\|, \sup_{v \in Y} \inf_{u \in X} \|u - v\|\},$$

for $X, Y \in CB(E)$. If X and Y are compact subset in E, then for each $u \in X$, there exists $v \in Y$ such that $\|u - v\| \leq H(X,Y)$.

Definition 5. *Let $\eta : E_1 \times E_1 \to E_1$ be a mapping and $N : C \to L(E_1, E_2)$ be a single valued mapping, where $L(E_1, E_2)$ be the space of all continuous linear mapping from E_1 to E_2. Suppose $A : C \to 2^{L(E_1, E_2)}$ be a nonempty compact set valued mapping, then*

(i) *N is said to be η-hemicontinuous, if*

$$\lim_{t\to 0^+}\langle N(u+t(v-u)),\eta(v,u)\rangle = \langle Nu,\eta(v,u)\rangle,\ \forall u,v\in C.$$

(ii) *A is said to be H-hemicontinuous, if for any $u,v\in C$, the mapping $t\to H(A(u+t(v-u)),Au)$ is continuous at 0^+, where H is a Hausdorff metric defined on* $\mathrm{CB}(L(E_1,E_2))$.

Definition 6. *A mapping $f: R^m\to R^n$ is lipschitz continuous on $D\subset R^m$ iff there is an $L\in R$ such that*

$$\|f(u)-f(v)\|\le L\|u-v\|,\ \forall u,v\in D. \tag{1}$$

Definition 7. *A mapping $F: E_1\to E_1$ is said to be affine if for any $u_i\in C$ and $\lambda_i\ge 0,\ (1\le i\le n)$ with $\sum_{i=1}^n\lambda_i=1$, we have $F(\sum_{i=1}^n\lambda_i u_i)=\sum_{i=1}^n\lambda_i F(u_i)$.*

Definition 8. *Let E_1 be an Euclidean space. A mapping $F: E_1\to R$ is a lower semicontinuous at $u_0\in E_1$ if $F(u_0)\le\liminf_n F(u_n)$, for any sequence $\{u_n\}\subset E_1$ such that $\{u_n\}$ converges to u_0.*

Definition 9. *Let E_1 be an Euclidean space. A mapping $F: E_1\to R$ is a weakly upper semi-continuous at $u_0\in E_1$ if $F(u_0)\ge\limsup_n F(u_n)$, for any sequence $\{u_n\}\subset E_1$ such that $\{u_n\}$ converges to u_0 weakly.*

Lemma 4. [2] *Let S be a nonempty compact convex subset of a finite dimensional space and $T: S\to S$ be a continuous mapping. Then there exists $x\in S$ such that $Tx=x$.*

In this paper, we introduce and study the following generalized mixed exponential type vector variational-like inequality problem (in short, GMEVVLIP). Let $C\subseteq E_1$ be a nonempty subset of an Euclidean space R^n and (E_2,K) be an ordered Euclidean space induces by a closed convex pointed cone K whose apex at origin. Let $K: C\to 2^{E_2}$ be a closed convex pointed cone valued mapping with $\mathrm{int}K\ne\emptyset$. Let γ be a nonzero real number, $\eta: C\times C\to E_1,\ g: C\to C,\ F: C\times C\to E_2$ and $N: L(E_1,E_2)\times L(E_1,E_2)\times L(E_1,E_2)\to L(E_1,E_2)$ be the mappings, where $L(E_1,E_2)$ be the space of all continuous linear mappings from E_1 to E_2 and $A_1,A_2,A_3: C\to 2^{L(E_1,E_2)}$ be set valued mappings then GMEVVLIP is to find $u_0 \in C$ and $x \in A_1(u_0),\ y \in A_2(u_0),\ z \in A_3(u_0)$ such that

$$\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) \nleq_{\text{int}K(u_0)} 0,\ \forall v \in C. \tag{2}$$

The following example is provided to illustrate problem (2)

Example 1. *Let* $E_1 = E_2 = R$, $C = [0,+\infty)$, $K(u_0) = [0,\infty)$, $\forall u_0 \in C$. *Define* $A_1,A_2,A_3 : C \to 2^{L(E_1,E_2)} \equiv 2^R$ *by*

For $u_0 \in C$

$$\begin{aligned} A_1(u_0) &= \{x \in R : \frac{1}{1+(x-1)^2} \geq \frac{1}{2}\} = [0,2] \\ A_2(u_0) &= \{y \in R : \frac{1}{1+(y-1)^2} \geq \frac{1}{2}\} = [0,2] \\ A_3(u_0) &= \{z \in R : \frac{1}{1+(z-1)^2} \geq \frac{1}{2}\} = [0,2]. \end{aligned}$$

Define $N : L(E_1,E_2) \times L(E_1,E_2) \times L(E_1,E_2) \to L(E_1,E_2)$ *by*

$$N(x,y,z) = \{x+y+z\},\ \forall x,y,z \in L(E_1,E_2) \equiv R,$$

$\eta : C \times C \to E_1 = R$ *such that*

$$\eta(u,v) = \ln(\frac{u}{2} - v + 1),\ \forall u,v \in C,$$

$g : C \to C$ *such that*

$$g(u) = \frac{u}{2},\ \forall u \in C,$$

and $F : C \times C \to E_2 = R$ *such that*

$$F(u,v) = \frac{v}{2} - u,\ \forall u,v \in C.$$

Consider $\gamma = 1$.

Now,

$$\begin{aligned} \langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) &= \langle x+y+z, e^{\ln(\frac{v}{2}-\frac{u_0}{2})} - 1\rangle + \frac{v}{2} - \frac{u_0}{2} \\ &= (x+y+z+1)(\frac{v}{2} - \frac{u_0}{2}). \end{aligned}$$

Thus,

$$\begin{aligned}(x+y+z+1)(\frac{v}{2}-\frac{u_0}{2}) &\geq 0\\ \Rightarrow u_0 &\leq v,\ \forall v\in C.\end{aligned}$$

This shows that $u_0 = 0$ is a solution of the GMEVVLIP(2).

Definition 10. *The mapping $A : C \to L(E_1,E_2)$ is said to be α_g-relaxed exponentially (γ,η)-monotone if for every pair of points $u,v \in C$, we have*

$$\langle Au-Av, \frac{1}{\gamma}(e^{\gamma\eta(u,g(v))}-1)\rangle \geq_{K(u)} \alpha_g(u-v), \tag{3}$$

where $\alpha_g : E_1 \to E_2$ with $\alpha_g(tu) = t^q\alpha_g(u)$, for all $t > 0$ and $u \in E_1$, where $q > 1$ is a real number.

Definition 11. *Let $N : L(E_1,E_2)\times L(E_1,E_2)\times L(E_1,E_2) \to L(E_1,E_2)$ be a single valued mappings. A multivalued mapping $A : C \to L(E_1,E_2)$ with compact valued is said to be α_g-relaxed exponentially (γ,η)-monotone with respect to first argument of N and g if for each pair of points $u,v,y,z \in C$, we have*

$$\langle N(x_1,y,z)-N(x_2,y,z), \frac{1}{\gamma}(e^{\gamma\eta(u,g(v))}-1)\rangle \geq_{K(u)} \alpha_g(u-v),\ \forall x_1 \in A(u),\ x_2 \in A(v), \tag{4}$$

where $\alpha_g : E_1 \to E_2$ with $\alpha_g(tu) = t^q\alpha_g(u)$, for all $t > 0$ and $u \in E_1$, where $q > 1$ is a real number.

Remark 2. *Some special cases:*

(i) *If $K(u) = K,\ g \equiv I$, identity mapping and $\alpha_g = 0$ then Definition 10 is called exponentially (γ,η)-monotone that is for each pair of points $u,v \in C$, we have*

$$\langle Au-Av, \frac{1}{\gamma}(e^{\gamma\eta(u,g(v))}-1)\rangle \geq_K 0. \tag{5}$$

(ii) *If $N(x,y,z) = N(x,y)$ then by Definition 11, we have for each pair of points $u,v,y \in C$,*

$$\langle N(x_1,y)-N(x_2,y), \frac{1}{\gamma}(e^{\gamma\eta(u,g(v))}-1)\rangle \geq_{K(u)} \alpha_g(u-v),\ \forall x_1 \in A(u),\ x_2 \in A(v), \tag{6}$$

where $\alpha_g : E_1 \to E_2$ with $\alpha_g(tu) = t^q\alpha_g(u)$, for all $t > 0$ and $u \in E_1$, where $q > 1$ is a real number.

(iii) *If* $N(x,y,z)=N(x)$ *then by Definition 11, we have for each pair of points* $u,v,y\in C$,

$$\langle N(x_1)-N(x_2),\frac{1}{\gamma}(e^{\gamma\eta(u,g(v))}-1)\rangle \geq_{K(u)} \alpha_g(u-v),\ \forall x_1\in A(u),\ x_2\in A(v), \tag{7}$$

where $\alpha_g : E_1\to E_2$ *with* $\alpha_g(tu)=t^q\alpha_g(u)$, *for all* $t>0$ *and* $u\in E_1$, *where* $q>1$ *is a real number.*

(iv) *If* $N(x,y,z)=N(x),\ K(u)=K,\ g\equiv I$, *identity mapping and* $\alpha_g=0$ *then Definition 11 is called* α*-relaxed exponentially* (γ,η)*-monotone with respect to* N *that is for each pair of points* $u,v\in C$,

$$\langle N(x_1)-N(x_2),\frac{1}{\gamma}(e^{\gamma\eta(u,v)}-1)\rangle \geq_K 0,\ \forall x_1\in A(u),\ x_2\in A(v). \tag{8}$$

3. Main results

Theorem 5. *Let* C *be a nonempty closed convex bounded subset of a real Euclidean space* E_1 *and* (E_2,K) *be an ordered Euclidean space induces by a pointed closed convex cone* K. *Let* $K : C\to 2^{E_2}$ *be a closed convex pointed cone valued mapping with* $\text{int}K(u)\neq\emptyset$. *Let* $g : C\to C$ *be a closed convex and continuous single valued mapping and* $\eta : C\times C\to E_1$ *be an affine in the first argument with* $\eta(u,u)=0$, *for all* $u\in C$. *Let* $F : C\times C\to E_2$ *be a* $K(u)$*-convex in the second argument with the condition* $F(u,u)=0$, *for all* $u\in C$. *Let* $N : L(E_1,E_2)\times L(E_1,E_2)\times L(E_1,E_2)\to L(E_1,E_2)$ *be a Lipschitz continuous mapping with all arguments,* $A_1,A_2,A_3 : C\to L(E_1,E_2)$ *be the nonempty compact valued mappings which are* H*-hemicontinuous and* α_g*-relaxed exponentially* (γ,η)*-monotone with respect to first argument of* N *and* g. *Then the following two statements (i) and (ii) are equivalent:*

(i) *there exists* $u_0\in C$ *and* $x\in A_1(u_0),\ y\in A_2(u_0),\ z\in A_3(u_0)$ *such that*

$$\langle N(x,y,z),\frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle+F(g(u_0),v)\not\leq_{\text{int}K(u_0)} 0,\ \forall v\in C,$$

(ii) *there exists* $u_0\in C$ *such that*

$$\begin{aligned}\langle N(r,s,t),\frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle \quad + \quad & F(g(u_0),v)\not\leq_{\text{int}K(u_0)} \alpha_g(v-u_0),\\ & \forall v\in C,\ r\in A_1(v),\ s\in A_2(v),\ t\in A_3(v).\end{aligned}$$

Proof. Let the statement (i) is true that is there exists $u_0 \in C$ and $x \in A_1(u_0),\ y \in A_2(u_0),\ z \in A_3(u_0)$ such that

$$\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) \nleq_{\text{int}K(u_0)} 0,\ \forall v \in C. \tag{9}$$

Since N is α_g-relaxed exponentially (γ,η)-monotone therefore $\forall v \in C,\ r \in A_1(v),\ s \in A_2(v),\ t \in A_3(v)$, we have

$$\begin{aligned}
\langle N(r,s,t)-N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) &\geq_{K(u_0)} \alpha_g(v-u_0)+F(g(u_0),v)\\
\langle N(r,s,t), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) &\geq_{K(u)} \langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle\\
&\quad +\alpha_g(v-u_0)+F(g(u_0),v)\\
\langle N(r,s,t), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v)-\alpha_g(v-u_0) &\geq_{K(u)} \langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle\\
&\quad +F(g(u_0),v). \qquad (10)
\end{aligned}$$

From (9), (10) and Lemma 1, we have

$$\begin{aligned}
\langle N(r,s,t), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle &+ F(g(u_0),v) \nleq_{\text{int}K(u_0)} \alpha_g(v-u_0),\\
&\forall v \in C,\ r \in A_1(v),\ s \in A_2(v),\ t \in A_3(v).
\end{aligned}$$

Conversely, consider the statement (ii) is correct, that is there exists $u_0 \in C$ such that

$$\begin{aligned}
\langle N(r,s,t), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle &+ F(g(u_0),v) \nleq_{\text{int}K(u_0)} \alpha_g(v-u_0),\\
&\forall v \in C,\ r \in A_1(v),\ s \in A_2(v),\ t \in A_3(v). \qquad (11)
\end{aligned}$$

Let $v \in C$ be an arbitrary element. Consider $v_\lambda = \lambda v + (1-\lambda)u_0$, $\lambda \in (0,1]$. As C is convex, $v_\lambda \in C$. Let $r_\lambda \in A_1(v_\lambda),\ s_\lambda \in A_2(v_\lambda),\ t_\lambda \in A_3(v_\lambda)$, we get from (11)

$$\langle N(r_\lambda,s_\lambda,t_\lambda), \frac{1}{\gamma}(e^{\gamma\eta(v_\lambda,g(u_0))}-1)\rangle + F(g(u_0),v_\lambda) \nleq_{\text{int}K(u_0)} \alpha_g(v_\lambda-u_0) = t^q\alpha_g(v-u_0). \tag{12}$$

Now,

$$\begin{aligned}\langle N(r_\lambda,s_\lambda,t_\lambda),\frac{1}{\gamma}(e^{\gamma\eta(v_\lambda,g(u_0))}-1)\rangle &+ F(g(u_0),v_\lambda)\\
&= \langle N(r_\lambda,s_\lambda,t_\lambda),\frac{1}{\gamma}(e^{\gamma\eta(\lambda v+(1-\lambda)u_0,g(u_0))}-1)\rangle\\
&\quad +F(g(u_0),\lambda v+(1-\lambda)u_0)\\
&= \langle N(r_\lambda,s_\lambda,t_\lambda),\frac{1}{\gamma}(e^{\gamma\eta\lambda(v,g(u_0))+(1-\lambda)\gamma\eta(u_0,g(u_0))}-1)\rangle\\
&\quad +\lambda F(g(u_0),v)+(1-\lambda)F(g(u_0),u_0)\\
&\leq_{K(u_0)} \langle N(r_\lambda,s_\lambda,t_\lambda),\frac{1}{\gamma}(\lambda(e^{\gamma\eta(v,g(u_0))}-1)\\
&\quad +(1-\lambda)(e^{\gamma\eta(v,g(u_0))}-1)\rangle+\lambda F(g(u_0),v)\\
&= \lambda\{\langle N(r_\lambda,s_\lambda,t_\lambda),\frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle+F(g(u_0),v)\}. \qquad (13)\end{aligned}$$

From (12), (13) and Lemma 1, we have

$$\langle N(r_\lambda,s_\lambda,t_\lambda),\frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle+F(g(u_0),v)\not\leq_{\text{int}K(u_0)} t^{q-1}\alpha_g(v-u_0) \qquad (14)$$

Since $A_1(v_\lambda)$, $A_2(v_\lambda)$, $A_3(v_\lambda)$, $A_1(u_0)$, $A_2(u_0)$ and $A_3(u_0)$ are compact, therefore by Lemma 3, for each fixed $r_\lambda\in A_1(v_\lambda),\ s_\lambda\in A_2(v_\lambda),\ t_\lambda\in A_3(v_\lambda)$ there exists $r_\lambda^{'}\in A_1(u_0),\ s_\lambda^{'}\in A_2(u_0),\ t_\lambda^{'}\in A_3(u_0)$ such that

$$\begin{aligned}\|r_\lambda-r_\lambda^{'}\| &\leq H(A_1(v_\lambda),A_1(u_0)),\\
\|s_\lambda-s_\lambda^{'}\| &\leq H(A_2(v_\lambda),A_2(u_0)),\\
\|t_\lambda-t_\lambda^{'}\| &\leq H(A_3(v_\lambda),A_3(u_0)). \qquad (15)\end{aligned}$$

Since $A_1(u_0)$, $A_2(u_0)$ and $A_3(u_0)$ are compact, therefore without loss of generality, we may assume that

$$\begin{aligned}&r_\lambda\to r_0\in A_1u_0 \text{ as } \lambda\to 0^+\\
&s_\lambda\to s_0\in A_2u_0 \text{ as } \lambda\to 0^+\\
&t_\lambda\to t_0\in A_3u_0 \text{ as } \lambda\to 0^+.\end{aligned}$$

Also, $A_1,\ A_2$ and A_3 are H-hemicontinuous, thus it follows that

$$\begin{aligned} &H(A_1(v_\lambda),A_1(u_0)) \to 0 \qquad \text{as } \lambda \to 0^+\\ &H(A_2(v_\lambda),A_2(u_0)) \to 0 \qquad \text{as } \lambda \to 0^+\\ &H(A_3(v_\lambda),A_3(u_0)) \to 0 \qquad \text{as } \lambda \to 0^+. \end{aligned}$$

By (15), we get

$$\begin{aligned} \|r_\lambda - r_0\| &\leq \|r_\lambda - r_\lambda^{'}\| + \|r_\lambda^{'} - r_0\|\\ &\leq H(A_1(v_\lambda),A_1(r_0)) + \|r_\lambda^{'} - r_0\| \to 0 \text{ as } \lambda \to 0^+, \end{aligned}$$

$$\begin{aligned} \|s_\lambda - v_0\| &\leq \|s_\lambda - s_\lambda^{'}\| + \|s_\lambda^{'} - v_0\|\\ &\leq H(A_2(v_\lambda),A_2(v_0)) + \|s_\lambda^{'} - v_0\| \to 0 \text{ as } \lambda \to 0^+, \end{aligned}$$

and

$$\begin{aligned} \|t_\lambda - t_0\| &\leq \|t_\lambda - t_\lambda^{'}\| + \|t_\lambda^{'} - t_0\|\\ &\leq H(A_3(v_\lambda),A_3(t_0)) + \|t_\lambda^{'} - t_0\| \to 0 \text{ as } \lambda \to 0^+. \end{aligned} \tag{16}$$

Since N is Lipschitz continuous with all arguments therefore we get

$$\begin{aligned} &\|\langle N(r_\lambda,s_\lambda,t_\lambda), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle - t^{q-1}\alpha_g(v-u_0) - \langle N(r_0,s_0,t_0), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle\|\\ &\leq \|\langle N(r_\lambda,s_\lambda,t_\lambda) - N(r_0,s_0,t_0), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle\| + \|t^{q-1}\alpha_g(v-u_0)\|\\ &\leq \frac{1}{\gamma}\{\|N(r_\lambda,s_\lambda,t_\lambda) - N(r_0,s_\lambda,t_\lambda)\| + \|N(r_0,s_\lambda,t_\lambda) - N(r_0,s_0,t_\lambda)\|\\ &+\|N(r_0,s_0,t_\lambda) - N(r_0,s_0,t_0)\|\}\|e^{\gamma\eta(v,g(u_0))}-1\| + t^{q-1}\|\alpha_g(v-u_0)\| \to 0 \text{ as } \lambda \to 0^+. \end{aligned} \tag{17}$$

By (12), we get

$$\langle N(r_\lambda,s_\lambda,t_\lambda), \frac{1}{\gamma}(e^{\gamma\eta(v_\lambda,g(u_0))}-1)\rangle + F(g(u_0),v_\lambda) - t^{q-1}\alpha_g(v-u_0) \in E_2 \setminus (\text{int}K(u_0)).$$

Since $E_2 \setminus (\text{int}K(u_0))$ is closed therefore from (17), we have

$$\langle N(r_0,s_0,t_0), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) \quad \in E_2 \setminus (\text{int}K(u_0))$$
$$\langle N(r_0,s_0,t_0), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) \quad \nleq_{\text{int}K(u_0)} 0, \ \forall\, v \in K.$$

□

Theorem 6. *Let C be a nonempty closed convex bounded subset of a real Euclidean space E_1 and (E_2,K) be an ordered Euclidean space induces by a pointed closed convex cone K. Let $K : C \to 2^{E_2}$ be a closed convex pointed cone valued mapping with* $\text{int}K(u) \neq \emptyset$*. Let $g : C \to C$ be a closed convex and continuous single valued mapping and $\eta : C \times C \to E_1$ be an affine in the first argument with $\eta(u,g(u)) = 0$, for all $u \in C$. Let $F : C \times C \to E_2$ be a completely continuous in the first argument and affine in the second argument with the condition $F(g(u),u) = 0$, for all $u \in C$. Let $\alpha_g : E_1 \to E_2$ be a weakly lower semicontinuous. Let $N : L(E_1,E_2) \times L(E_1,E_2) \times L(E_1,E_2) \to L(E_1,E_2)$ be a Lipschitz continuous mapping with all arguments, $A_1,A_2,A_3 : C \to L(E_1,E_2)$ be the nonempty compact valued mappings which are H-hemicontinuous and α_g-relaxed exponentially (γ,η)-monotone with respect to first argument of N and g. Then* (2) *is a solvable, that is there exists $u \in C$ and $x \in A_1(u)$, $y \in A_2(u)$, $z \in A_3(u)$ such that*

$$\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) \nleq_{\text{int}K(u)} 0, \ \forall v \in C.$$

Proof. Consider the set valued mapping $S : C \to 2^{E_1}$ such that $\forall v \in C$

$$S(v) = \{u \in C : \langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) \nleq_{\text{int}K(u)} 0, \ \forall x \in A_1(u),\ y \in A_2(u),\ z \in A_3(u)\}.$$

First, we claim that S is a KKM-mapping. If S is not a KKM-mapping then there exists $\{u_1,u_2,u_3,...,u_m\} \subset C$ such that $\text{co}\{u_1,u_2,u_3,...,u_m\} \nsubseteq \bigcup_{i=1}^m S(u_i)$ that means there exists at least $u \in \text{co}\{u_1,u_2,u_3,...,u_m\}$, $u = \sum_{i=1}^m \lambda_i u_i$, where $\lambda_i \geq 0$, $i = 1,2,3,...,m$, $\sum_{i=1}^m \lambda_i = 1$ but $u \notin \bigcup_{i=1}^m S(u_i)$. From the construction of S, for any $x \in A_1(u)$, $y \in A_2(u)$, $z \in A_3(u)$, we have

$$\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(u_i,g(u))}-1)\rangle + F(g(u),u_i) \nleq_{\text{int}K(u)} 0, \text{ for } i = 1,2,3,...,m. \tag{18}$$

From (18) and since η and F are affine in first and second argument respectively, it follows that

$$\begin{aligned}
0 &= \langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(u,g(u))}-1)\rangle + F(g(u),u) \\
&= \langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(\sum_{i=1}^m \lambda_i u_i, g(u))}-1)\rangle + F(g(u),\sum_{i=1}^m \lambda_i u_i) \\
&= \langle N(x,y,z), \frac{1}{\gamma}(e^{\sum_{i=1}^m \lambda_i \gamma\eta(u_i,g(u))}-1)\rangle + \sum_{i=1}^m \lambda_i F(g(u),u_i) \\
&\leq_{K(u)} \langle N(x,y,z), \frac{1}{\gamma}(e^{\sum_{i=1}^m \lambda_i \gamma\eta(u_i,g(u))}-1)\rangle + \sum_{i=1}^m \lambda_i F(g(u),u_i) \\
&= \sum_{i=1}^m \lambda_i \{\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(u_i,g(u))}-1)\rangle + F(g(u),u_i)\} \leq_{\mathrm{int}K(u)} 0,
\end{aligned}$$

this shows that $0 \in \mathrm{int}K(u)$, which contradicts the fact that $K(u)$ is proper. Hence, S is a KKM-mapping. Define another set valued mapping $W : C \to 2^{E_1}$ such that

$$\begin{aligned}
W(v) \quad = \quad & \{u \in C : \langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) \\
\not\leq_{\mathrm{int}K(u)} \quad & \alpha_g(v-u),\ \forall p \in A_1(v),\ q \in A_2(v),\ r \in A_3(v)\},\ \forall v \in C.
\end{aligned}$$

Now, we will prove that $S(v) \subset W(v),\ \forall v \in C$.

Let $u \in S(v)$, there exists some $x \in A_1(u),\ y \in A_2(u),\ z \in A_3(u)$, such that

$$\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) \not\leq_{\mathrm{int}K(u)} 0. \tag{19}$$

Since N is α_g-relaxed exponentially (γ,η)-monotone therefore $\forall v \in C,\ p \in A_1(v),\ q \in A_2(v),\ r \in A_3(v)$ we have

(20)

$$\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) \leq_{\mathrm{int}K(u)} \langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) - \alpha_g(v-u).$$

Using (19), (20) and Lemma 1, we have

$$\begin{aligned}
& \langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))}-1)\rangle + F(g(u),v) \not\leq_{\mathrm{int}K(u)} \alpha_g(v-u), \\
& \forall v \in C,\ p \in A_1(v),\ q \in A_2(v),\ r \in A_3(v).
\end{aligned}$$

Therefore $u \in W(v)$ that is $S(v) \subset W(v),\ \forall v \in C$. This implies that W is also a KKM-mapping. We claim that for each $v \in C$, $W(v) \subset C$ is closed in the weak topology of E_1. Let us suppose that $\bar{u} \in \overline{W(v)}^w$, the weak closure of $W(v)$. Since E_1 is reflexive, there is a sequence $\{u_n\}$ in

$W(v)$ such that $\{u_n\}$ converges weakly to $\overline{u} \in C$. Then, for each $p \in A_1(v),\ q \in A_2(v),\ r \in A_3(v)$, we have

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_n))}-1)\rangle + F(g(u_n),v) \quad \nleq_{\text{int}K(u_n)} \quad \alpha_g(v-u_n)$$

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_n))}-1)\rangle + F(g(u_n),v) - \alpha_g(v-u_n) \qquad \in \qquad E_2 \setminus (-\text{int}K(u_n)).$$

Since N and F are completely continuous and $E_2 \setminus (-\text{int}K(u_n))$ is closed, α_g is weakly lower semicontinuous therefore the sequence

$$\{\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_n))}-1)\rangle + F(g(u_n),v) - \alpha_g(v-u_n)\}$$

converges to

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(\overline{u}))}-1)\rangle + F(g(\overline{u}),v) - \alpha_g(v-\overline{u})$$

and

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(\overline{u}))}-1)\rangle + F(g(\overline{u}),v) - \alpha_g(v-\overline{u}) \in E_2 \setminus (-\text{int}K(\overline{u})).$$

Therefore

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(\overline{u}))}-1)\rangle + F(g(\overline{u}),v) \nleq_{\text{int}K(u_n)} \alpha_g(v-\overline{u}).$$

Thus, $\overline{u} \in W(v)$. This shows that $W(v)$, $\forall v \in C$ is weakly closed. Furthermore, E_1 is reflexive and $C \subset E_1$ is a nonempty closed convex and bounded. Therefore, C is weakly compact subset of E_1 and so $W(v)$ is also weakly compact. Therefore from Lemma 2 and Theorem 5, it follows that

$$\bigcap_{v\in C} W(v) \neq \emptyset.$$

Thus, there exists $\overline{u} \in C$ such that

$$\langle N(p,q,r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(\overline{u}))}-1)\rangle + F(g(\overline{u}),v) \nleq_{\text{int}K(u_n)} \alpha_g(v-\overline{u}),\ \forall v \in C,\ p \in A_1(v),\ q \in A_2(v),\ r \in A_3(v).$$

Hence from Theorem 5, we can conclude that there exists $\overline{u} \in C$ and $\overline{x} \in A_1(\overline{u}),\ \overline{y} \in A_2(\overline{u}),\ \overline{z} \in A_3(\overline{u})$ such that

$$\langle N(\overline{x},\overline{y},\overline{z}), \frac{1}{\gamma}(e^{\gamma\eta(v,g(\overline{u}))}-1)\rangle + F(g(\overline{u}),v) \nleq_{\text{int}K(\overline{u})} 0,\ \forall v \in C,$$

that is (2) is solvable. □

Theorem 7. *Let C be a nonempty closed convex bounded subset of a real Euclidean space E_1 with $0 \in C$ and (E_2, K) be an ordered Euclidean space induces by a pointed closed convex cone $K(u)$. Let $K : C \to 2^{E_2}$ be a closed convex pointed cone valued mapping with $\mathrm{int}K(u) \neq \emptyset$. Let $g : C \to C$ be a closed convex and continuous single valued mapping and $\eta : C \times C \to E_1$ be an affine in the first argument with $\eta(u,u) = 0$, for all $u \in C$. Let $F : C \times C \to E_2$ be a completely continuous in the first argument and affine in the second argument with the condition $F(u,u) = 0$, for all $u \in C$. Let $\alpha_g : E_1 \to E_2$ be a weakly lower semicontinuous. Let $N : L(E_1,E_2) \times L(E_1,E_2) \times L(E_1,E_2) \to L^c(E_1,E_2)$ be a Lipschitz continuous mapping with all arguments, where $L^c(E_1,E_2)$ be a space of all completely continuous linear mapping from E_1 to E_2, $A_1, A_2, A_3 : C \to L(E_1,E_2)$ be the nonempty compact valued mappings which are H-hemicontinuous and α_g-relaxed exponentially (γ,η)-monotone with respect to first argument of N and g. If there exists one $r > 0$ such that*

$$\langle N(p,q,s), \frac{1}{\gamma}(e^{\gamma\eta(g(0),v)} - 1)\rangle + F(v, g(0)) \nleq_{\mathrm{int}K(0)} 0,$$

$$\forall v \in C,\ p \in A_1(v),\ q \in A_2(v),\ s \in A_3(v) \text{ with } \|v\| = r. \tag{21}$$

Then (2) *is solvable that is there exists $u \in C$ and $x \in A_1(u),\ y \in A_2(u),\ z \in A_3(u)$ such that*

$$\langle N(x,y,z), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u))} - 1)\rangle + F(g(u), v) \nleq_{\mathrm{int}K(u)} 0,\ \forall v \in C.$$

Proof. For $r > 0$, assume that $C_r = \{u \in E_1 : \|u\| \leq r\}$. From Theorem 6, we know that (2) is solvable over C_r that is there exist $u_r \in C \bigcap C_r$ and $x_r \in A_1(u_r),\ y_r \in A_2(u_r),\ z_r \in A_3(u_r)$ such that

$$\langle N(x_r,y_r,z_r), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_r))} - 1)\rangle + F(g(u_r), v) \nleq_{\mathrm{int}K(u_r)} 0,\ \forall v \in C \cap C_r. \tag{22}$$

Putting $v = 0$ in (22), we get

$$\langle N(x_r,y_r,z_r), \frac{1}{\gamma}(e^{\gamma\eta(0,g(u_r))} - 1)\rangle + F(g(u_r), 0) \nleq_{\mathrm{int}K(u_r)} 0. \tag{23}$$

If $\|u_r\| = r$, for all r then it contradicts to (21). Hence $\|u_r\| < r$. For any $w \in C$, let us choose $\lambda \in (0,1)$ small enough such that $(1-\lambda)u_r + \lambda w \in C \cap C_r$. Putting $v = (1-\lambda)u_r + \lambda w$ in (22), we get

$$\langle N(x_r,y_r,z_r), \frac{1}{\gamma}(e^{\gamma\eta((1-\lambda)u_r+\lambda w, g(u_r))} - 1)\rangle + F(g(u_r), (1-\lambda)u_r + \lambda w) \nleq_{\mathrm{int}K(u_r)} 0. \tag{24}$$

Since η and F are affine in the first and second variable, we have

$$\langle N(x_r,y_r,z_r),\frac{1}{\gamma}(e^{\gamma\eta((1-\lambda)u_r+\lambda w,g(u_r))}-1)\rangle+F(g(u_r),(1-\lambda)u_r+\lambda w)$$

$$=\langle N(x_r,y_r,z_r),\frac{1}{\gamma}(e^{(1-\lambda)\gamma\eta(u_r,g(u_r))+\lambda\gamma\eta(w,g(u_r))}-1)\rangle+\lambda F(g(u_r),w)$$

$$\leq_{K(u_r)}\langle N(x_r,y_r,z_r),\frac{1}{\gamma}(1-\lambda)(e^{\gamma\eta(u_r,g(u_r))}-1)+\frac{1}{\gamma}\lambda e^{\gamma\eta(w,g(u_r))}-1)\rangle+\lambda F(g(u_r),w)$$

$$=\lambda\{\langle N(x_r,y_r,z_r),\frac{1}{\gamma}e^{\gamma\eta(w,g(u_r))}-1)\rangle+F(g(u_r),w)\}. \tag{25}$$

Hence from (24), (25) and Lemma 1, we get

$$\langle N(x_r,y_r,z_r),\frac{1}{\gamma}e^{\gamma\eta(w,g(u_r))}-1)\rangle+F(g(u_r),w)\nleq_{\mathrm{int}K(u_r)}0,\ \forall w\in C. \tag{26}$$

Thus, (2) is solvable. □

If $N(x,y,z)=N(x,y)$ and $A_3\equiv 0$, a zero mapping, then Theorem 5 reduces to the following corollary:

Corollary 8. *Let C be a nonempty closed convex bounded subset of a real Euclidean space E_1 and (E_2,K) be an ordered Euclidean space induces by a pointed closed convex cone K. Let $K:C\to 2^{E_2}$ be a closed convex pointed cone valued mapping with* $\mathrm{int}K(u)\neq\emptyset$*. Let $g:C\to C$ be a closed convex and continuous single valued mapping and $\eta:C\times C\to E_1$ be an affine in the first argument with $\eta(u,u)=0$, for all $u\in C$. Let $F:C\times C\to E_2$ be a $K(u)$-convex in the second argument with the condition $F(u,u)=0$, for all $u\in C$. Let $N:L(E_1,E_2)\times L(E_1,E_2)\to L(E_1,E_2)$ be a Lipschitz continuous mapping with all arguments, $A_1,A_2:C\to L(E_1,E_2)$ be the nonempty compact valued mappings which are H-hemicontinuous and α_g-relaxed exponentially (γ,η)-monotone with respect to first argument of N and g. Then the following two statements (i) and (ii) are equivalent:*

(i) *there exists $u_0\in C$ and $x\in A_1(u_0),\ y\in(A_2(u_0))$ such that*

$$\langle N(x,y),\frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle+F(g(u_0),v)\nleq_{\mathrm{int}K(u_0)}0,\ \forall v\in C,$$

(ii) *there exists* $u_0 \in C$ *such that*

$$\langle N(r,s), \frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle + F(g(u_0),v) \nleq_{\text{int}K(u_0)} \alpha_g(v-u_0),$$
$$\forall v \in C,\ r \in A_1(v),\ s \in A_2(v).$$

If $N(x,y,z) = N(x,y)$ and $A_3 \equiv 0$, a zero mapping, and $g \equiv I$, an identity mapping then Theorem 5 reduces to the following corollary:

Corollary 9. *Let* C *be a nonempty closed convex bounded subset of a real Euclidean space* E_1 *and* (E_2,K) *be an ordered Euclidean space induces by a pointed closed convex cone* K*. Let* $K : C \to 2^{E_2}$ *be a closed convex pointed cone valued mapping with* $\text{int}K(u) \neq \emptyset$*. Let* $\eta : C \times C \to E_1$ *be an affine in the first argument with* $\eta(u,u) = 0$*, for all* $u \in C$*. Let* $F : C \times C \to E_2$ *be a* $K(u)$*-convex in the second argument with the condition* $F(u,u) = 0$*, for all* $u \in C$*. Let* $N : L(E_1,E_2) \times L(E_1,E_2) \to L(E_1,E_2)$ *be a Lipschitz continuous mapping with all arguments,* $A_1, A_2 : C \to L(E_1,E_2)$ *be the nonempty compact valued mappings which are* H*-hemicontinuous and* α*-relaxed exponentially* (γ,η)*-monotone with respect to first argument of* N*. Then the following two statements (i) and (ii) are equivalent:*

(i) *there exists* $u_0 \in C$ *and* $x \in A_1(u_0),\ y \in A_2(u_0)$ *such that*

$$\langle N(x,y), \frac{1}{\gamma}(e^{\gamma\eta(v,u_0)}-1)\rangle + F(u_0,v) \nleq_{\text{int}K(u_0)} 0,\ \forall v \in C,$$

(ii) *there exists* $u_0 \in C$ *such that*

$$\langle N(r,s), \frac{1}{\gamma}(e^{\gamma\eta(v,u_0)}-1)\rangle + F(u_0,v) \nleq_{\text{int}K(u_0)} \alpha(v-u_0),$$
$$\forall v \in C,\ r \in A_1(v),\ s \in A_2(v).$$

If $N(x,y,z) = N(x)$ and $A_2, A_3 \equiv 0$, a zero mapping then Theorem 5 reduces to the following corollary:

Corollary 10. *Let* C *be a nonempty closed convex bounded subset of a real Euclidean space* E_1 *and* (E_2,K) *be an ordered Euclidean space induces by a pointed closed convex cone* K*. Let* $K : C \to 2^{E_2}$ *be a closed convex pointed cone valued mapping with* $\text{int}K(u) \neq \emptyset$*. Let* $g : C \to C$ *be a closed convex and continuous single valued mapping and* $\eta : C \times C \to E_1$ *be an affine in the first argument with* $\eta(u,u) = 0$*, for all* $u \in C$*. Let* $F : C \times C \to E_2$ *be a* $K(u)$*-convex in the* *second argument with the condition $F(u,u)=0$, for all $u\in C$. Let $N:L(E_1,E_2)\to L(E_1,E_2)$ be a Lipschitz continuous mapping, $A_1:C\to L(E_1,E_2)$ be the nonempty compact valued mapping which is H-hemicontinuous and α_g-relaxed exponentially (γ,η)-monotone with respect to first argument of N and g. Then the following two statements (i) and (ii) are equivalent:*

(i) *there exists $u_0\in C$ and $x\in A_1(u_0)$ such that*

$$\langle N(x),\frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle+F(g(u_0),v)\not\leq_{\mathrm{int}K(u_0)}0,\ \forall v\in C,$$

(ii) *there exists $u_0\in C$ such that*

$$\langle N(r),\frac{1}{\gamma}(e^{\gamma\eta(v,g(u_0))}-1)\rangle+F(g(u_0),v)\not\leq_{\mathrm{int}K(u_0)}\alpha_g(v-u_0),\ \forall v\in C,\ r\in A_1(v).$$

If $N(x,y,z)=N(x)$, $A_2,A_3\equiv 0$, zero mappings and $g\equiv I$, an identity mapping then Theorem 5 reduces to the following corollary:

Corollary 11. *Let C be a nonempty closed convex bounded subset of a real Euclidean space E_1 and (E_2,K) be an ordered Euclidean space induces by a pointed closed convex cone K. Let $K:C\to 2^{E_2}$ be a closed convex pointed cone valued mapping with $\mathrm{int}K(u)\neq\emptyset$. Let $\eta:C\times C\to E_1$ be an affine in the first argument with $\eta(u,u)=0$, for all $u\in C$. Let $F:C\times C\to E_2$ be a $K(u)$-convex in the second argument with the condition $F(u,u)=0$, for all $u\in C$. Let $N:L(E_1,E_2)\to L(E_1,E_2)$ be a Lipschitz continuous mapping, $A_1:C\to L(E_1,E_2)$ be the nonempty compact valued mapping which is H-hemicontinuous and α-relaxed exponentially (γ,η)-monotone with respect to first argument of N. Then the following two statements (i) and (ii) are equivalent:*

(i) *there exist $u_0\in C$ and $x\in A_1(u_0)$ such that*

$$\langle N(x),\frac{1}{\gamma}(e^{\gamma\eta(v,u_0)}-1)\rangle+F(u_0,v)\not\leq_{\mathrm{int}K(u_0)}0,\ \forall v\in C,$$

(ii) *there exists $u_0\in C$ such that*

$$\langle N(r),\frac{1}{\gamma}(e^{\gamma\eta(v,u_0)}-1)\rangle+F(u_0,v)\not\leq_{\mathrm{int}K(u_0)}\alpha(v-u_0),\ \forall v\in C,\ r\in A_1(v).$$

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Conflict of Interests

The author declares that there is no conflict of interests.

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