QUADRUPLED BEST PROXIMITY POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES

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Abstract: In this paper, we prove some quadrupled best proximity point theorems in partially ordered metric space by using \((\psi, \phi)\) contraction. Our results generalise the results of Kumam et.al. (Coupled best proximity points in ordered metric spaces, Fixed Point Theory and Application 2014, 2014:107). An example is also given to verify the results obtained.

Keywords: partial ordered set; best proximity point; quadrupled fixed point; quadrupled best proximity point.

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1. INTRODUCTION

In a metric space \((X, d)\) a self mapping \(T\) on \(X\) is said to posses fixed point if the equation \(Tx = x\) has at least one solution. In this case \(x\) is said to be the fixed point of \(T\). The existence of fixed point of a mapping may or may not be unique. The study of the existence of fixed point of mapping has been an area of extensive research for the past many years and Banach
contraction principle has the most important tool in order to obtain fixed point of a mapping. Now, the question arises if the equation $T \cdot x = x$ has no solution. In order to investigate further the study of best proximity started.

Let $A$ and $B$ be non-empty closed subsets of a metric space $(X, d)$ and $T: A \rightarrow B$ be a non-self mapping. A point $x$ in $A$ for which $d(x, Tx) = d(A, B)$ is called a best proximity point of $T$. It can be noted that the best proximity point becomes a fixed point if the underlying mapping is assumed to be self mapping.

The best approximation theorem given by Ky Fan [1] stated as follows:

If $K$ is a non-empty compact convex subset of a Hausdorff locally convex topological vector space $E$ with a continuous seminorm $p$ and $T: K \rightarrow E$ is a single valued continuous function, then there exists an element $x \in K$ such that

\[ p(x - Tx) = d_p(Tx, K) := \inf\{p(Tx - y) : y \in K\} \]

The result of Ky Fan was generalised by a large number of authors in various direction (see [2-19] and references there in).

The concept of coupled fixed point was introduced by Guo and Lakshmikantham [20] in the year 1987. Further, Bhaskar and Lakshmikantham [21] introduced the concept of mixed monotone mapping and established some coupled fixed point theorems for mapping satisfying mixed monotone property.

The concept of coupled fixed point was further extended to triple fixed point by Berinde and Borcut [22] and quadrupled fixed point by Karapinar and Luong [23]. These concepts of Tripled and quadrupled fixed points were presented in a more generalised form by Wu and Liu [24]. For more results on coupled, tripled, quadrupled fixed point, one can see the research articles ([25-50] and references there in).

In this note we prove some quadrupled best proximity point theorems in partially ordered metric space by using proximally quadrupled weak $(\psi, \phi)$ contraction on the line of proximally coupled weak $(\psi, \phi)$ contractions introduced by Kumam et.al. [34].

Following definition was given in Karapinar and Luong [23].
**DEFINITION 1.** [23] Let $X$ be a non empty set and $F:X^4 \to X$ be a given mapping. An element $(x,y,z,w) \in X^4$ is called a quadrupled fixed point of the mapping $F$ if

$$x = F(x,y,z,w), \ y = F(y,z,w,x), \ z = F(z,w,x,y) \text{ and } w = F(w,x,y,z).$$

The authors mentioned above also introduced the notion of mixed monotone mapping. If $(X,\leq)$ is a partially ordered set, the mapping $F$ is said to have the mixed monotone property, if

- $x_1,x_2 \in X, x_1 \leq x_2 \implies F(x_1,y,z,w) \leq F(x_2,y,z,w), y,z,w \in X,$
- $y_1,y_2 \in X, y_1 \leq y_2 \implies F(x,y_1,z,w) \geq F(x,y_2,z,w), x,z,w \in X,$
- $z_1,z_2 \in X, z_1 \leq z_2 \implies F(x,y,z_1,w) \leq F(x,y,z_2,w), x,y,w \in X,$
- $w_1,w_2 \in X, w_1 \leq w_2 \implies F(w,y,z,w_1) \geq F(w,y,z,w_2), x,y,z,w \in X.$

Let $A$ and $B$ be non empty subsets of a metric space $(X,d)$. We use the following notions in the sequel:

$$d(A,B) = \inf\{d(x,y): x \in A \text{ and } y \in B\},$$

- $A_0 = \{x \in A: d(x,y) = d(A,B), \text{ for some } y \in B\}$
- $B_0 = \{y \in B: d(x,y) = d(A,B), \text{ for some } x \in A\}$

Now we give the following definition.

**DEFINITION 2.** Let $(X,d,\leq)$ be a partially ordered metric space and $A$, $B$ are nonempty subsets of $X$. A mapping $F:A \times A \times A \times A \to B$ is said to have proximal mixed monotone property if $F(x,y,z,w)$ is proximally non decreasing in $x$ and $z$, and is proximally non increasing in $y$ and $w$, that is for all $x,y,z,w \in A$,

$$x_1 \leq x_2 \leq x_3 \leq x_4$$

$$d(u_1,F(x_1,y,z,w)) = d(A,B)$$

$$d(u_2,F(x_2,y,z,w)) = d(A,B)$$

$$d(u_3,F(x_3,y,z,w)) = d(A,B)$$

$$d(u_4,F(x_4,y,z,w)) = d(A,B)$$
\[ u_1 \leq u_2 \leq u_3 \leq u_4 \]

\[ y_1 \leq y_2 \leq y_3 \leq y_4 \]

\[ d \left( u_5, F(x, y_1, z, w) \right) = \text{dist}(A, B) \]

\[ d \left( u_6, F(x, y_2, z, w) \right) = \text{dist}(A, B) \]

\[ d \left( u_7, F(x, y_3, z, w) \right) = \text{dist}(A, B) \]

\[ d \left( u_8, F(x, y_4, z, w) \right) = \text{dist}(A, B) \]

\[ u_5 \geq u_6 \geq u_7 \geq u_8 \]

\[ z_1 \leq z_2 \leq z_3 \leq z_4 \]

\[ d \left( u_9, F(x, y, z_1, w) \right) = \text{dist}(A, B) \]

\[ d \left( u_{10}, F(x, y, z_2, w) \right) = \text{dist}(A, B) \]

\[ d \left( u_{11}, F(x, y, z_3, w) \right) = \text{dist}(A, B) \]

\[ d \left( u_{12}, F(x, y, z_4, w) \right) = \text{dist}(A, B) \]

\[ u_9 \leq u_{10} \leq u_{11} \leq u_{12} \]

and

\[ w_1 \leq w_2 \leq w_3 \leq w_4 \]

\[ d \left( u_{13}, F(x, y, z, w_1) \right) = \text{dist}(A, B) \]

\[ d \left( u_{14}, F(x, y, z, w_2) \right) = \text{dist}(A, B) \]

\[ d \left( u_{15}, F(x, y, z, w_3) \right) = \text{dist}(A, B) \]

\[ d \left( u_{16}, F(x, y, z, w_4) \right) = \text{dist}(A, B) \]

\[ u_{13} \geq u_{14} \geq u_{15} \geq u_{16} \]

where \( x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4, u_1, u_2, \ldots, u_{16} \in A \)
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If we take \( A = B \) the above definition, the notion of the proximal mixed monotone property becomes the conditions of mixed monotone property. The following lemmas are essential for our results.

**LEMMA 1.** Let \((X, d, \leq)\) be a partially ordered metric space and \( A, B \) are nonempty subsets of \( X \). Assume that \( A_0 \) is nonempty. A mapping \( F: A \times A \times A \times A \to B \) has the proximal mixed monotone property with \( F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0 \) whenever

\[
\begin{aligned}
x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, z_0, z_1, z_2, z_3, w_0, w_1, w_2, w_3 & \in A \\
& \text{such that } \\
x_0 \leq x_1 \leq x_2 \leq x_3; y_0 \geq y_1 \geq y_2 \geq y_3; z_0 \leq z_1 \leq z_2 \leq z_3; w_0 \geq w_1 \geq w_2 \geq w_3 \\
d\left(x_1, F(x_0, y_0, z_0, w_0)\right) &= \text{dist}(A, B). \\
d\left(x_2, F(x_1, y_1, z_1, w_1)\right) &= \text{dist}(A, B). \\
d\left(x_3, F(x_2, y_2, z_2, w_2)\right) &= \text{dist}(A, B). \\
d\left(x_4, F(x_3, y_3, z_3, w_3)\right) &= \text{dist}(A, B). \\
\Rightarrow x_1 \leq x_2 \leq x_3 \leq x_4
\end{aligned}
\]

(1)

**Proof:** By hypothesis we have \( F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0 \), therefore \( F(x_3, y_0, z_0, w_0) \in B_0 \). Hence, there exist \( x_1^* \in A \) such that

\[
d\left(x_1^*, F(x_3, y_0, z_0, w_0)\right) = \text{dist}(A, B).
\]

(2)

Since \( F \) is proximal mixed monotone (in particular \( F \) is proximally non decreasing in \( x \)) from (1) and (2), we get

\[
\begin{aligned}
x_0 \leq x_1 \leq x_2 \leq x_3; \\
d\left(x_1, F(x_0, y_0, z_0, w_0)\right) &= \text{dist}(A, B). \\
d\left(x_2, F(x_1, y_1, z_1, w_1)\right) &= \text{dist}(A, B). \\
d\left(x_3, F(x_2, y_2, z_2, w_2)\right) &= \text{dist}(A, B). \\
d\left(x_1^*, F(x_3, y_0, z_0, w_0)\right) &= \text{dist}(A, B). \\
\Rightarrow x_1 \leq x_2 \leq x_3 \leq x_1^*
\end{aligned}
\]

(3)

Similarly, using the fact that \( F \) is proximal mixed monotone (in particular, \( F \) is proximally non increasing in \( y \)) from (1) and (2), we get
From (3) and (4), we have $x_1 \leq x_2 \leq x_3 \leq x_4$. This completes the proof.

**LEMMA 2.** Let $(X,d,\leq)$ be a partially ordered metric space and $A, B$ are non empty subsets of $X$. Assume $A_0$ is nonempty. A mapping $F: A \times A \times A \times A \to B$ has the proximal mixed monotone property with $F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0$ whenever

$$x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, z_0, z_1, z_2, z_3, w_0, w_1, w_2, w_3 \in A$$ such that

$$\begin{cases}
  x_0 \leq x_1 \leq x_2 \leq x_3; y_0 \geq y_1 \geq y_2 \geq y_3; z_0 \leq z_1 \leq z_2 \leq z_3; w_0 \geq w_1 \geq w_2 \geq w_3 \\
  d\left(y_1, F(y_0, z_0, w_0, x_0)\right) = dist(A, B). \\
  d\left(y_2, F(y_1, z_1, w_1, x_1)\right) = dist(A, B). \\
  d\left(y_3, F(y_2, z_2, w_2, x_2)\right) = dist(A, B). \\
  d\left(y_4, F(y_3, z_3, w_3, x_3)\right) = dist(A, B). \\
  \Rightarrow y_1 \geq y_2 \geq y_3 \geq y_4
\end{cases}$$

**LEMMA 3.** Let $(X,d,\leq)$ be a partially ordered metric space and $A, B$ are non empty subsets of $X$. Assume $A_0$ is nonempty. A mapping $F: A \times A \times A \times A \to B$ has the proximal mixed monotone property with $F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0$ whenever

$$x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, z_0, z_1, z_2, z_3, w_0, w_1, w_2, w_3 \in A$$ such that

$$\begin{cases}
  x_0 \leq x_1 \leq x_2 \leq x_3; y_0 \geq y_1 \geq y_2 \geq y_3; z_0 \leq z_1 \leq z_2 \leq z_3; w_0 \geq w_1 \geq w_2 \geq w_3 \\
  d\left(z_1, F(z_0, w_0, x_0, y_0)\right) = dist(A, B). \\
  d\left(z_2, F(z_1, w_1, x_1, y_1)\right) = dist(A, B). \\
  d\left(z_3, F(z_2, w_2, x_2, y_2)\right) = dist(A, B). \\
  d\left(z_4, F(z_3, w_3, x_3, y_3)\right) = dist(A, B). \\
  \Rightarrow z_1 \leq z_2 \leq z_3 \leq z_4
\end{cases}$$
LEMMA 4. Let \((X, d, \leq)\) be a partially ordered metric spaces and \(A, B\) are non empty subsets of \(X\). Assume \(A_0\) is nonempty. A mapping \(F: A \times A \times A \times A \to B\) has the proximal mixed monotone property with \(F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0\) whenever

\[
\begin{align*}
  x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, z_0, z_1, z_2, z_3, w_0, w_1, w_2, w_3 & \in A \\
  x_0 \leq x_1 \leq x_2 \leq x_3; y_0 \geq y_1 \geq y_2 \geq y_3; z_0 \leq z_1 \leq z_2 \leq z_3; w_0 \geq w_1 \geq w_2 \geq w_3 \\
  d(w_1, F(w_0, x_0, y_0, z_0)) &= \text{dist}(A, B) \\
  d(w_2, F(w_1, x_1, y_1, z_1)) &= \text{dist}(A, B) \\
  d(w_3, F(w_2, x_2, y_2, z_2)) &= \text{dist}(A, B) \\
  d(w_4, F(w_3, x_3, y_3, z_3)) &= \text{dist}(A, B) \\
  \Rightarrow w_1 \geq w_2 \geq w_3 \geq w_4
\end{align*}
\]

Proof of lemma 2, 3, 4 are omitted.

Following definition was given by Luong and Thuan [37]

DEFINITION 3. [37] Let \(\phi\) be the class of all functions \(\phi: [0, \infty) \to [0, \infty)\) which satisfy:

1. \(\phi\) is continuous and non decreasing,
2. \(\phi(t) = 0\) if and only if \(t = 0\),
3. \(\phi(t + s) \leq \phi(t) + \phi(s), \forall t, s \in [0, \infty)\).

And let \(\psi\) be the class of all functions \(\psi: [0, \infty) \to [0, \infty)\) which satisfy \(\lim_{t \to r} \psi(t) > 0\) for all \(r > 0\) and \(\lim_{t \to 0^+} \psi(t) = 0\).

Now we give the following definition of proximally quadrupled weak \((\psi, \phi)\) contraction.

DEFINITION 4. Let \((X, d, \leq)\) be a partially ordered metric space and \(A\) and \(B\) are non empty subsets of \(X\). Assume \(A_0\) is non empty. A mapping \(F: A \times A \times A \times A \to B\) is said to be proximally quadrupled weak \((\psi, \phi)\) contraction on \(A\), whenever

\[
\begin{align*}
  x_1 \leq x_2; y_1 \geq y_2; z_1 \leq z_2; w_1 \geq w_2 \\
  d(u, F(x_1, y_1, z_1, w_1)) &= \text{dist}(A, B) \\
  d(v, F(x_2, y_2, z_2, w_2)) &= \text{dist}(A, B)
\end{align*}
\]
\[ \Rightarrow \phi(d(u, v)) \leq \frac{1}{4} \phi(d(x_1, x_2) + d(y_1, y_2) + d(z_1, z_2) + d(w_1, w_2)) - \psi \left( \frac{d(x_1, x_2) + d(y_1, y_2) + d(z_1, z_2) + d(w_1, w_2)}{4} \right) \]

where \( x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2 \in A. \)

If \( A = B \) in the above definition, the notion of proximally quadrupled weak \((\psi, \phi)\) contraction on \( A \) reduces to that of a quadrupled weak \((\psi, \phi)\) contraction.

### 3. MAIN RESULT

Let \((X, d, \leq)\) be a partially ordered complete metric space endowed with the product space \(A \times A \times A \times A\) satisfying the following conditions for \((x, y, z, w), (p, q, r, s) \in A \times A \times A \times A,\)

\[(p, q, r, s) \leq (x, y, z, w) \iff x \geq p, y \leq q, z \geq r, w \leq s.\]

**Theorem 1.** Let \((X, d, \leq)\) be a partially ordered complete metric space. Let \(A, B\) are nonempty closed subsets of the metric space \((X, d)\) such that \(A_0 \neq \phi.\) Let \(F: A \times A \times A \times A \to B\) satisfy the following conditions:

1. \(F\) is continuous proximally quadrupled weak \((\psi, \phi)\) contraction on \(A\) having the proximal mixed monotone property on \(A\) such that \(F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0.\)

2. There exist elements \((x_0, y_0, z_0, w_0)\) and \((x_1, y_1, z_1, w_1)\) in \(A_0 \times A_0 \times A_0 \times A_0\) such that

\[
\begin{align*}
d(x_1, F(x_0, y_0, z_0, w_0)) &= \text{dist}(A, B) \text{ with } x_0 \leq x_1, \\
d(y_1, F(y_0, z_0, w_0, x_0)) &= \text{dist}(A, B) \text{ with } y_0 \geq y_1, \\
d(z_1, F(z_0, w_0, x_0, y_0)) &= \text{dist}(A, B) \text{ with } z_0 \leq z_1 \text{ and} \\
d(w_1, F(w_0, x_0, y_0, z_0)) &= \text{dist}(A, B) \text{ with } w_0 \geq w_1,
\end{align*}
\]
Then there exists \((x, y, z, w)\) in \(A \times A \times A \times A\) such that \(d(x, F(x, y, z, w)) = \text{dist}(A, B)\), \(d(y, F(y, z, w, x)) = \text{dist}(A, B)\), \(d(x, F(z, w, x, y)) = \text{dist}(A, B)\) and \(d(w, F(w, x, y, z)) = \text{dist}(A, B)\).

Proof: By hypothesis, there exists elements \((x_0, y_0, z_0, w_0)\) and \((x_1, y_1, z_1, w_1)\) in \(A_0 \times A_0 \times A_0 \times A_0\) such that

\[
d \left( x_0, F(x_0, y_0, z_0, w_0) \right) = \text{dist}(A, B) \text{ with } x_0 \leq x_1,
\]

\[
d \left( y_0, F(y_0, z_0, w_0, x_0) \right) = \text{dist}(A, B) \text{ with } y_0 \geq y_1,
\]

\[
d \left( z_0, F(z_0, w_0, x_0, y_0) \right) = \text{dist}(A, B) \text{ with } z_0 \leq z_1, \text{ and}
\]

\[
d \left( w_0, F(w_0, x_0, y_0, z_0) \right) = \text{dist}(A, B) \text{ with } w_0 \geq w_1.
\]

Since \((A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0\), then there exists element \((x_2, y_2, z_2, w_2)\) in \(A_0 \times A_0 \times A_0 \times A_0\) such that

\[
d \left( x_2, F(x_1, y_1, z_1, w_1) \right) = \text{dist}(A, B)
\]

\[
d \left( y_2, F(y_1, z_1, w_1, x_1) \right) = \text{dist}(A, B)
\]

\[
d \left( z_2, F(z_1, w_1, x_1, y_1) \right) = \text{dist}(A, B) \text{ and}
\]

\[
d \left( w_2, F(w_1, x_1, y_1, z_1) \right) = \text{dist}(A, B).
\]

And also there exists element \((x_3, y_3, z_3, w_3)\) in \(A_0 \times A_0 \times A_0 \times A_0\)

\[
d \left( x_3, F(x_2, y_2, z_2, w_2) \right) = \text{dist}(A, B)
\]

\[
d \left( y_3, F(y_2, z_2, w_2, x_2) \right) = \text{dist}(A, B)
\]

\[
d \left( z_3, F(z_2, w_2, x_2, y_2) \right) = \text{dist}(A, B)
\]

\[
d \left( w_3, F(w_2, x_2, y_2, z_2) \right) = \text{dist}(A, B).
\]
Thus by lemma 1, 2, 3, 4 we have $x_1 \leq x_2 \leq x_3; \ y_1 \geq y_2 \geq y_3; \ z_1 \leq z_2 \leq z_3$ and $w_1 \geq w_2 \geq w_3. \ 4$ Continuing in this way, we can construct sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ in $A_0$ such that

$$d \left( x_{n+1}, F(x_n, y_n, z_n, w_n) \right) = \text{dist}(A, B), \forall n \in \mathbb{N} \text{ with } x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$$  \hspace{1cm} (8)

$$d \left( y_{n+1}, F(y_n, z_n, w_n, x_n) \right) = \text{dist}(A, B), \forall n \in \mathbb{N} \text{ with } y_0 \geq y_1 \geq \cdots \geq y_n \geq y_{n+1} \geq \cdots$$  \hspace{1cm} (9)

$$d \left( z_{n+1}, F(z_n, w_n, x_n, y_n) \right) = \text{dist}(A, B), \forall n \in \mathbb{N} \text{ with } z_0 \leq z_1 \leq \cdots \leq z_n \leq z_{n+1} \leq \cdots$$  \hspace{1cm} (10)

$$d \left( w_{n+1}, F(w_n, x_n, y_n, z_n) \right) = \text{dist}(A, B), \forall n \in \mathbb{N} \text{ with } w_0 \geq w_1 \geq \cdots \geq w_n \geq w_{n+1} \geq \cdots$$  \hspace{1cm} (11)

Then

$$d \left( x_n, F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \right) = \text{dist}(A, B), \ d \left( x_{n+1}, F(x_n, y_n, z_n, w_n) \right) = \text{dist}(A, B) \text{ and we also have } x_{n-1} \leq x_n, y_{n-1} \geq y_n, z_{n-1} \leq z_n \text{ and } w_{n-1} \geq w_n, \forall n \in \mathbb{N}.$$  

Since $F$ is proximally quadrupled weak $(\psi, \phi)$ contraction on $A$, we have

$$\phi(d(x_n, x_{n+1}))$$

$$\leq \frac{1}{4} \phi(d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n))$$

$$- \psi \left( \frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n)}{4} \right)$$  \hspace{1cm} (12)

Similarly,

$$\phi(d(y_n, y_{n+1}))$$

$$\leq \frac{1}{4} \phi(d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n) + d(x_{n-1}, x_n))$$

$$- \psi \left( \frac{d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n) + d(x_{n-1}, x_n)}{4} \right)$$  \hspace{1cm} (13)

$$\phi(d(z_n, z_{n+1}))$$

$$\leq \frac{1}{4} \phi(d(z_{n-1}, z_n) + d(w_{n-1}, w_n) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n))$$

$$- \psi \left( \frac{d(z_{n-1}, z_n) + d(w_{n-1}, w_n) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{4} \right)$$  \hspace{1cm} (14)

and
\[ \phi(d(w_n, w_{n+1})) \]
\[ \leq \frac{1}{4} \phi(d(w_{n-1}, w_n) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n)) \]
\[ - \psi\left(\frac{d(w_{n-1}, w_n) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n)}{4}\right) \]  
\[ \quad (15) \]

From (12), (13), (14) and (15), we get
\[ \phi(d(x_n, x_{n+1})) + \phi(d(y_n, y_{n+1})) + \phi(d(z_n, z_{n+1})) + \phi(d(w_n, w_{n+1})) \]
\[ \leq \phi(d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n)) \]
\[ - 4\psi\left(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n)}{4}\right) \]  
\[ \quad (16) \]

By property (iii) of \( \phi \), we have
\[ \phi(d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1})) \]
\[ \leq \phi(d(x_n, x_{n+1})) + \phi(d(y_n, y_{n+1})) + \phi(d(z_n, z_{n+1})) + \phi(d(w_n, w_{n+1})) \]  
\[ \quad (17) \]

From (16) and (17), we get
\[ \phi(d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1})) \]
\[ \leq \phi(d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n)) \]
\[ - 4\psi\left(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n)}{4}\right) \]  
\[ \quad (18) \]

Since \( \phi \) is non-decreasing from (18), we get
\[ d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1}) \]
\[ \leq d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n) \]  
\[ \quad (19) \]

Let \( \delta_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1}) \), then the sequence \{\delta_n\} is decreasing. Therefore, there is some \( \delta \geq 0 \) such that
\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \left[ d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1}) \right] = \delta
\]  \hspace{1cm} (20)

Now we have to show that \( \delta = 0 \). On the contrary, let \( \delta > 0 \). Taking limit as \( n \to \infty \) on both sides of (18) and using the fact that \( \phi \) is continuous, we have

\[
\phi(\delta) \leq \lim_{n \to \infty} \phi(\delta_{n-1}) - 4 \lim_{n \to \infty} \psi \left( \frac{\delta_{n-1}}{4} \right)
= \phi(\delta) - 4 \lim_{n \to \infty} \psi \left( \frac{\delta_{n-1}}{4} \right) < \phi(\delta),
\]

Since \( \lim_{t \to r} \psi(t) > 0, \forall r > 0 \), which is a contradiction and hence we conclude that \( \delta = 0 \).

Thus,

\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \left[ d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1}) \right] = 0
\]  \hspace{1cm} (21)

Next we have to prove that \( \{x_n\}, \{y_n\}, \{z_n\} \text{ and } \{w_n\} \) are Cauchy sequences. On the contrary, let us assume that at least one of the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \text{ and } \{w_n\} \) is not a Cauchy sequence. This means that at least one of

\[
\lim_{n,m \to \infty} d(x_n, x_m) = 0, \quad \lim_{n,m \to \infty} d(y_n, y_m) = 0
\]

\[
\lim_{n,m \to \infty} d(z_n, z_m) = 0, \quad \lim_{n,m \to \infty} d(w_n, w_m) = 0
\]

is not true and consequently,

\[
\lim_{n,m \to \infty} \left[ d(x_n, x_m) + d(y_n, y_m) + d(z_n, z_m) + d(w_n, w_m) \right] = 0
\]

is not true.

Then, there exists \( \varepsilon > 0 \) for which we can find subsequences \( \{x_{n_k}\}, \{x_{m_k}\} \text{ of } \{x_n\}; \{y_{n_k}\}, \{y_{m_k}\} \text{ of } \{y_n\}; \{z_{n_k}\}, \{z_{m_k}\} \text{ of } \{z_n\} \text{ and } \{w_{n_k}\}, \{w_{m_k}\} \text{ of } \{w_n\} \) such that \( n_k \) is the smallest index for which \( n_k > m_k > k \),

\[
d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k}) \geq \varepsilon
\]  \hspace{1cm} (22)

This means that

\[
d(x_{n_k-1}, x_{m_k}) + d(y_{n_k-1}, y_{m_k}) + d(z_{n_k-1}, z_{m_k}) + d(w_{n_k-1}, w_{m_k}) < \varepsilon
\]  \hspace{1cm} (23)
Using triangle inequality from (19) and (20), we get
\[
\varepsilon \leq d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})
\]
\[
\leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) + d(y_{n_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{m_k}) + d(z_{n_k}, z_{n_k-1}) + d(z_{n_k-1}, z_{m_k}) + d(w_{n_k}, w_{n_k-1}) + d(w_{n_k-1}, w_{m_k})
\]
\[
\leq d(x_{n_k}, x_{n_k-1}) + d(y_{n_k}, y_{n_k-1}) + d(z_{n_k}, z_{n_k-1}) + d(w_{n_k}, w_{n_k-1}) + \varepsilon
\]

Taking limit as \( k \to \infty \) and using (18), we obtain
\[
\lim_{n,m \to \infty} [d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})] = \varepsilon \quad (24)
\]

By triangle inequality, we have
\[
d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})
\]
\[
\leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k}) + d(y_{n_k}, y_{n_k+1}) + d(y_{n_k+1}, y_{m_k+1}) + d(y_{m_k+1}, y_{m_k}) + d(z_{n_k}, z_{n_k+1}) + d(z_{n_k+1}, z_{m_k+1}) + d(z_{m_k+1}, z_{m_k}) + d(w_{n_k}, w_{n_k+1}) + d(w_{n_k+1}, w_{m_k+1}) + d(w_{m_k+1}, w_{m_k})
\]
\[
= [d(x_{n_k}, x_{n_k+1}) + d(y_{n_k}, y_{n_k+1}) + d(z_{n_k}, z_{n_k+1}) + d(w_{n_k}, w_{n_k+1})] + [d(x_{m_k+1}, x_{m_k}) + d(y_{m_k+1}, y_{m_k}) + d(z_{m_k+1}, z_{m_k}) + d(w_{m_k+1}, w_{m_k})] + [d(x_{n_k+1}, x_{m_k+1}) + d(y_{n_k+1}, y_{m_k+1}) + d(z_{n_k+1}, z_{m_k+1}) + d(w_{n_k+1}, w_{m_k+1})]
\]
\[
(25)
\]

By property of \( \phi \), we obtain
\[
\phi(y_k) \leq \phi(\delta_{n_k}) + \phi(\delta_{m_k}) + \phi(d(x_{n_k+1}, x_{m_k+1})) + \phi(d(y_{n_k+1}, y_{m_k+1})) + \phi(d(z_{n_k+1}, z_{m_k+1})) + \phi(d(w_{n_k+1}, w_{m_k+1}))
\]
\[
(26)
\]
Where \( y_k = d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k}) \)

\[
\delta_{n_k} = d(x_{n_k}, x_{n_k+1}) + d(y_{n_k}, y_{n_k+1}) + d(z_{n_k}, z_{n_k+1}) + d(w_{n_k}, w_{n_k+1})
\]

\[
\delta_{m_k} = d(x_{m_k+1}, x_{m_k}) + d(y_{m_k+1}, y_{m_k}) + d(z_{m_k+1}, z_{m_k}) + d(w_{m_k+1}, w_{m_k})
\]

Since \( x_{n_k} \geq x_{m_k}, \ y_{n_k} \leq y_{m_k}, \ z_{n_k} \geq z_{m_k} \) and \( w_{n_k} \leq w_{m_k} \), using the fact that \( F \) is a proximally quadrupled weak \((\psi, \phi)\) contraction on \( A \), we get

\[
\phi\left(d(x_{n_k+1}, x_{m_k+1})\right) 
\leq \frac{1}{4} \phi\left(d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})\right) 
- \psi\left(\frac{d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})}{4}\right) 
= \frac{1}{4} \phi(y_k) - \psi\left(\frac{y_k}{4}\right) \tag{27}
\]

Similarly,

\[
\phi\left(d(y_{n_k+1}, y_{m_k+1})\right) 
\leq \frac{1}{4} \phi\left(d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k}) + d(x_{n_k}, x_{m_k})\right) 
- \psi\left(\frac{d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})}{4}\right) 
= \frac{1}{4} \phi(y_k) - \psi\left(\frac{y_k}{4}\right) \tag{28}
\]

\[
\phi\left(d(z_{n_k+1}, z_{m_k+1})\right) 
\leq \frac{1}{4} \phi\left(d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k}) + d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k})\right) 
- \psi\left(\frac{d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})}{4}\right) 
= \frac{1}{4} \phi(y_k) - \psi\left(\frac{y_k}{4}\right) \tag{29}
\]
and
\[
\phi \left( d(w_{n_{k+1}}, w_{m_{k+1}}) \right) \\
\leq \frac{1}{4} \phi \left( d(w_{n_{k}}, w_{m_{k}}) + d(x_{n_{k}}, x_{m_{k}}) + d(y_{n_{k}}, y_{m_{k}}) + d(z_{n_{k}}, z_{m_{k}}) \right) \\
- \psi \left( \frac{d(x_{n_{k}}, x_{m_{k}}) + d(y_{n_{k}}, y_{m_{k}}) + d(z_{n_{k}}, z_{m_{k}}) + d(w_{n_{k}}, w_{m_{k}})}{4} \right) \\
= \frac{1}{4} \phi(y_{k}) - \psi \left( \frac{y_{k}}{4} \right) \tag{30}
\]

Using (27), (28), (29) and (30), we get
\[
\phi(y_{k}) \leq \phi(\delta_{n_{k}}) + \phi(\delta_{m_{k}}) + \phi(y_{k}) - 4\psi \left( \frac{y_{k}}{4} \right) \tag{31}
\]
Letting \( k \to \infty \) and using (21), (24) and (31), we have
\[
\phi(\varepsilon) \leq \phi(0) + \phi(0) + \phi(\varepsilon) - 4\psi \left( \frac{\varepsilon}{4} \right) < \phi(\varepsilon),
\]
which is a contradiction and hence we can conclude that \( \{x_n\}, \{y_n\}, \{z_n\} \) and \( \{w_n\} \) are Cauchy sequences. Since \( A \) is a closed subset of a complete metric space \( X \), these sequences have limits. Hence, there exists \( x, y, z, w \in A \) such that \( x_n \to x, y_n \to y, z_n \to z \) and \( w_n \to w \). Then \( (x_n, y_n, z_n, w_n) \to (x, y, z, w) \) in \( A \times A \times A \times A \).

Since \( F \) is continuous, we have that \( F(x_n, y_n, z_n, w_n) \to F(x, y, z, w) \), \( F(y_n, z_n, w_n, x_n) \to F(y, z, w, x) \), \( F(z_n, w_n, x_n, y_n) \to F(z, w, x, y) \) and \( F(w_n, x_n, y_n, z_n) \to F(w, x, y, z) \).

But from (8), (9), (10) and (11), we know that the sequences \( \{d(x_{n+1}, F(x_n, y_n, z_n, w_n))\}, \{d(y_{n+1}, F(y_n, z_n, w_n, x_n))\}, \{d(z_{n+1}, F(z_n, w_n, x_n, y_n))\} \) and \( \{d(w_{n+1}, F(w_n, x_n, y_n, z_n))\} \) are constant sequences with the value \( \text{dist}(A,B) \). Therefore, \( d(x, F(x, y, z, w)) = \text{dist}(A,B) \), \( d(y, F(y, z, w, x)) = \text{dist}(A,B) \), \( d(z, F(z, w, x, y)) = \text{dist}(A,B) \) and \( d(w, F(w, x, y, z)) = \text{dist}(A,B) \). This completes our proof.

**Corollary 1.** Let \((X, d, \preceq)\) be a partially ordered complete metric space. Let \( A \) be a non empty
closed subset of the metric space \((X,d)\). Let \(F: A \times A \times A \times A \to A\) satisfy the following conditions:

1. \(F\) is continuous having proximal mixed monotone property and proximally quadrupled weak \((\psi, \phi)\) contraction on \(A\).

2. There exists \((x_0, y_0, z_0, w_0)\) and \((x_1, y_1, z_1, w_1)\) in \(A \times A \times A \times A\) such that \(x_1 = F(x_0, y_0, z_0, w_0)\) with \(x_0 \leq x_1\), \(y_1 = F(y_0, z_0, w_0, x_0)\) with \(y_0 \geq y_1\), \(z_1 = F(z_0, w_0, x_0, y_0)\) with \(z_0 \leq z_1\) and \(w_1 = F(w_0, x_0, y_0, z_0)\) with \(w_0 \geq w_1\).

Then there exists \((x, y, z, w) \in A \times A \times A \times A\) such that \(d(x, F(x, y, z, w)) = 0\),

\[d(y, F(y, z, w, x)) = 0, \quad d(z, F(z, w, x, y)) = 0 \quad \text{and} \quad d(w, F(w, x, y, z)) = 0.\]

We also note that theorem 1 is still valid for \(F\) not necessarily continuous if \(A\) has the following property that

\[
\{x_n\} \quad \text{is a non decreasing sequence in } A \text{ such that } x_n \to x, \quad x_n \leq x, \quad (32)
\]

\[
\{y_n\} \quad \text{is a non decreasing sequence in } A \text{ such that } y_n \to y, \quad y_n \geq y, \quad (33)
\]

\[
\{z_n\} \quad \text{is a non decreasing sequence in } A \text{ such that } z_n \to z, \quad z_n \leq z, \quad (34)
\]

and

\[
\{w_n\} \quad \text{is a non decreasing sequence in } A \text{ such that } w_n \to w, \quad w_n \geq w, \quad (35)
\]

**Theorem 2.** Assume the conditions (32), (33), (34) and (35) and \(A_0\) is closed in \(X\) instead of continuity of \(F\) in Theorem 1, then the conclusion of Theorem 1 holds.

Proof. Proceeding similar to Theorem 1, we claim that there exists sequences \(\{x_n\}, \{y_n\}, \{z_n\}\) and \(\{w_n\}\) in \(A\) satisfying the following conditions:

\[
\begin{align*}
&d(x_{n+1}, F(x_n, y_n, z_n, w_n)) = \text{dist}(A, B) \quad \text{with } x_n \leq x_{n+1}, \forall \ n \in \mathbb{N}, \quad (36) \\
&d(y_{n+1}, F(y_n, z_n, w_n, x_n)) = \text{dist}(A, B) \quad \text{with } y_n \geq y_{n+1}, \forall \ n \in \mathbb{N}, \quad (37) \\
&d(z_{n+1}, F(z_n, w_n, x_n, y_n)) = \text{dist}(A, B) \quad \text{with } z_n \leq z_{n+1}, \forall \ n \in \mathbb{N}, \quad (38) \\
&d(w_{n+1}, F(w_n, x_n, y_n, z_n)) = \text{dist}(A, B) \quad \text{with } w_n \leq w_{n+1}, \forall \ n \in \mathbb{N}, \quad (39)
\end{align*}
\]
Moreover, \( x_n \to x, y_n \to y, z_n \to z \) and \( w_n \to w \). From (32), (33), (34) and (35), we get \( x_n \leq x, y_n \geq y, z_n \leq z \) and \( w_n \geq w \) respectively. Note that the sequence \( \{x_n\}, \{y_n\}, \{z_n\} \) and \( \{w_n\} \) are in \( A_0 \) and \( A_0 \) is closed. Therefore, \( (x, y, z, w) \in A_0 \times A_0 \times A_0 \times A_0 \). Since \( F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0 \), there exists \( F(x, y, z, w), F(y, z, w, x), F(z, w, x, y) \) and \( F(w, x, y, z) \) in \( B_0 \). Therefore, there exists \( (x^*, y^*, z^*, w^*) \in A_0 \times A_0 \times A_0 \times A_0 \) such that

\[
d(x^*, F(x, y, z, w)) = \text{dist}(A, B), d(y^*, F(y, z, w, x)) = \text{dist}(A, B), d(z^*, F(z, w, x, y)) = \text{dist}(A, B) \text{ and } d(w^*, F(w, x, y, z)) = \text{dist}(A, B).
\]

Since \( x_n \leq x, y_n \geq y, z_n \leq z \) and \( w_n \geq w \), by using the fact that \( F \) is a proximally quadrupled weak \( (\psi, \phi) \) contraction on \( A \) and using (36), (37), (38) and (39), we get

\[
\phi\left(d(x_{n+1}, x^*)\right) \\
\leq \frac{1}{4} \phi\left(d(x_n, x) + d(y_n, y) + d(z_n, z) + d(w_n, w)\right) \\
- \psi\left(\frac{d(x_n, x) + d(y_n, y) + d(z_n, z) + d(w_n, w)}{4}\right), \text{ for all } n
\]

Similarly,

\[
\phi\left(d(y_{n+1}, y^*)\right) \\
\leq \frac{1}{4} \phi\left(d(y_n, y) + d(z_n, z) + d(w_n, w) + d(x_n, x)\right) \\
- \psi\left(\frac{d(y_n, y) + d(z_n, z) + d(w_n, w) + d(x_n, x)}{4}\right), \text{ for all } n
\]

\[
\phi\left(d(z_{n+1}, z^*)\right) \\
\leq \frac{1}{4} \phi\left(d(z_n, z) + d(w_n, w) + d(x_n, x) + d(y_n, y)\right) \\
- \psi\left(\frac{d(z_n, z) + d(w_n, w) + d(x_n, x) + d(y_n, y)}{4}\right), \text{ for all } n
\]

and
\[
\phi(d(w_n, w^*)) \leq \frac{1}{4} \phi(d(w_n, w) + d(x_n, x) + d(y_n, y) + d(z_n, z)) - \psi\left(\frac{d(w_n, w) + d(x_n, x) + d(y_n, y) + d(z_n, z)}{4}\right)
\]

Since \( x_n \to x, y_n \to y, z_n \to z \) and \( w_n \to w \) by taking the limit on the above inequalities, we get \( x = x^*, y = y^*, z = z^* \) and \( w = w^* \). Hence, we get that \( d(x, F(x, y, z, w)) = \) dist\((A, B), d(y, F(y, z, w, x)) = \) dist\((A, B), d(z, F(z, w, x, y)) = \) dist\((A, B) \) and \( d(w, F(w, x, y, z)) = \) dist\((A, B) \). This complete the proof.

**EXAMPLE 1.** Let \( X=\{(0,0,0,1), (1,0,0,0), (-1,0,0,0), (0,0,0,-1)\} \subseteq R^4 \) and consider the usual order

\[
(x, y, z, w) \leq (p, q, r, d) \iff x \leq p, y \leq q, z \leq r, w \leq s.
\]

Thus, \((X, \leq)\) is a partially ordered set. Besides, \((X, d)\) is a complete metric space. Considering \( d \) the Euclidean metric. Let \( A = \{(0,0,0,1), (1,0,0,0)\} \) and \( B = \{(0,0,0,-1), (-1,0,0,0)\} \) be closed subsets of \( X \).

Then \( \text{dist}(A, B) = \sqrt{2} \), \( A = A_0 \) and \( B = B_0 \). Let \( F: A \times A \times A \times A \to B \) be defined as

\[
F((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4), (w_1, w_2, w_3, w_4)) = (-x_4, -x_3, -x_2, -x_1).
\]

Then, one can see that \( F \) is continuous such that \( F(A_0 \times A_0 \times A_0) \subseteq B_0 \). Only comparable pairs of points in \( A \) are \( x \leq x \) for \( x \in A \). Hence the proximal mixed monotone property and the proximally quadrupled weak \((\psi, \phi)\) contraction on \( A \) are obviously satisfied. However, \( F \) has many quadrupled best proximity points such as \((0,0,0,1), (0,0,0,1), (0,0,0,1), (0,0,0,1), (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,0,0,0), (1,0,0,0), (1,0,0,0), (1,0,0,0) \) etc. Hence not unique.

However, we can prove that the quadrupled best proximity points is in fact unique, provided that the product space \( A \times A \times A \times A \) endowed with the partial ordered mentioned above has the following property:
Every pair of elements has either a lower bound or an upper bound.

This condition is equivalent to the following statement.

For every pair of \((x, y, z, w), (x^*, y^*, z^*, w^*)\) ∈ \(A \times A \times A\), there exists

\((a, b, c, d) \in A \times A \times A \times A\) which is comparable to \((x, y, z, w)\) and \((x^*, y^*, z^*, w^*)\).

**Theorem 3.** In addition to the hypothesis of Theorem 1 (resp. Theorem 2), suppose that for any two elements \((x, y, z, w)\) and \((x^*, y^*, z^*, w^*)\) in \(A_0 \times A_0 \times A_0 \times A_0\) there exists \((z_1, z_2, z_3, z_4) \in A_0 \times A_0 \times A_0 \times A_0\) such that \((z_1, z_2, z_3, z_4)\) is comparable to \((x, y, z, w)\) and \((x^*, y^*, z^*, w^*)\), then \(F\) has a unique quadrupled best proximity point.

**Proof:** From Theorem 1 (resp. Theorem 2), the set of quadrupled best proximity points of \(F\) is non empty. Suppose that there exists \((x, y, z, w)\) and \((x^*, y^*, z^*, w^*)\) in \(A \times A \times A \times A\) which are quadrupled best proximity points. That is

\[
d(x, F(x, y, z, w)) = dist(A, B)
\]

\[
d(y, F(y, z, w, x)) = dist(A, B)
\]

\[
d(z, F(z, w, x, y)) = dist(A, B)
\]

\[
d(w, F(w, x, y, z)) = dist(A, B)
\]

and

\[
d(x^*, F(x^*, y^*, z^*, w^*)) = dist(A, B)
\]

\[
d(y^*, F(y^*, z^*, w^*, x^*)) = dist(A, B)
\]

\[
d(z^*, F(z^*, w^*, x^*, y^*)) = dist(A, B)
\]

\[
d(w^*, F(w^*, x^*, y^*, z^*)) = dist(A, B)
\]

We consider two cases:

Case I: Suppose \((x, y, z, w)\) is comparable. Let \((x, y, z, w)\) is comparable to \((x^*, y^*, z^*, w^*)\) with respect to the ordering \(A \times A \times A \times A\). Using the fact that \(F\) is a proximally quadrupled weak \((\psi, \phi)\) contraction on \(A\) to

\[
d(x, F(x, y, z, w)) = dist(A, B)
\]
and \((x^*, F(x^*, y^*, z^*, w^*)) = \text{dist}(A, B)\), we get

\[
\phi(d(x, x^*)) \leq \frac{1}{4} \phi(d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*)) \\
- \psi \left( \frac{d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*)}{4} \right)
\] (40)

Similarly, we get

\[
\phi(d(y, y^*)) \leq \frac{1}{4} \phi(d(y, y^*) + d(z, z^*) + d(w, w^*) + d(x, x^*)) \\
- \psi \left( \frac{d(y, y^*) + d(z, z^*) + d(w, w^*) + d(x, x^*)}{4} \right)
\] (41)

Also,

\[
\phi(d(z, z^*)) \leq \frac{1}{4} \phi(d(z, z^*) + d(w, w^*) + d(x, x^*) + d(y, y^*)) \\
- \psi \left( \frac{d(z, z^*) + d(w, w^*) + d(x, x^*) + d(y, y^*)}{4} \right)
\] (42)

And

\[
\phi(d(w, w^*)) \leq \frac{1}{4} \phi(d(w, w^*) + d(x, x^*) + d(y, y^*) + d(z, z^*)) \\
- \psi \left( \frac{d(w, w^*) + d(x, x^*) + d(y, y^*) + d(z, z^*)}{4} \right)
\] (43)

Adding (40), (41), (42) and (43), we get

\[
\phi(d(x, x^*)) + \phi(d(y, y^*)) + \phi(d(z, z^*)) + \phi(d(w, w^*)) \\
\leq \phi(d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*)) \\
- 4\psi \left( \frac{\phi(d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*)}{4} \right)
\] (44)

By the property (iii) of \(\phi\), we obtain

\[
\phi(d(x, x^*)) + \phi(d(y, y^*)) + \phi(d(z, z^*)) + \phi(d(w, w^*)) \\
\leq \phi(d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*)) \\
\] (45)
From (44) and (45), we get
\[
\phi(d(x, x^*)) + \phi(d(y, y^*)) + \phi(d(z, z^*)) + \phi(d(w, w^*)) \\
\leq \phi(d(x, x^*), d(y, y^*), d(z, z^*), d(w, w^*)) \\
- 4\psi\left(\frac{\phi(d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*))}{4}\right)
\]
(46)

This implies that
\[
4\psi\left(\frac{d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*)}{4}\right) \leq 0. \text{ Using the property of } \psi, \text{ we get}
\]
d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*) = 0. \text{ Hence } d(x, x^*) = d(y, y^*) = d(z, z^*) = d(w, w^*) = 0. \text{ So, } x = x^*, y = y^*, z = z^* \text{ and } w = w^*.

Case II: Suppose (x, y, z, w) is not comparable. Let (x, y, z, w) be not comparable to (x^*, y^*, z^*, w^*), then there exists (p_1, q_1, r_1, s_1) \in A_0 \times A_0 \times A_0 \times A_0 \text{ which is comparable to (x, y, z, w) and (x^*, y^*, z^*, w^*) . Since } F(A_0 \times A_0 \times A_0) \subseteq B_0, \text{ there exists}
\]
(p_2, q_2, r_2, s_2) \in A_0 \times A_0 \times A_0 \times A_0 \text{ such that}
\[
d(p_2, F(p_1, q_1, r_1, s_1)) = \text{dist}(A, B) \\
d(q_2, F(q_1, r_1, s_1, p_1)) = \text{dist}(A, B) \\
d(r_2, F(r_1, s_1, p_1, q_1)) = \text{dist}(A, B) \\
d(s_2, F(s_1, p_1, q_1, r_1)) = \text{dist}(A, B)
\]
Without loss of generality, assume that (p_1, q_1, r_1, s_1) \leq (x, y, z, w) \text{ (i.e } x \geq p_1, y \leq q_1, z \geq r_1, w \leq s_1) \text{ . Note that } (p_1, q_1, r_1, s_1) \leq (x, y, z, w) \text{ implies that } (y, z, w, x) \leq (q_1, r_1, s_1, p_1).

From Lemma 1, 2, 3 and 4, we get
\[
p_1 \leq x, q_1 \geq y, r_1 \leq z \text{ and } s_1 \geq w \\
d(p_2, F(p_1, q_1, r_1, s_1)) = \text{dist}(A, B), d(x, F(x, y, z, w)) = \text{dist}(A, B) \text{ implies that } p_2 \leq x.
\]
p_1 \leq x \text{ and } q_1 \geq y
\[ d(q_2, F(q_1, r_1, s_1, p_1)) = \text{dist}(A, B), \quad d(y, F(y, z, w, x)) = \text{dist}(A, B) \]

implies that \( q_2 \geq y \), \( q_1 \geq y \) and \( r_1 \leq z \)

\[ d(r_2, F(r_1, s_1, p_1, q_1)) = \text{dist}(A, B), \quad d(z, F(z, w, x, y)) = \text{dist}(A, B) \]

implies that \( r_2 \leq z \), \( r_1 \leq z \) and \( s_1 \geq w \)

\[ d(s_2, F(s_1, p_1, q_1, r_1)) = \text{dist}(A, B), \quad d(w, F(w, x, y, z)) = \text{dist}(A, B) \]

implies that \( s_2 \geq w \).

From the above four inequalities, we obtain \((p_2, q_2, r_2, s_2) \leq (x, y, z, w)\). Continuing this process, we get sequences \( \{p_n\}, \{q_n\}, \{r_n\} \) and \( \{s_n\} \) such that

\[ d(p_{n+1}, F(p_n, q_n, r_n, s_n)) = \text{dist}(A, B) \]

\[ d(q_{n+1}, F(q_n, r_n, s_n, p_n)) = \text{dist}(A, B) \]

\[ d(r_{n+1}, F(r_n, s_n, p_n, q_n)) = \text{dist}(A, B) \]

\[ d(s_{n+1}, F(s_n, p_n, q_n, r_n)) = \text{dist}(A, B) \]

with \((p_n, q_n, r_n, s_n) \leq (x, y, z, w), \forall n \in N\). By using the fact that \( F \) is a proximally quadrupled weak \((\psi, \phi)\) contraction on \( A \), we get \( p_n \leq x, \; q_n \geq y, \; r_n \leq z \) and \( s_n \geq w \).

\[ d(p_{n+1}, F(p_n, q_n, r_n, s_n)) = d(A, B), \quad d(x, F(x, y, z, w)) = d(A, B) \]

implies that

\[ \phi(d(p_{n+1}, x) \]

\[ \leq \frac{1}{4} \phi(d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w)) \]

\[ - \psi \left( \frac{d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w)}{4} \right) \]  

(47)

Similarly, we can have

\[ y \leq q_n, \; z \geq r_n, \; w \leq s_n \] and \( x \geq p_n \).

\[ d(q_{n+1}, F(q_n, r_n, s_n, p_n)) = \text{dist}(A, B), \quad d(y, F(y, z, w, x)) = \text{dist}(A, B) \]

implies that
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\[
\phi(d(q_{n+1}, y) \leq \frac{1}{4} \phi(d(q_n, y) + d(r_n, z) + d(s_n, w) + d(p_n, x)) - \psi \left( \frac{d(q_n, y) + d(r_n, z) + d(s_n, w) + d(p_n, x)}{4} \right) \tag{48}
\]

\[r_n \leq z, \ s_n \geq w, \ p_n \leq x \text{ and } q_n \geq y.\]

\[d(r_{n+1}, F(r_n, s_n, p_n, q_n)) = \text{dist}(A, B), d(z, F(z, w, x, y)) = \text{dist}(A, B) \text{ implies that}
\]

\[
\phi(d(z, r_{n+1}) \leq \frac{1}{4} \phi(d(z, r_n) + d(w, s_n) + d(x, p_n) + d(y, q_n)) - \psi \left( \frac{\phi(d(z, r_n) + d(w, s_n) + d(x, p_n) + d(y, q_n))}{4} \right) \tag{49}
\]

\[\text{and } w \leq s_n, \ x \geq p_n, \ y \leq q_n \text{ and } z \geq r_n.
\]

\[d(s_{n+1}, F(s_n, p_n, q_n, r_n)) = \text{dist}(A, B), d(w, F(w, x, y, z)) = \text{dist}(A, B) \text{ implies that}
\]

\[
\phi(d(w, s_{n+1}) \leq \frac{1}{4} \phi(d(w, s_n) + d(x, p_n) + d(y, q_n) + d(z, r_n)) - \psi \left( \frac{\phi(d(w, s_n) + d(x, p_n) + d(y, q_n) + d(z, r_n))}{4} \right) \tag{50}
\]

Adding (47), (48), (49) and (50), we obtain

\[
\phi(d(p_{n+1}, x) + \phi(d(q_{n+1}, y) + \phi(d(r_{n+1}, z) + \phi(d(s_{n+1}, w))
\leq \phi(d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w))
- 4\psi \left( \frac{d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w)}{4} \right) \tag{51}
\]

By the property (iii) of \(\phi\), we get

\[
\phi(d(p_{n+1}, x) + d(q_{n+1}, y) + d(r_{n+1}, z) + d(s_{n+1}, w))
\leq \phi(d(p_{n+1}, x)) + \phi(d(q_{n+1}, y)) + \phi(d(r_{n+1}, z)) + \phi(d(s_{n+1}, w)) \tag{52}
\]

From (51) and (52), we obtain
\[ \phi(d(p_{n+1}, x)) + \phi(d(q_{n+1}, y)) + \phi(d(r_{n+1}, z)) + \phi(d(s_{n+1}, w)) \]
\[ \leq \phi(d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w)) \]
\[ - 4\psi\left(\frac{d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w)}{4}\right) \] (53)

This implies that
\[ \phi(d(p_{n+1}, x)) + \phi(d(q_{n+1}, y)) + \phi(d(r_{n+1}, z)) + \phi(d(s_{n+1}, w)) \]
\[ \leq \phi(d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w)) \] (54)

Using the fact that \( \phi \) is non-decreasing, we get
\[ d(p_{n+1}, x) + d(q_{n+1}, y) + d(r_{n+1}, z) + d(s_{n+1}, w) \]
\[ \leq d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w) \] (55)

This means that the sequence \( \{d(p_{n+1}, x) + d(q_{n+1}, y) + d(r_{n+1}, z) + d(s_{n+1}, w)\} \) is decreasing. Therefore, there exists \( \alpha \geq 0 \) such that
\[ \lim_{n \to \infty} [d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w)] = \alpha \] (56)

We show that \( \alpha = 0 \). Suppose, to the contrary, that \( \alpha > 0 \). Taking the limit as \( n \to \infty \) in (53), we have that
\[ \phi(\alpha) \leq \phi(\alpha) - 4 \lim_{n \to \infty} \psi\left(\frac{d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w)}{4}\right) < \phi(\alpha). \]

This is a contradiction. Thus \( \alpha = 0 \), that is
\[ \lim_{n \to \infty} [d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w)] = 0 \] (57)

So we have that
\[ p_n \to x, q_n \to y, r_n \to z \text{ and } s_n \to w. \] Analogously, we can prove that \( p_n \to x^*, q_n \to y^*, r_n \to z^* \text{ and } s_n \to w^* \). Therefore, \( x = x^*, y = y^*, z = z^* \text{ and } w = w^* \).

**Conflict of Interests**

The authors declare that there is no conflict of interests.
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