Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 4, 906-921
https://doi.org/10.28919/jmcs/4387
ISSN: 1927-5307

A THREE-PHASE SIMPLEX METHOD FOR INFEASIBLE AND UNBOUNDED LINEAR PROGRAMMING PROBLEMS

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Abstract. The paper presents a modified artificial basis method MODART, which combine a big-M method with two-phase method. Unlike previous works, the sum of artificial variables in the objective function has not been used, but an additional constraint has been composed for this sum. In contrast with the classical implementation of the simplex method, in the Phase-0, we take into account the objective function of the initial problem and use the big-M method idea for the enlarged problem. If the solution found is infeasible for the initial problem, then Phase-1 finds the feasible solution and Phase-2 the optimal solution. In this work, unbounded, infeasible and degenerate problems are mainly considered. Finally, in part five, suggested formulation and solution to the problems is given together with some computational experience. The main ideas are explained by simple examples. Over 60 years of age problem with M has been finally solved in present paper.

Keywords: M-method; artificial basis; degenerate and infeasible linear programming problems.

2010 AMS Subject Classification: 90C05, 65K05.

1. Introduction

Linear programming (LP) is a part of optimization techniques for solving constrained optimization problems. LP is used every day in the organization and allocation of resources, for

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Received November 23, 2019

minimizing transportations cost, for maximizing profit of operations or improving one aspect of product quality. These problems are characterized by the large number of solutions that satisfy the conditions of each problem. The selection of a particular solution as the optimal solution depends on the statement of the problem.

Many algorithms for linearly constrained optimization problems maintain sets of basic variables. The textbook LP algorithms do not pay much attention to determining the starting basis. They usually create a Phase-I problem, consisting artificial variables. The objective at this phase is to drive all artificial variables to zero and make them non-basic in order to obtain feasibility in terms of the original variables of the problem. Unfortunately, the process of finding a feasible solution has the same complexity bound as the linear programming problem does. All the more, the main point is that a basis found at Phase-I is rather unrelated to the original objective function and therefore usually a large number of iterations are needed to solve the problem. When the big-M method is used, the issue of determining the order of magnitude that M has, is problematic. If M is chosen 1000 times larger than the largest coefficient in the original objective function, then roundoff errors and other computational difficulties arise, see [1]. For this reason, most computer codes solve LPs using the two-phase simplex method.

Various papers have tried to tackle the problem of avoiding the use of big M. For example, general Phase-I method in which the sum of infeasibilities is reduced without regard to the feasibility of individual variables has been suggested by Maros [1, 2, 3]. As result, the number of extreme points to visit is reduced. In order to improve pivoting algorithms, Paparrizos [2] proposed an exterior point simplex algorithm that avoids the feasible region. H.V.Junior and M.P.E.Lins [4] presented a new approach to the problem of improving the pivot algorithms. They suggested as initial basis a vertex of the feasible region that is much closer to the optimal vertex than the initial solution adopted by Phase-I. G.Dantzig and M.Thapa [5] solved least squares subproblems and guarantees strict improvement on degenerate problems at each step. A similar algorithm based on the least squares method was described by S. Kong [6] and E. Ubi [7].

The classical setting up of the LP problem is challenged by E. Ubi in his work [8], where he replaced the objective function with the inequality of this function's guaranteed optimal value.

With this, the LP problem becomes to the system of inequalities $Ax \leq b$, in which the solution of minimum norm could be found in two ways. Firstly, using the least squares method. Secondly, the system of inequalities can be reduced to the linear complementary problem $-AA^T y + s = b$, where $x = -A^T y, ys = 0, y, s \geq 0$.

In the current work, a modified artificial basis method MODART is described. This method combines a big-M method with a two-phase method. The main emphasis is on solving infeasible (unbounded) problems, where for any M the enlarged problem is infeasible (unbounded.) In contrast to the classical two-phase approach, while solving with our version of the simplex method, the objective function of the initial problem is at once taken into account. If the solution that we obtain in Phase-0 will turn out to be feasible, then we have arrived to the final solution, see examples 3.1 and 5.1 for $b_1 = 32$. In the other case, the Phase-I commenses (see Example 4.1), wherein we will use the obtained basis solution as the point of departure and determine the feasible solution of the initial problem. Phase-II follows the classical version, thus stemming from the feasible basis solution obtained and calculating the optimal solution.

Firstly, in the current article we describe a new method for finding a non-negative solution to the underdetermined system of linear equations. An additional constraint for the sum of artificial variables is added. That constraint is used also in linear programming, see the description of algorithm MODART in the third part. In the fourth part infeasible, unbounded and cycling examples are solved. In the fifth part two examples are presented.

2. NONNEGATIVE SOLUTION TO THE UNDERDETERMINED SYSTEM

Let us consider finding a nonnegative solution to the system

(1)
$$Ax = b, b \neq 0$$
$$x \geq 0,$$

where A is $m \times n$ matrix, b is $m-$vector, c and x are $n-$vectors, $m \leq n$. We would first learn how to solve problem (1) and then worry about minimizing over the set of solutions. The need to find solutions to such problems arise in both linear as well as quadratic programming.

Problem (1) may be solved also using least-squares method, see [5]. Their algorithm solves least-squares subproblems and guarantees strict improvement at each step for non-generate problems.

A feasible solution may be found by solving a LP problem

(2)
$$-t \to min$$
$$v_1 + ... + v_m + t = 1$$
$$Ax + v - tb = 0$$
$$t, v, x \geq 0,$$

where v is a vector of artificial variables. If and only if the optimal solution contains $t_{max} = 1$, then the problem (1) has a solution. If $t_{max} < 1$, then the system (1) does not have a solution, see Example 4.1 and 5.1. Variable t can also be used for characterizing the feasibility of the constraints of LP problems.

3. A Modified Artificial Basis Method

Let us compose a LP problem

(3)
$$z = (c, x) \to min$$
$$Ax = b$$
$$x \geq 0,$$

where A is $m \times n$ matrix, $c, x \in R^n, b \in R^m$. We assume that the right hand side $b \neq 0$, $b \geq 0$ and matrix A is of full row rank. In example 4.3, lets consider the case $b = 0$. In the same vein with the previous part we compose an enlarged problem

(4)
$$z_1 = (c, x) - Mt \to min$$
$$v_1 + ... + v_m + Mt = M$$
$$Ax + v - tb = 0$$

$$t, v, x \geq 0,$$

In the Phase-0 we solve the problem for fixed $M > 0$. If in the optimal solution variable $t = 1$, then the initial problem (3) is solved. If $t_{max} < 1$, then the Phase-I commenses, wherein we continue with the basis solution obtained, and solve problem (5).

(5)
$$z_2 = -t \to min$$

$$v_1 + ... + v_m + Mt = M$$

$$Ax + v - tb = 0$$

$$t, v, x \geq 0.$$

.

If the maximum of the problem (5) is less than 1, then we have arrived at the infeasibility of the initial problem (3), see Example 4.1. In the other case, Phase-II commenses and we remove the artificial variables that are zero and continue with the problem (4).

Algorithm MODART

0. Initiate M, find basic solution $t = M/(M + \sum_{i=1}^{i=m} b_i), v_i = tb_i, i = 1, ..., m$;

1. Transform $z1-$ and $t-$rows;
2. If $t - row \geq 0$ and $t < 1$ then stop: problem is infeasible;
3. Solve LP problem (4) by using the simplex method;
4. If t=1 then
5. Remove the columns that correspond to artificial variables;
6. If z_1 is unbounded, then stop: z is unbounded;
7. If $z_1 - row \leq 0$, then stop: problem is solved;
8. End if;
9. If $t < 1$, then
10. solve LP problem (5) by using the simplex method;
11. If $t_{max} < 1$, then stop: problem (3) is infeasible;
12. End if;

13. Remove the columns that correspond to artificial variables;

14. Goto step 3;

Problem is solved.

Remark 3.1. If artificial variables are eliminated from the first constraint of problem (4) and $t - row \geq 0$, then problem is infeasible, see Example 4.1.

Remark 3.2. The Phase-I starts at step 9. If at step 10 we have $t = 1$, then Phase-II commenses.

Example 3.1.

$$z = x_1 - 2x_2 \rightarrow min$$

$$x_1 + x_2 \geq 2$$

$$-x_1 + x_2 \geq 1$$

$$x_2 \leq 3$$

$$x \geq 0.$$

This problem has been solved with the two-phase simplex method that is put forth in textbook [10], where Phase-I and Phase-II both consist of three steps. A feasible solution at Phase-I is $x_1 = 1/2, x_2 = 3/2$ (steps 1,2,3,). Phase-II moved to the feasible point (0,2) and finally to the optimal point (0,3).The table 1 below puts forth the solution of the problem by utilizing the three-phase simplex method- by taking only four steps. Introducing the slack variables x_3, x_4 and x_5, artificial variables v_1 and v_2, we get the following problem

$$z_1 = x_1 - 2x_2 - Mt \rightarrow min$$

$$v_1 + v_2 + Mt = M$$

$$x_1 + x_2 - x_3 + v_1 - 2t = 0$$

$$-x_1 + x_2 - x_4 + v_2 - t = 0$$

$$x_2 + x_5 - 3t = 0$$

$$t, v, x \geq 0.$$

The basic variable $t = 0.25$ corresponds to the first constraint. The results of the calculations are given in the following table. The solution found at the Phase-0 is optimal..

$x1$	x2	x3	x4	x5	v1	v2	t
0	0	0	0	0,75	0,5	0,25	0,25
0	0,5	0	0	1	0,5	0	0,5
0,5	1,5	0	0	1,5	0	0	1
0	2	0	1	1	0	0	1
0	3	1	2	0	0	0	1

Table 1. Computing results for $M = 1$, example 3.1.

Therefore at $x = (0,;3;1;2;0)^T$ we have found the minimum of z.

Remark 3.3 Solving a feasible problem is not the subject of this article. To increase the efficiency of the MODART algorithm, the LP problem can be transformed so that Phase-0 will find the optimal solution.

4. Infeasible, Unbounded and Degenerate Problems

Example 4.1.

$$z = 2x_1 - x_2 + 3x_3 \rightarrow min$$

$$x_1 + x_2 + x_3 = 2$$

$$2x_1 + x_2 = 7$$

$$x \geq 0.$$

The results of the calculations are put forth in the table 2. The solution found in the Phase-I is optimal.

Remark 4.1. In the beginning we determine from the 3 x 3 system the values of the basic variables v_1, v_2 and t. At the second step we introduce variable x_2 and drop v_1 according to the simplex rule. At the third step in the z_1- row the criteria of optimality is fulfilled. At the fourth step the Phase-I begins. We introduce variable x_1 to the basis according to $t-$row and find that $t_{max} = 0,25 < 1$, problem is infeasible. In Example 4.1, we used M = 1 to explain all the steps of the algorithm. If we take M = 20, then the problem solves already during Phase-0, we get $t = 0.25, x = (0.5; 0; 0)^T, v_2 = 0.75$, t-row is nonnegative.

Remark 4.2. At the third step of the algorithm MODART, the objective function of the problem (4) may be unbounded for some $v_i > 0$. Then we have to start Phase-1, solve problem (5).

.	x1	x2	x3	v1	v2	t	RHS
z1	-2	1	-3	0	0	1	0
.	0	0	0	1	1	1	1
.	1	1	1	1	0	-2	0
.	2	1	0	0	1	-7	0
z1	-1,7	1,2	-2,9	0	0	0	-0,1
t	-0,3	-0,2	-0,1	0	0	1	0,1
v1	0,4	0,6	0,8	1	0	0	0,2
v2	-0,1	-0,4	-0,7	0	1	0	0,7
z1	-2,5	0	-4,5	-2	0	0	-0,5
t	-0,17	0	0,17	0,33	0	1	0,17
x2	0,67	1	1,33	1,67	0	0	0,33
v2	0,16	0	-0,17	0,67	1	0	0,83
z1	.	.	.	.	.	.	.
t	-0,17	0	0,17	0,33	0	1	0,17
x2	0,67	1	1,33	1,67	0	0	0,33
v2	0,17	0	-0,17	0,67	1	0	0,83
.	.	.	.	.	.	.	.
t	0	0,25	0,5	0,75	0	1	0,25
x1	1	1,5	2	2,5	0	0	0,5
v2	0	-0,25	-0,5	0,25	1	0	0,75

Table 2. Computing results for $M = 1$, example 4.1.

Example 4.2.

$$z = -4x_1 + 3x_2 \to min$$

$$x_1 - x_2 + x_3 + x_4 = 1$$

$$x_1 - x_2 + x_3 + 2x_4 = 2$$

$$x \geq 0.$$

.	x1	x2	x3	x4	v1	v2	t	RHS
z1	4	-3	0	0	0	0	17	0
t	0	0	0	0	1	1	17	17
.	1	-1	1	1	1	0	-1	0
.	1	-1	1	2	0	1	-2	0
z1	5,7	-4,7	1,7	2,55	0	0	0	-14,45
t	-0,1	0,1	-0,1	-0,15	0	0	1	0,85
v1	0,9	-0,9	0,9	0,85	1	0	0	8,25
v2	0,8	-0,8	0,8	1,7	0	1	0	1,7
z1	0	1	-4	-2,83	-6,33	0	0	-19,83
t	0	0	0	-0,06	0,11	0	1	0,94
x1	1	-1	1	0,94	1,11	0	0	0,94
v2	0	0	0	0,94	-0,89	1	0	0,94
z1	0	1	-4	0	-9	3	0	-17
t	0	0	0	0	0,06	0,06	1	1
x1	1	-1	1	0	2	-1	0	0
x4	0	0	0	1	-0,94	1,06	0	1

Table 3. Computing results for $M = 17$, example 4.2.

At the last step we obtain a feasible solution to the problem (5), $t_{max} = 1$, and the objective function is unbounded.

Example 4.3.

Cycles may occur, while solving this problem using the simplex method, see Solow [5,10].

$$z = -2x_1 - 2x_2 + 8x_3 + 2x_4 \rightarrow min$$

$$-7x_1 - 3x_2 + 7x_3 + 2x_4 = 0$$

$$2x_1 + x_2 - 3x_3 - x_4 = 0$$

$$x \geq 0.$$

Introducing the artificial variables v_1 , v_2 and v_3 , $v_1 + v_2 + v_3 = 0$. In the case of this right hand side, there is no need for variable t and parameter M.

.	x1	x2	x3	x4	v1	v2	v3	RHS
z1	2	2	-8	-2	0	0	0	0
v3	0	0	0	0	1	1	1	0
.	-7	-3	7	2	1	0	0	0
.	2	1	-3	-1	0	1	0	0
z1	2	2	-8	-2	0	0	0	0
v3	5	2	-4	-1	0	0	1	0
v1	-7	-3	7	2	1	0	0	0
v2	2	1	-3	-1	0	1	0	0
z1	0	1	-5	-1	0	-1	0	0
v3	0	-0,5	3,5	1,5	0	-2,5	1	0
v1	0	0,5	-3,5	-1,5	1	3,5	0	0
x1	1	0,5	-1,5	-0,5	0	0,5	0	0
z1	0	0	2	2	-2	-8	0	0
v3	0	0	0	0	1	1	1	0
x2	0	1	-7	-3	2	7	0	0
x1	1	0	2	1	-1	-3	0	0
z1	-1	0	0	1	-1	-5	0	0
v3	0	0	0	0	1	1	1	0
x2	3,5	1	0	0,5	-1,5	-3,5	0	0
x3	0,5	0	1	0,5	-0,5	-1,5	0	0
z1	-2	0	-2	0	0	-2	0	0
v3	0	0	0	0	1	1	1	0
x2	3	1	-1	0	-1	-2	0	0
x4	1	0	2	1	-1	-3	0	0

Table 4. Computing results for Example 4.3

Therefore at $x = (0;0;0;0)^T$ we have found the minimum of the objective function. Solving cycling examples is described in details in [1, 3, 9,10]. We solved this problem by Dantzig, Bland and other rules can also be used.

5. Two Applications of Modified Artificial Basis Method

Example 5.1. An advertising problem

It is planned to spend a total of 31 000 euros for advertising, whereas three requirements should be met. The advertisement must be seen by at least 80 000 men, 120 000 pensioners and 70 000 women. The advertisement can be shown during a football game and opera, with the cost of 1 min being, respectively, 5000 and 3000 euros. The number of potential viewers per 1 min is given in thousands: 14 000 men during the football game and 6000 men during opera, while the respective numbers are 20 000 and 10 000 for pensioners and 10 000 and 8000 for women. Draw up a plan where a maximum of advertising is done during the football game.

Let x_1 and x_2 denote the length of advertising time in minutes during football game and opera, respectively. Let as compose the corresponding linear programming problem

$$z = x_1 \rightarrow max$$

$$5x_1 + 3x_2 \leq 31(budget)$$

$$14x_1 + 6x_2 \geq 80(men)$$

$$20x_1 + 10x_2 \geq 120(pensioner)$$

$$10x_1 + 8x_2 \geq 70(women)$$

$$x \geq 0.$$

Introducing the slack variables x_3, x_4, x_5, x_6 and artificial variables v_2, v_3, v_4, we get the following problem

$$z_1 = -x_1 - Mt \rightarrow min$$

$$v_2 + v_3 + v_4 + Mt = M$$

$$5x_1 + 3x_2 + x_3 - 31t = 0(budget)$$

$$14x_1 + 6x_2 - x_4 + v_2 - 80t = 0(men)$$

$$20x_1 + 10x_2 - x_5 + v_3 - 120t = 0(pensioner)$$

$$10x_1 + 8x_2 - x_6 + v_4 - 70t = 0(women)$$

$$t, x, v \geq 0.$$

We shall solve the problem for $M = 1$ using the algorithm MODART. We obtain an optimal solution to the enlarged problem (4), $t = 0,2; x = (1;0,4;0;0,4;0;0)^T$, which does not satisfy the last constraint of the initial problem, the artificial variable v_4 is in optimal basis, $v_4 = 0,8$, the woman constraint is not fulfilled. The "almost feasible" solution will be found by the formula $x := x/t, x = (5;2;0;2;0;0)^T$. It means that only 66 000 women see the advertisement instead of 70 000. Thus the algorithm MODART provides an additional possibility of changing the conditions of the problem if these are contradictory. Let us change the problem by assuming that 32 000 and not 31 000 euros can be used for advertising. In that case the initial problem has a solution - the optimal solution for $M = 1$ is $x = (4,6;3;0;2,4;2;0)^T, t = 1$, all artificial variables drop the basis in the Phase-0. The advertising time should be 4,6 min during the football game and 3 min during the opera.

Example 5.2. The farmer's problem, see [11]. Consider a European farmer who specializes in raising grain, corn and sugar beets on his 500 acres of land. During the winter, he wants to decide how much land to devote to each crop. The farmer knows that at last 200 tons of wheat and 240 T of corn are needed for cattle feed. These amounts can be raised on the farm or bought from a wholesaler. Any production in excess of the feeding requirement would be sold. Selling prices are 170 and 150 per ton of wheat and corn, respectively. The purchase prices are 40 percent more than this due to the wholesaler's margin and transportations costs. Another profitable crop is sugar beet, which sells at $36/T$; however, the EU imposes a quota on sugar beet production. Any amount in excess of the quota can be sold only at $10/T$. The farmer's quota for next year is 6000T. Based on past experience, the farmer knows that mean yield on his land is roughly 2.5T,3T and 20T per acre for wheat, corn and sugar beet, resp. Let us compose the following model. Acres devoted to wheat, corn and beets are x_1, x_2, x_3 resp. tons of wheat, corn and sugar beets sold are w_1, w_2, w_3 and w_4 resp. stands for the tons of sugar beets sold at the lower price, y_1, y_2 stand for tons of wheat and corn are purchased. The problem reads as follows:

$$z = -150x_1 - 230x_2 - 260x_3 + 170w_1 - 210y_2 + 150w_2 + 36w_3 + 10w_4 \to max$$

$$x_1 + x_2 + x_3 \leq 500,\ \ 2.5x_1 + y_1 - w_1 \geq 200 \tag{6}$$

$$3x_2 + y_2 - w_2 \geq 240$$

$$w_3 + w_4 \leq 20x_3$$

$$w_3 \leq 6000$$

$$x, y, w \geq 0.$$

The results of computations are given in table 5, the profit is 118 600.

culture	wheat	corn	sugar beets
surface	120	80	300
yield	300	240	6000
sales	100	.	6000
purchase	.	.	.

Table 5 Solution for the average values.

We will assume some correlation among the yields of the different crops. The years are good, fair or bad for all crops, resulting in above average (+20 percent), average and below average(-20) yields for all crops. Assuming that the probability of a good, fair and bad weather are 1/3, 1/3 and 1/3, in the textbook [11] second and third constraints are replaced by six constraints. For example, in the case of good weather, an inequality $3x_1(1) + y_1(1) - w_1(1) \geq 200$ is added to the second constraint and $2x_1(3) + y_1(3) - w_1(3) \geq 200$ in the case of bad weather. Similarly, two constraints are added to the third constraint. We get a linear programming problem with 40 variables (including 6 artificial) and 53 constraints. The average profit is 108 390, see table 6.

culture	wheat	corn	sugar beets
surface	170	80	250
yield	510	288	6000
sales	310	48	6000
purchase	.	.	.
culture	wheat	corn	sugar beets
fair	.	.	.
yield	425	240	5000
sales	225	.	5000
purchase	.	.	.
culture	wheat	corn	sugar beets
bad	.	.	.
yield	340	192	4000
sales	140	.	4000
purchase	.	48	.

Table 6 Solution to the problem (6) for equal probabilities

Finally we solve this problem, assuming that the probabilities of good, fair and bad weather are 1/5, 3/5, 1/5. The first tree members of the objective function of the problem (6) do not depend on the weather, the following members have the coefficients 1/5, 3/5, 1/5. The objective function is

$$z = -150x_1 - 230x_2 - 260x_3 +$$

$$+1/5[170w_1(1) - 238y_1(1) + 150w_2(1) - 210y_2(1) + 36w_3(1) + 10w_4(1)] +$$

$$+3/5[170w_1(2) - 238y_1(2) + 150w_2(2) - 210y_2(1) + 36w_3(2) + 10w_4(2)] +$$

$$+1/5[170w_1(3) - 238y_1(3) + 150w_2(3) - 210y_2(3) + 36w_3(3) + 10w_4(3)] \to max.$$

Table 7 shows the solution of the problem for M = 100, the average profit z = 122 686,5.

culture	wheat	corn	sugar beets
surface	145.5	80	274.5
yield	436.5	288	6588
sales	236.5	48	6588
purchase	.	.	.
culture	wheat	corn	sugar beets
fair	.	.	.
yield	363.7	240	5490
sales	163.7	.	5490
purchase	.	.	.
culture	wheat	corn	sugar beets
bad	.	.	.
yield	291	192	4392
sales	91	.	4392
purchase	.	48	.

Table 7 Solution to the problem (6) for different probabilities and M=100.

To sum up, we can say that the average yield increases as the probability of fair weather increases.

6. Conclusion and Future Works

The purpose of the work is to combine the two-phase simplex method and the big-M method. By the presented method MODART, there are no disadvantages of these two methods, the parameter M has to be selected for medium size and during the Phase-0 and Phase-1 we take into account the objective function of the initial problem. Next we continue as in the case of a classical two-phase simplex method. The method presented has been used to solve two practical examples. In the last example there are 53 variables and 40 constraints.

In the future, it is necessary to consider the criteria for comparing the classical implementation of the simplex method with the three-phase method when solving the test problems. It must be taken into account that the number of steps in the three-phase method depends on the value of the selected parameter M. If the problem is feasible, it can be transformed before constructing the extended problem so that it is solved in Phase -0 for the medium parameter M.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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