Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 6, 2891-2906
https://doi.org/10.28919/jmcs/4429
ISSN: 1927-5307

A NEW EXTENSION OF THE NADARAJAH HAGHIGHI MODEL: MATHEMATICAL PROPERTIES AND APPLICATIONS

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Abstract: In the present study, we introduced a new extended Nadarajah-Haghighi distribution with four parameters. The current density function is computed with linear combination of Nadarajah Haghighi exponentiated density. For several statistical and mathematical quantities, we extracted analytical formula. The method of maximum likelihood is used for estimating the model's unknown parameters. Finally, the flexibility and utility of current distribution is demonstrated empirically with a real data set.

Keywords: Nadarajah-Haghighi model; exponentiated Weibull-H family; generating function; order statistic; maximum likelihood; probability weighted moment.

1. INTRODUCTION

Let $h_{\alpha,\lambda}(x)$ be probability density function (PDF) and $H_{\alpha,\lambda}(x)$ be cumulative distribution function (CDF) of a random variable (rv) X that follows Nadarajah Haghighi (NH) distribution, then

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Received December 13, 2019

$$h_{\alpha,\lambda}(x) = \alpha\lambda(1+\lambda x)^{\alpha-1}\, exp[1-(1+\lambda x)^{\alpha}]\,|_{(x>0)}, \tag{1}$$

and

$$H_{\alpha,\lambda}(x) = 1 - exp[1-(1+\lambda x)^{\alpha}]\,|_{(x>0)}, \tag{2}$$

For which, $\alpha > 0$ and $\lambda > 0$ are respectively the shape and scale parameters. When $\alpha = 1$, we get the standard exponential (E) model. Nadarajah and Haghighi [1] noted that the PDF (1) has the important characteristic of providing the zero mode at all times. They further revealed that higher α values would result in faster upper tail decay. We will use (1), (2) and exponentiated Weibull-H (EW-H) family which was obtained by Cordeiro et al. [2]. The CDF of EW-H family with parameter vector φ is given by

$$F_{a,b,\varphi}(x) = \left\{1 - exp\left[-\left(\frac{H_{\varphi}(x)}{\bar{H}_{\varphi}(x)}\right)^{b}\right]\right\}^{a} \tag{3}$$

The PDF of the EW-H family reduces to

$$f_{a,b,\varphi}(x) = abh_{\varphi}(x)\frac{H_{\varphi}(x)^{b-1}}{\bar{H}_{\varphi}(x)^{b+1}} exp\left[-\left(\frac{H_{\varphi}(x)}{\bar{H}_{\varphi}(x)}\right)^{b}\right]\left\{1 - exp\left[-\left(\frac{H_{\varphi}(x)}{\bar{H}_{\varphi}(x)}\right)^{b}\right]\right\}^{a-1} \tag{4}$$

To this end, we shall use (3) and (4) to generate a new version of the NH model called exponentiated Weibull Nadarajah Haghighi (EWNH) model with following CDF

$$F(x) = F_{a,b,\alpha,\lambda}(x) = \left[1 - exp\left(-\left\{\frac{1-exp[1-(1+\lambda x)^{\alpha}]}{exp[1-(1+\lambda x)^{\alpha}]}\right\}^{b}\right)\right]^{a} \tag{5}$$

Therefore, the PDF of the EWNH model reduces to

$$f(x) = f_{a,b,\alpha,\lambda}(x) = ab\alpha\lambda(1+\lambda x)^{\alpha-1}\frac{\{1-exp[1-(1+\lambda x)^{\alpha}]\}^{b-1}}{\{exp[1-(1+\lambda x)^{\alpha}]\}^{b}}$$

$$\times\, exp\left(-\left\{\frac{1-exp[1-(1+\lambda x)^{\alpha}]}{exp[1-(1+\lambda x)^{\alpha}]}\right\}^{b}\right) \times \left[1 - exp\left(-\left\{\frac{1-exp[1-(1+\lambda x)^{\alpha}]}{exp[1-(1+\lambda x)^{\alpha}]}\right\}^{b}\right)\right]^{a-1} \tag{6}$$

The additional parameter a and b will allow us with more flexibility to explore the tail activity of the density (6). Furthermore, due to the versatility of the EWNH to handle all aspects of HRF (increasing, decreasing, constant, bath, upside-down bath), as in Figure 2, the latest model becomes more vital to use for real data in a variety of applications. The PDF referenced by equation (6) has a random variable $X{\sim}EWHN(a,b,\alpha,\lambda)$. In Table 1, several different forms of the EWNH model are mentioned.

Table 1: Some special cases of the EWNH model.

Reduced Model	a	b	α	λ	Reduced CDF
W-NH	1	b	α	λ	$1-exp\left(-\left\{\frac{1-exp[1-(1+\lambda x)^{\alpha}]}{exp[1-(1+\lambda x)^{\alpha}]}\right\}^{b}\right)$
W-E	1	b	1	λ	$1-exp\left\{-\left[\frac{1-exp(-\lambda x)}{exp(-\lambda x)}\right]^{b}\right\}$
Burr X-NH	a	2	α	λ	$\left[1-exp\left(-\left\{\frac{1-exp[1-(1+\lambda x)^{\alpha}]}{exp[1-(1+\lambda x)^{\alpha}]}\right\}^{2}\right)\right]^{a}$
R-NH	1	2	α	λ	$1-exp\left(-\left\{\frac{1-exp[1-(1+\lambda x)^{\alpha}]}{exp[1-(1+\lambda x)^{\alpha}]}\right\}^{2}\right)$
Burr X-E	a	2	1	λ	$1-exp\left\{-\left[\frac{1-exp(-\lambda x)}{exp(-\lambda x)}\right]^{2}\right\}$
R-E	1	2	1	λ	$\left(1-exp\left\{-\left[\frac{1-exp(-\lambda x)}{exp(-\lambda x)}\right]^{2}\right\}\right)^{a}$
E-NH	1	1	α	λ	$1-exp\left\{-\frac{1-exp[1-(1+\lambda x)^{\alpha}]}{exp[1-(1+\lambda x)^{\alpha}]}\right\}$
E-E	1	1	1	λ	$1-exp\left[-\frac{1-exp(\lambda x)}{exp(\lambda x)}\right]$
EE-NH	a	1	α	λ	$\left(1-exp\left\{-\frac{1-exp[1-(1+\lambda x)^{\alpha}]}{exp[1-(1+\lambda x)^{\alpha}]}\right\}\right)^{a}$
EE-E	a	1	1	λ	$\left\{1-exp\left[-\frac{1-exp(\lambda x)}{exp(\lambda x)}\right]\right\}^{a}$
EE-E	a	1	1	1	$\left\{1-exp\left[-\frac{1-exp(x)}{exp(x)}\right]\right\}^{a}$

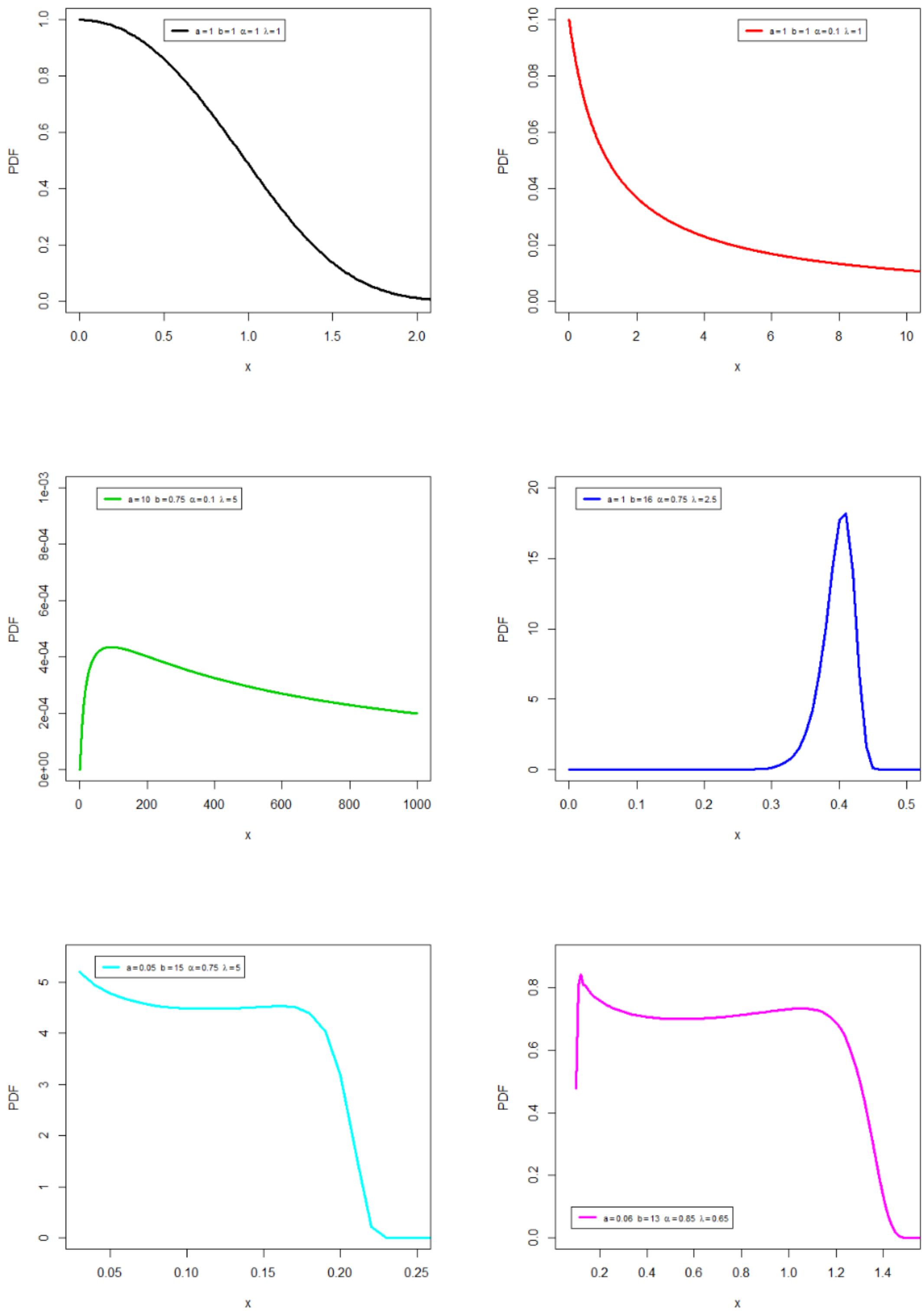

Figure 1: PDF plots for the EWNH model.

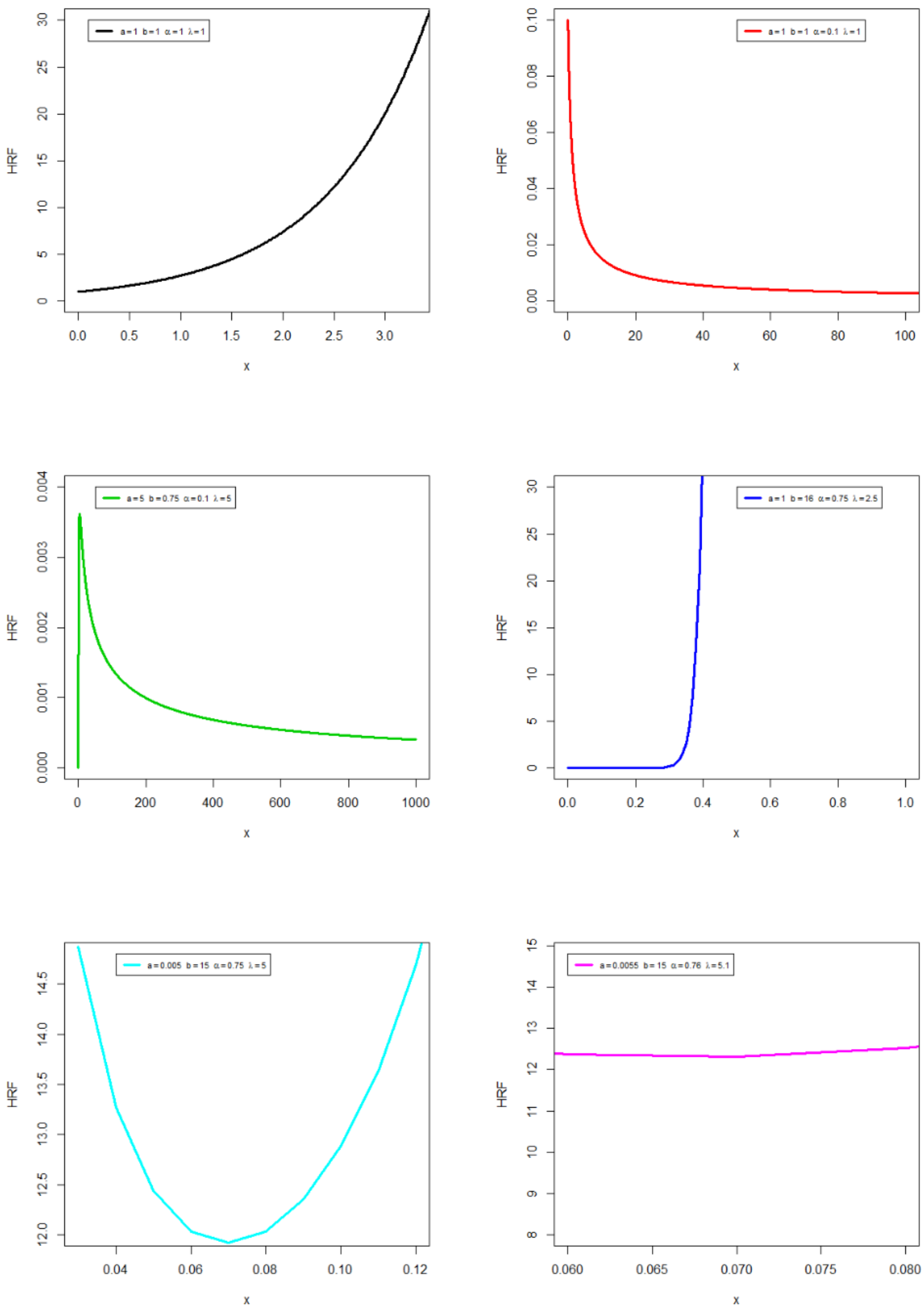

Figure 2: HRF plots for the EWNH model.

2. LINEAR REPRESENTATION

For EWNH density, we offer a useful linear representation. The following power series holds for all $|z| < 1$ and $b > 0$ (a real non-integer)

$$(1-z)^{b-1} = \sum_{w=0}^{\infty} \frac{(-1)^w \ \Gamma(b)}{w! \, \Gamma\,(b-w)} \ z^w. \tag{7}$$

Applying (7) to the last term in (4) gives

$$f(x) = ba \ \alpha\lambda(1+\lambda x)^{\alpha-1} \, exp[1-(1+\lambda x)^{\alpha}] \times \left(\frac{\{1-exp[1-(1+\lambda x)^{\alpha}]\}^{b-1}}{\{exp[1-1+\lambda x)^{\alpha}]\}^{b+1}}\right)$$

$$\times \ \left(\sum_{m=0}^{\infty} \frac{(-1)^m \ \Gamma(a)}{m!\Gamma\,(a-m)}\right) \times \underbrace{exp\left[-(m+1)\left(\frac{\{1-exp[1-(1+\lambda x)^{\alpha}]\}}{exp[1-(1+\lambda x)^{\alpha}]}\right)^{b}\right]}_{A_m}. \tag{8}$$

If we extend the quantity of A_m in power series, then

$$A_m = \sum_{K=0}^{\infty} \frac{(-1)^k (m+1)^k}{k!} \frac{\{1-exp[1-(1+\lambda x)^{\alpha}]\}^{kb}}{\{exp[1-(1+\lambda x)^{\alpha}]\}^{kb}} \, .$$

Inserting the above expression of A_m in (8), the EWNH density reduces to

$$f(x) = \alpha\lambda(1+\lambda x)^{\alpha-1} \, exp[1-(1+\lambda x)^{\alpha}]$$

$$\times \sum_{m,k=0}^{\infty} \frac{(-1)^{k+m} \ ab\Gamma\,(a)(m+1)^k}{m!k!\Gamma\,(a-m)} \times \frac{\{1-exp[1-(1+\lambda x)^{\alpha}]\}^{(k+1)b-1}}{\{exp[1-(1+\lambda x)^{\alpha}]\}^{(k+1)b+1}}. \tag{9}$$

Implementing the generalized binomial expansion to $\{exp[1-(1+\lambda x)^{\alpha}]\}^{-[(k+1)b+1]}$ in eq. (9), EWNH density is further portrayed as an infinite linear combination of the density of exp-H as

$$f(x) = \sum_{k,w=0}^{\infty} \upsilon_{k,w} \ \pi_{(k+1)b+w}(x;\alpha,\lambda) \tag{10}$$

where

$$\pi_{(k+1)b+w}(x;\alpha,\lambda) = [(k+1)b+w] \times \underbrace{\alpha\lambda(1+\lambda x)^{\alpha-1} exp[1-(1+\lambda x)^{\alpha}]}_{h_{\alpha,\lambda}(x)}$$

$$\times \underbrace{\{1-exp[1-(1+\lambda x)^{\alpha}]\}^{[(k+1)b+w]-1}}_{[H_{\alpha,\lambda}(x)]^{[(k+1)b+w]-1}}.$$

which is the PDF of exp-NH for the power parameter $[(k+1)b+w]$ and

$$\upsilon_{k,w} = \sum_{m=0}^{\infty} \frac{(-1)^{k+m} \ ab(m+1)^k \ \Gamma(a) \ \Gamma([k+1]b+w+1)}{k!m!w![(k+1)b+w] \ \Gamma(a-m) \ \Gamma([k+1]b+1)}.$$

Equation (10) clearly demonstrates that density of X are being characterized as a linear combination of exp-H densities. Therefore, by understanding the exp-NH distribution, several mathematical characteristics of the new family are being generated. Likewise, the EWNH model's CDF can often be viewed as a linear combination of CDFs of exp-NH defined as

$$F(x) = \sum_{k,w=0}^{\infty} \upsilon_{k,w} \, \Pi_{[(\mathrm{k}+1)\mathrm{b}+\mathrm{w}]}(x\colon \alpha, \lambda),$$

where

$$\Pi_{(k+1)b+w}(x\colon \alpha, \lambda) = \underbrace{\{1 - exp[1-(1+\lambda x)^{\alpha}]\}^{[(k+1)b+w]}}_{[H_{\alpha,\lambda}(x)]^{[(k+1)b+w]}}$$

which is the CDF of exp-NH for the power parameter $[(k+1)b+w]$.

3. Mathematical Properties

3.1 Moments

The r^{th} moment of X, say μ_r', derived through equation (10) as

$$\mu_r' = E(X)^r = \sum_{k,w,j=0}^{\infty} \sum_{i=0}^{r} \upsilon_{k,w} \zeta_{j,i}^{\{[(k+1)b+w],r\}} \Gamma\left(1+\frac{i}{\alpha}, 1+j\right)$$

where

$$\zeta_{j,i}^{\{[(k+1)b+w],r\}} = [(k+1)b+w]\lambda^{-r} \, exp(1+j) \, (-1)^{r+j-i} \times (1+j)^{-\left(\frac{i}{\alpha}+1\right)} \binom{[(k+1)b+w]-1}{j} \binom{r}{i}$$

or

$$\mu_r' = \sum_{k,w=0}^{\infty} \sum_{j=0}^{[(k+1)b+w]-1} \sum_{i=0}^{r} \upsilon_{k,w} \zeta_{j,i}^{\{[(k+1)b+w],r\}} \times \Gamma\left(1+\frac{i}{\alpha}, 1+j\right).$$

The n^{th} central moment of X, say M_n, is defined by

$$M_n = E(X-\mu_r')^n = \sum_{r=0}^{n} \binom{n}{r} (-\mu_r')^{n-r} E(X^r)$$

$$= \sum_{r=0}^{n} \sum_{k,w,j=0}^{\infty} \sum_{i=0}^{n} \binom{n}{r} (-\mu_r')^{n-r} \times \upsilon_{k,w} \zeta_{j,i}^{\{[(k+1)b+w],r\}} \Gamma\left(1+\frac{i}{\alpha}, 1+j\right).$$

3.2 Quantile function (QF) and generating function (MGF)

Let $Q(u)$ is the QF of X which is specified by inverting (3) as

$$Q(u) = F^{-1}(u) = H^{-1}\left(\left\{1+\left[-log\left(1-u^{1/a}\right)\right]^{-1/b}\right\}^{-1}\right), \; 0 < u < 1.$$

The MGF can follow from equation (10) as

$$M_X(t) = E(e^{tX}) = \sum_{k,w,j,r=0}^{\infty} \sum_{i=0}^{r} \frac{t^r}{r!} \upsilon_{k,w} \zeta_{j,i}^{\{[(k+1)b+w],r\}} \Gamma\left(1+\frac{i}{\alpha}, 1+j\right)$$

or

$$M_X(t) = \sum_{k,w,r=0}^{\infty} \sum_{j=0}^{[(k+1)b+w]-1} \sum_{i=0}^{r} \frac{t^r}{r!} \upsilon_{k,w} \zeta_{j,i}^{\{[(k+1)b+w],r\}} \times \Gamma\left(1+\frac{i}{\alpha}, 1+j\right) |_{([(k+1)b+w]>0 \text{ and integer})}.$$

3.3 Incomplete moments

The primary objectives of the first incomplete moment are to describe the Bonferroni and Lorenz curves as well as to be considered very useful in calculations of mean deviation. In insurance, economics, demography, reliability and medicine these curves are quite significant. From equation (10) the s^{th} incomplete moment, say $\varphi_s(t)$, of X is being defined as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)dx = \sum_{k,w,j=0}^{\infty} \sum_{i=0}^{s} \upsilon_{k,w} \zeta_{j,i}^{\{[(k+1)b+w],s\}} \times \begin{bmatrix} \Gamma\left(1+\frac{i}{\alpha}, 1+j\right) \\ -\Gamma\left(1+\frac{i}{\alpha}, (1+j)(1+\lambda t)^{\alpha}\right) \end{bmatrix} \quad (11)$$

or

$$\varphi_s(t) = \sum_{k,w,=0}^{\infty} \sum_{j=0}^{[(k+1)b+w]-1} \sum_{i=0}^{s} \upsilon_{k,w} \zeta_{j,i}^{\{[(k+1)b+w],s\}}$$
$$\times \begin{bmatrix} \Gamma\left(1+\frac{i}{\alpha}, 1+j\right) \\ -\Gamma\left(1+\frac{i}{\alpha}, (1+j)(1+\lambda t)^{\alpha}\right) \end{bmatrix} |_{([(k+1)b+w]>0 \text{ and integer})}. \quad (12)$$

The mean deviations about mean $[\delta_1 = E(|X-\mu_1'|)]$ and about median $[\delta_2 = E(|X-M|)]$ of X are defined by $\delta_1 = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\varphi_1(M)$, respectively, where $\mu_1' = E(X)$, $M = Median(X) = Q(0.5)$ is the median, $F(\mu_1')$ is conveniently evaluated from eq. (5) and φ_1 (t) is the first incomplete moment described by (11) (or (12)) with $s = 1$. The $\varphi_1(t)$ can be obtained as

$$\varphi_1(t) = \sum_{k,w,j=0}^{\infty} \sum_{i=0}^{1} \upsilon_{k,w} \zeta_{j,i}^{\{[(k+1)b+w],1\}} \begin{bmatrix} \Gamma\left(1+\frac{i}{\alpha}, 1+j\right) \\ -\Gamma\left(1+\frac{i}{\alpha}, (1+j)(1+\lambda t)^{\alpha}\right) \end{bmatrix}$$

or

$$\varphi_1(t) = \sum_{k,w,=0}^{\infty} \sum_{j=0}^{[(k+1)b+w]-1} \sum_{i=0}^{1} \upsilon_{k,w} \zeta_{j,i}^{\{[(k+1)b+w],1\}}$$
$$\times \begin{bmatrix} \Gamma\left(1+\frac{i}{\alpha}, 1+j\right) \\ -\Gamma\left(1+\frac{i}{\alpha}, (1+j)(1+\lambda t)^{\alpha}\right) \end{bmatrix} |_{([(k+1)b+w]>0 \text{ and integer})}.$$

4. ORDER STATISTICS

In several areas of statistical principles and application, order statistics keep emerging. Let $X_1, X_2, \ldots, X_n$ be a random sample drawn through EWNH model. Let $X_{i:n}$ be the i^{th} order statistic then the PDF is being written as

$$f_{i:n}(x) = \frac{f(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \tag{13}$$

in which B (.,.) is the beta funsction. On the basis of equations (3) and (4), we obtained

$$f(x)F^{j+i-1}(x) = ab\alpha\lambda(1+\lambda x)^{\alpha-1} exp[1-(1+\lambda x)^{\alpha} \times \frac{\{1-exp[1-(1+\lambda x)^{\alpha}]\}^{b-1}}{\{exp[1-(1+\lambda x)^{\alpha}]\}^{b+1}}$$

$$\times\, exp\left(-\left\{\frac{1-exp[1-(1+\lambda x)^{\alpha}]}{exp[1-(1+\lambda x)^{\alpha}]}\right\}^{b}\right) \times \left[1-exp\left(-\left\{\frac{1-exp[1-(1+\lambda x)^{\alpha}]}{exp[1-(1+\lambda x)^{\alpha}}\right\}^{b}\right)\right]^{a(j+i)-1}$$

Along with the same linear representation steps (10) we get

$$f(x)F^{j+i-1}(x) = \alpha\lambda(1+\lambda x)^{\alpha-1} exp[1-(1+\lambda x)^{\alpha}$$

$$\times \sum_{i,k=0}^{\infty} \left\{\frac{(-1)^{l+k} ab(1+l)^k \Gamma([j+i]a)}{l!k!\Gamma([j+i]a-l)} \times \frac{\{1-exp[1-(1+\lambda x)^{\alpha}]\}^{(k+1)b-1}}{\{exp[1-(1+\lambda x)^{\alpha}]\}^{(k+1)b+1}}\right\}.$$

Then

$$f(x)F^{j+i-1}(x) = \sum_{k,m=0}^{\infty} t_{k,m}^{j+i-1}\, \Pi_{[(k+1)b+m]}(x\colon \alpha,\lambda), \tag{14}$$

where

$$t_{k,m}^{(j+i-1)} = \sum_{l=0}^{\infty} \frac{(-1)^{k+m}\, ab(1+m)^k\, \Gamma([j+i]a)\, \Gamma([k+1]b+1+m)}{l!k!m![(k+1)b+w]\, \Gamma([j+i]a-l)\, \Gamma([k+1]b+1)}.$$

Substituting eq. (14) in eq. (13), the PDF of $X_{i:n}$ is being represented as

$$f_{i:n}(x) = \frac{1}{B(i,n-i+1)} \sum_{k,m=0}^{\infty} q_{k,m}\, \pi_{[(k+1)b+m]}(x\colon \alpha,\lambda)$$

in which $\pi_{(k+1)b+m}(x)$ is the density of exp-H for the power parameter $[(k+1)b+m]$ and

$$q_{k,m} = \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} t_{k,m}^{j+i-1}.$$

A linear combination of exp-H densities is the density of EWNH order statistics. The moments of $X_{i:n}$ are attained by

$$E(X_{i:n}^{s}) = \sum_{k,h,m=0}^{\infty} \sum_{d=0}^{r} \frac{q_{k,m}}{B(i,n-i+1)} \zeta_{j,i}^{\{[(k+1)b+w],s\}} \Gamma\left(1+\frac{d}{\alpha}, 1+h\right), \tag{15}$$

or

$$E(X_{i:n}^{s}) = \sum_{k,m=0}^{\infty} \sum_{h=0}^{[(k+1)b+w]-1} \sum_{d=0}^{r} \frac{q_{k,m}}{B(i,n-i+1)} \zeta_{j,i}^{\{[(k+1)b+w],s\}}$$

$$\times\, \Gamma\left(1+\frac{d}{\alpha}, 1+h\right) |_{([(k+1)b+w]>0 \text{ and integer})}. \tag{16}$$

L-moments are strongly connected with ordinary moments, and are obtained through linear combinations of order statistics. Although, the mean of the distribution takes place, they occur, but there may not be any higher moments, and they are reasonably resilient to the impacts of outliers. Explicit representations about the L-moments of X can be extracted as infinite weighted linear combinations of means of appropriate EWNH order statistics, based on the moments in eq. (15) and (16).

5. PROBABILITY WEIGHTED MOMENTS (PWMS)

The PWMs are expectations of a random variable for certain functions and thus should be specified by each random variable about which ordinary moments are meaningful. The $(s,r)^{th}$ PWM, say $\rho_{s,r}$ of the EWNH distribution is being explicitly specified by

$$\rho_{s,r} = E\{X^s F(X)^r\} = \int_{-\infty}^{\infty} x^s \; F\,(x)^r f(x)dx.$$

We can write from equations (3) and (4)

$$f(x)\;F(X)^r = \sum_{k,w=0}^{\infty} c_{k,w}^{(r)} \;\; \pi_{(k+1)b+w}\,(x),$$

where

$$c_{k,w}^{(r)} = \sum_{m=0}^{\infty} \frac{(-1)^{k+m} ab(m+1)^k \Gamma([r+1]a)\Gamma([k+1]b+w+1)}{m!k!w![(k+1)b+w]\Gamma([r+1]a-m)\Gamma([k+1]b+1)}.$$

Then, $\rho_{s,r}$ can be presented as

$$\rho_{s,r} = \sum_{k,w=0}^{\infty} d_{k,w}^{(r)} \int_{-\infty}^{\infty} x^{\delta} \pi_{(k+1)b+w}\,(x)dx.$$

After that, the $(s,r)^{th}$ PWM of X are being generated through an infinite linear combination of exp-NH moments concluded by

$$\rho_{s,r} = \sum_{k,w,j=0}^{\infty} \sum_{i=0}^{s} c_{k,w}^{(r)} \zeta_{j,i}^{\{[(k+1)b+w],s\}} \Gamma\left(1+\frac{i}{a}, 1+j\right),$$

or

$$\rho_{s,r} = \sum_{k,w=0}^{\infty} \sum_{j=0}^{[(k+1)b+w]-1} \sum_{i=0}^{s} c_{k,w}^{(r)} \zeta_{j,i}^{\{[(k+1)b+w],s\}}$$

$$\times \Gamma\left(1+\frac{i}{a}, 1+j\right) |_{([(k+1)b+w]>0 \text{ and integer})}.$$

6. APPLICATIONS

An example of real data set from Klein and Moeschberger [3] of 26 psychiatric patients' death times is presented in this section.

We compare EWNH distribution with some well-established four & five parameter distributions such as extended exponentiated NH (EENH) by Alizadeh et al. [4], Beta-Weibull (BW) by Lee et al. [5], Gamma Modified Weibull (GaMW) by Cordeiro et al. [6], Beta-Exponentiated Weibull (BEW) by Cordeiro et al. [7] and many other NH-distribution extensions including Kw-Nadarajah-Haghighi (KwNH) by Lima [8], beta-Nadarajah-Haghighi (BNH) by Dias [9] and exponentiated generalized Nadarajah-Haghighi (EGNH) by VedoVatto [10]. Other useful versions of the NH model can be found in Lemonte [11], Yousof and Korkmaz [12], Yousof et al. [13], Abdul-Moniem [14], Cordeiro et al. [15], Marcelo, et al. [16] and Ortega et al. [17].

For comparison, we will use the statistics LL (log-likelihood), AIC (Akaike information criterion), BIC (Bayesian information criterion), HQIC (Hannan-Quinn information criterion), CAIC (Consistent Akaike information criterion). In all the goodness of fit test criteria mentioned, EWNH model shows better fit for given data sets. Tables 2 displays the estimated values of the parameters based on MLE procedure and standard deviation (in parenthesis). Tables 3 record the measurements of the goodness-of-fit statistics. The perception of the hazard shape will lead to the selection of a particular model in the applications. A technique called Total Time-on-Test (TTT) plot described by Aarset [18] is very effective for this purpose. The TTT plot is constructed by evaluating

$$G(r/n) = \frac{\left(\sum_{m=1}^{r} y_{(m)}\right)+(n-r)y_{(r)}}{\sum_{m=1}^{r} y_{(m)}}$$

where $r = 1,2, \dots, n$ and $y_{(m)}$ $(m = 1,2, \dots, n)$ represent the order statistics of the sample, verses r/n. Table 3 reveals that the new EWNH model becomes preferable over EENH, BW, GaMW, BEW, KwNH, BNH and EGNH models. Figure 3 illustrates the estimated CDF, estimated PDF, estimated HRF, Kaplan-Meier survival plot, P-P plot for the times of death data. For the data set used, Figure 4 shows that it is a good fit for the current model. Figure 4 and Figure 5 represents TTT plot and PP plot respectively for the data set.

Table 2: Parameters estimates and standard deviation in parenthesis

Model	Estimators
EWNH (a, b, α, λ)	0.152(2.9× e^{-2}), 7.636(0.001), 3.155(1.567), 0.005(7.8× e^{-5})
BW (a, b, β, c)	0.018(0.006), 40.624(0.325), 0.023(0.145), 78.531(17.531)
GaMW $(a, \alpha, \lambda, \beta)$	1.341(0.449), 0.043(0.038), 0.092(0.022), 0.159(0.158)
KwNH (λ, α, a, b)	1.844(0.509), 1.042(0.901), 0.002(0.0006), 11.072(4.406)
BNH (λ, α, a, b)	3.384(.528), 0.0487(0.009), 0.237(0.001), 1.540(0.002)
EGNH (λ, α, a, b)	0.061(0.012), 2.056(0.574), 0.915(0.003), 0.991(0.002)
BEW $(a, b, \alpha, c, \lambda)$	0.187(0.07), 0.04(0.007), 5.49(0.037), 1.5(0.002), 0.259(0.002)

Table 3: Goodness of fit criteria

Model	**-LL**	**AIC**	**BIC**	**HQIC**	**CAIC**
EWNH	**94.070**	**196.141**	**201.173**	**197.590**	**198.045**
EENH	94.372	196.745	201.778	198.195	198.650
BW	95.035	198.071	203.103	199.520	199.976
GaMW	95.081	198.162	203.195	199.612	200.067
BEW	104.908	219.816	226.106	221.627	222.816
KwNH	103.923	215.846	220.879	217.296	217.751
BNH	108.591	225.183	230.215	226.632	227.087
EGNH	108.997	225.995	231.027	227.444	227.899

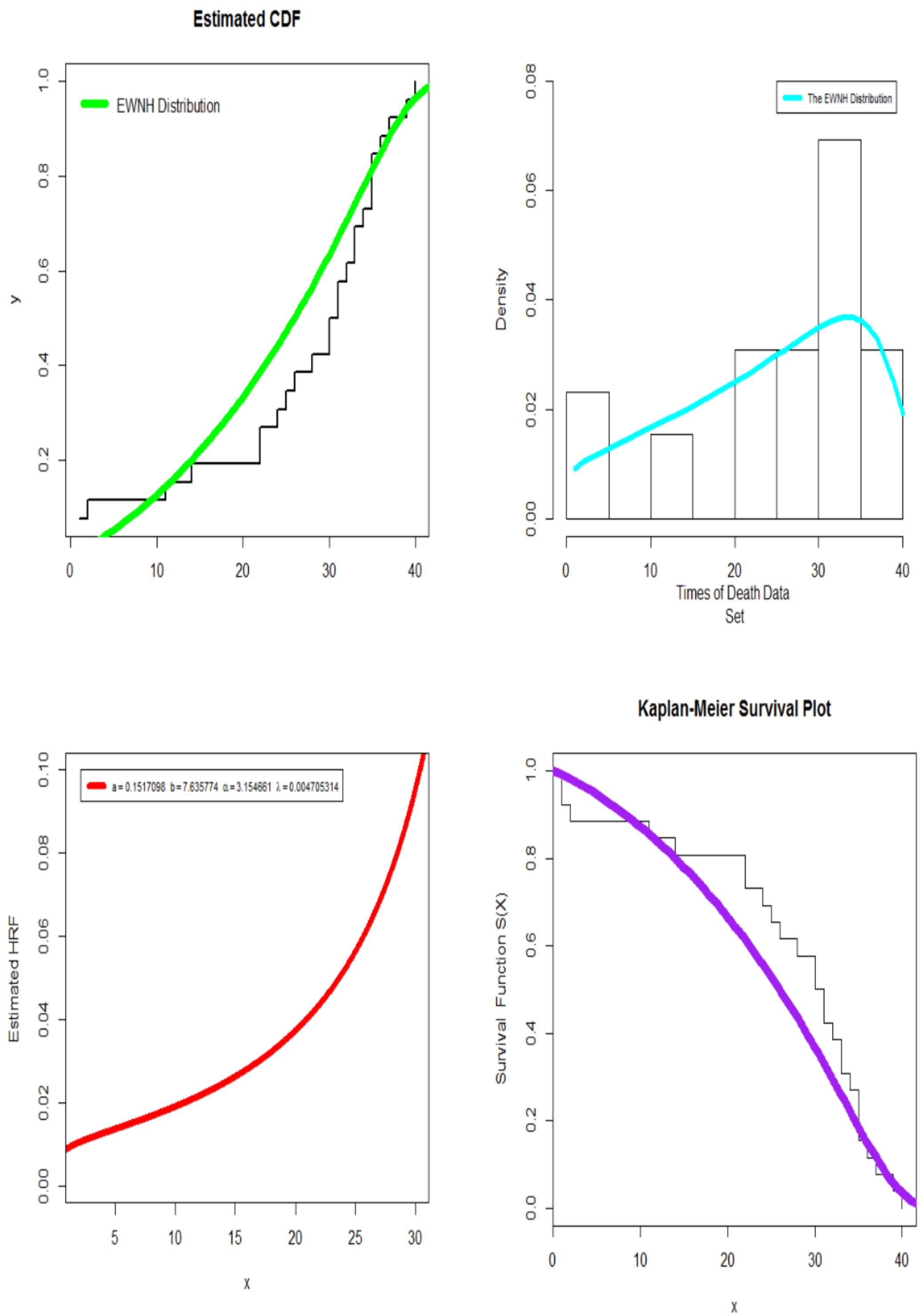

Figure 3: Estimated CDF, estimated PDF, estimated HRF, Kaplan-Meier survival plot for the times of death data.

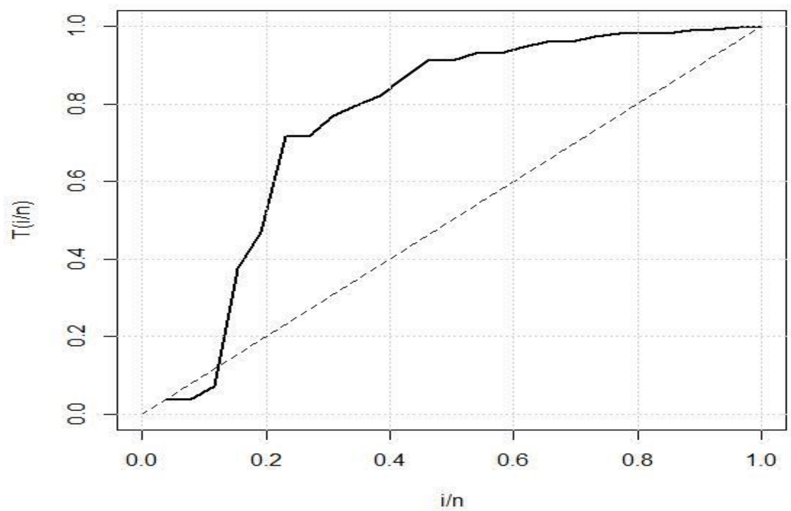

Figure 4: TTT plot.

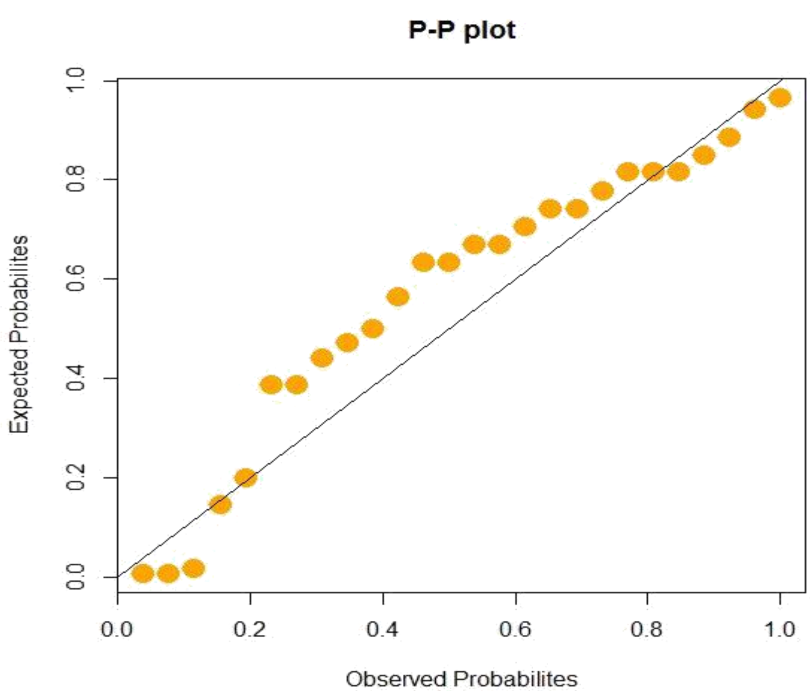

Figure 5: P-P plot for the times of death data.

CONCLUSIONS

This paper introduced a new Nadarajah-Haghighi (NH) four parameters model, named exponentiated Weibull NH (EWNH) distribution which expanded the NH distribution. The EWNH density is being presented as a linear combination of exponentiated NH density. We extracted analytical expressions like ordinary moments, moment generating function, incomplete moments, moments of residual life and reversed residual life for its mathematical and statistical quantities. For the estimation of the model's unknown parameters, the approach of maximum likelihood is considered. When modeling a data set, we factually demonstrated the flexibility and significance of the EWNH model. We believe that EWNH model can promote broader applications in various disciplines, including engineering, economics, meteorological hydrology, survival and lifetime data among others.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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