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FITTED DIFFERENCE APPROACH FOR DIFFERENTIAL EQUATIONS WITH DELAY AND ADVANCED PARAMETERS

G. SANGEETHA[1], G. MAHESH[2], AND K. PHANEENDRA[3,*]

[1]Department of Mathematics, Bhoj Reddy Engineering College for Women, Hyderabad, India

[2]Department of Mathematics, Geethanjali College of Engineering and Technology, Cheeryal, India

[3]Department of Mathematics, University College of Engineering, Osmania University, Hyderabad, India

Abstract: A difference scheme involving acceptable fitting parameters is suggested for differential equations with delay and advanced terms, the solutions of which show boundary layer behaviour. First, the original problem is reshaped into asymptotically comparable second order singular perturbation problem using Taylor series approximation for the retarded terms. In order to obtain precise solution, fitting parameters are introduced in difference scheme using modified upwind differences for the first order derivatives. Thomas procedure is used to solve the resulting tri-diagonal difference system. The method is tested on numerical examples for various values of the perturbation, delay and advance parameters. Computed maximum absolute errors are tabulated. Numerical experiments are shown in graphs and the effects of small shifts have been studied on the boundary layer region. Also, convergence has been established of the proposed method.

Keywords: boundary layer; delay and advance parameters; modified upwind; singular perturbation problem.

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[*]Corresponding author
E-mail address: kollojuphaneendra@yahoo.co.in
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1. INTRODUCTION

Modelling of many practical phenomena such as, thermo-elasticity [2], hybrid optical system [3], in population dynamics [10], in models for physiological processes [14], red blood cell system [13], predator-prey models [15] and in the potential in nerve cells by random synaptic inputs in dendrites [18] causes differential-difference problems.

Bellman and Cooke [1], Doolan et al. 5], Driver [6], El'sgol'tsand Norkin [7], Kokotovic [9], Miller et al. [16] and Smith [17] can be found in the collection of books for further study of mathematical aspects of the above class of models and singular perturbation problems. Lange and Miura [11-12] provided an overview of equations with small shifts, layers having turning points and rapid oscillations. In [4], for the solution of the singularly perturbed differential-difference equations with mixed shifts, a fourth order difference method with a fitting factor is proposed. The researchers in [8], proposed a fitted piecewise-uniform mesh method with analysis for differential difference equation having mixed small shifts having boundary layer. With this inspiration, in the next section, we define the problem and derivation of the of the numerical scheme using modified upwind differences with two fitting parameters.

2. DESCRIPTION OF THE METHOD

Consider the differential-difference equation with small delay as well as advance terms having the layer structure of the form:

$$\varepsilon\theta''(s)+\alpha(s)\theta'(s)+\beta(s)\theta(s-\delta)+C(s)\theta\left(s\right)+D(s)\theta\left(s+\eta\right)=F(s),\ 0<s<1 \tag{1}$$

with the boundary conditions

$$\theta(s)=\phi(s)\ ,\quad \text{over}\ \ -\delta\le s\le 0 \tag{2}$$

$$\theta(s)=\gamma(s),\quad \text{over}\quad 1\le s\le 1+\eta \tag{3}$$

where $\alpha(s),\ \beta(s),\ C\left(s\right),\ D\left(s\right),\ \phi(s)$ and $\gamma(s)$ are differentiable functions over (0, 1), perturbation parameter is $\varepsilon\ (0<\varepsilon<<1)$, $\delta\ (0<\delta=o(\varepsilon))$ the delay parameter and η the advance parameter respectively ($0<\eta=o(\varepsilon)$).

Using Taylor's expansion for the terms having delay and advanced parameters, we have

$$\theta(s-\delta) \approx \theta(s) - \delta\theta'(s) \tag{4}$$

$$\theta(s+\eta) \approx \theta(s) + \eta\theta'(s) \tag{5}$$

Using Eq. (4) and Eq.(5) in Eq. (1), we have the following an asymptotically similar problem

$$\varepsilon\theta''(s) + \tilde{\alpha}(s)\theta'(s) + \tilde{\beta}(s)\theta(s) = F(s) \tag{6}$$

$$\theta(0) = \phi(0) \tag{7}$$

$$\theta(1) = \gamma(1) \tag{8}$$

where $\tilde{\alpha}(s) = \alpha(s) + D(s)\eta - \beta(s)\delta$ and $\tilde{\beta}(s) = \beta(s) + C(s) + D(s)$.

Since $0 < \delta << 1$ and $0 < \eta << 1$, the conversion from Eq. (1) to Eq. (6) shall be admitted (El'sgolt's and Norkin [7]).

The roots for the characteristic equationof Eq. (6) may be described by

$$\varepsilon\xi(s)^2 + \tilde{\alpha}(s)\xi(s) - \tilde{\beta}(s) = 0$$

The two continuous functions of the above equation are given by

$$\xi_1(s) = -\frac{\tilde{\alpha}(s)}{2\varepsilon} - \sqrt{\left(\frac{\tilde{\alpha}(s)}{2\varepsilon}\right)^2 + \frac{\tilde{\beta}(s)}{\varepsilon}} \tag{9}$$

$$\xi_2(s) = -\frac{\tilde{\alpha}(s)}{2\varepsilon} + \sqrt{\left(\frac{\tilde{\alpha}(s)}{2\varepsilon}\right)^2 + \frac{\tilde{\beta}(s)}{\varepsilon}} \tag{10}$$

The function $\xi_1 < 0$ characterizes the layer on the left-end *s* = 0, while $\xi_2 > 0$ describes layer on the right-end *s* = 1.

3. NUMERICAL METHOD

Discretize the space [0, 1] in *N* equivalent sub-intervals of mesh size $h = \frac{1}{N}$, so that $s_i = s_0 + ih,\ \ i = 0, 1, 2, \ldots, N$ are the nodal points with $0 = s_0, 1 = s_N$.

As there are two boundary layers at s =0 and s =1 for the given problem, the space [0, 1] split into two sub-intervals $\left[0,\ s_p\right]$ and $\left[s_p,\ 1\right]$ where $s_p = \frac{1}{2}$. In $\left[0,\ s_p\right]$ the layer at the left end $s = 0$ and the layer is at right end $s = 1$ in $\left[s_p,\ 1\right]$.

We consider the difference scheme

$$\varepsilon\chi_i(\rho)\Omega_{\pm}\theta_i + \alpha(s_i)\mu_i(\rho)\Omega'_{+}\theta_i - \beta(s_i)\theta_i = F(s_i) \quad \text{for } i = 1, 2, \ldots p \tag{11}$$

$$\varepsilon\chi_i(\rho)\Omega_{\pm}\theta_i + \alpha(s_i)\mu_i(\rho)\Omega'_{-}\theta_i - \beta(s_i)\theta_i = F(s_i) \text{ for } i = p+1, p+2, \ldots, N\text{-}1 \tag{12}$$

with

$$\theta_o = \phi, \quad \theta_N = \gamma \tag{13}$$

where $\chi_i(\rho)$ and $\mu_i(\rho)$ are defined in such way that the solution of the related homogeneous differential equation is the exact solution of the related homogeneous difference of Eq. (11), Eq. (12).

Here $\Omega_{\pm}\theta_i \approx \dfrac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{h^2}$, $\Omega'_{+}\theta_i \approx \dfrac{\theta_{i+1} - \theta_i}{h} - \dfrac{h}{2}\theta''_i$, $\Omega'_{-}\theta_i \approx \dfrac{\theta_i - \theta_{i-1}}{h} + \dfrac{h}{2}\theta''_i$ and $\rho = \dfrac{h}{\varepsilon}$.

Substituting Eq. (9) and Eq. (10) in the related homogeneous difference Eq. (11) and Eq. (12), we can determine the fitting factors

$$\chi_i(\rho) = -\frac{\beta(s_i)h}{4\left[\frac{1}{\rho} - \frac{\alpha_i}{2}\right]}\left(\frac{e^{-\left(\frac{\alpha(s_i)h}{2\varepsilon}\right)}}{Sinh\left(\frac{\xi_1(s_i)h}{2}\right)Sinh\left(\frac{\xi_2(s_2)h}{2}\right)}\right) \text{for } i = 1, 2, \ldots, p \tag{14a}$$

$$\chi_i(\rho) = -\frac{\beta(s_i)h}{4\left[\frac{1}{\rho} + \frac{\alpha_i}{2}\right]}\left(\frac{e^{-\left(\frac{\alpha(s_i)h}{2\varepsilon}\right)}}{Sinh\left(\frac{\xi_1(s_i)h}{2}\right)Sinh\left(\frac{\xi_2(s_2)h}{2}\right)}\right) \text{for } i = p+1, p+2, \ldots, N\text{-}1 \tag{14b}$$

and

$$\tau_i(\rho) = \frac{\beta(s_i)h}{2\alpha(s_i)}\left(Coth\left(\frac{\xi_1(s_i)h}{2}\right) + Coth\left(\frac{\xi_2(s_i)h}{2}\right)\right) \quad \text{for } i = 1, 2, \ldots N\text{-}1. \tag{15}$$

The Eq. (11) and Eq. (12) reduces to below tridiagonal systems of equations

$$\left(\frac{\chi_i}{h^2}\left[\varepsilon - \frac{\alpha_i h}{2}\right]\right)\theta_{i-1} - \left(\left(\frac{2\sigma_i}{h^2}\left[\varepsilon - \frac{\alpha_i h}{2}\right]\right) + \frac{\alpha_i\tau_i}{h} + \beta_i\right)\theta_i + \left(\frac{\sigma_i}{h^2}\left[\varepsilon - \frac{\alpha_i h}{2}\right] + \frac{\alpha_i\tau_i}{h}\right)\theta_{i+1} = F_i \tag{16}$$

for $i = 1, 2, \ldots, p$

$$\left(\frac{\chi_i}{h^2}\left[\varepsilon+\frac{\alpha_i h}{2}\right]-\frac{\alpha_i\tau_i}{h}\right)\theta_{i-1}-\left(\left(\frac{2\chi_i}{h^2}\left[\varepsilon-\frac{\alpha_i h}{2}\right]\right)-\frac{\alpha_i\tau_i}{h}+\beta_i\right)\theta_i+\left(\frac{\chi_i}{h^2}\left[\varepsilon+\frac{\alpha_i h}{2}\right]\right)\theta_{i+1}=F_i \tag{17}$$

$i = p+1, p+2, \ldots N-1$

To solve the above system of equations, Thomas algorithm is used with the boundary conditions Eq. (13).

4. CONVERGENCE ANALYSIS

The matrix vector form of tridiagonal system Eq. (16) can be expressed as

$$MZ = R \tag{18}$$

where $M = \left(p_{ij}\right)$, $1 \le i \le p$-1 and $1 \le j \le p$-1, such that

$$p_{i\ i-1}=\frac{\chi_i}{h^2}\left[\varepsilon-\frac{\alpha_i h}{2}\right],\ p_{i\ i}=-\left(\left(\frac{2\chi_i}{h^2}\left[\varepsilon-\frac{\alpha_i h}{2}\right]\right)+\frac{\alpha_i\tau_i}{h}+\beta_i\right),\ p_{i\ i+1}=\left(\frac{\varphi_i}{h^2}\left[\varepsilon+\frac{\alpha_i h}{2}\right]\right)+\frac{\alpha_i\tau_i}{h}$$

and $R=\left(F_i\right)$ is a column matrix for $i=1,\ 2,\ldots,\ p\text{-}1$ with local truncation error

$$T_i(h)=h\left(\frac{\tau\alpha_i}{2}\right)\theta_i''+h^2\left(\frac{\tau\alpha_i}{6}\theta_i'''+\frac{\chi_i}{12}\left[\varepsilon-\frac{\alpha_i h}{2}\right]\theta_i''''\right)+O(h^3) \tag{19}$$

i.e., truncation error in the scheme is of $O\left(h\right)$.

The matrix-vector form of tridiagonal system Eq. (17) can be expressed as

$$MZ = R \tag{20}$$

where $M = \left(p_{i\ j}\right)$, $p+1 \le i, j \le N$-1, with

$$p_{i\ i-1}=\left(\frac{\chi_i}{h^2}\left[\varepsilon+\frac{\alpha_i h}{2}\right]-\frac{\alpha_i\tau_i}{h}\right),\ p_{i\ i}=-\left(\left(\frac{2\chi_i}{h^2}\left[\varepsilon-\frac{\alpha_i h}{2}\right]\right)-\frac{\alpha_i\tau_i}{h}+\alpha_i\right),\ p_{i\ i+1}=\left(\frac{\chi_i}{h^2}\left[\varepsilon+\frac{\alpha_i h}{2}\right]\right)$$

and $\mathrm{R}=\left(F_i\right)$ is a column matrix for $i=p+1,\ p+2,\ldots,N\text{-}1$ with local truncation error

$$T_i(h)=h\left(-\frac{\tau\alpha_i}{2}\right)\theta_i''+h^2\left(\frac{\tau\alpha_i}{6}\theta_i'''+\frac{\chi_i}{12}\left[\varepsilon+\frac{\alpha_i h}{2}\right]\theta_i''''\right)+O(h^3)$$

We also have $$M\,\overline{Z}-T_E(h)=R \tag{21}$$

where $\bar{Z} = \left(\bar{\theta}_0, \bar{\theta}_1, ..., \bar{\theta}_N\right)^t$ symbolizes the exact solution and the local truncation error is denoted by $T_E\left(h\right) = \left(T_0(h), T_1(h), ..., T_N(h)\right)^t$.

Using Eq. (18), Eq. (20) and Eq. (21), we get

$$M\left(\bar{Z} - Z\right) = T_E(h) \tag{22}$$

Thus the error equation is

$$ME = T_E(h) \tag{23}$$

Here $E = \bar{Z} - Z = \left(e_0, e_1, e_2, ..., e_N\right)^t$.

Clearly, we have

$$S_i = \sum_{j=1}^{N-1} p_{i\,j} = -\frac{\chi}{h^2}\left(\varepsilon - \frac{\alpha_i h}{2}\right) + \beta_i \quad \text{for } i = 1$$

$$S_i = \sum_{j=1}^{N-1} p_{i\,j} = 2\beta_i = \tilde{\beta}_{i_0} \quad \text{for } i = 2, 3, ..., N-2$$

$$S_i = \sum_{j=1}^{N-1} p_{i\,j} = -\frac{\chi}{h^2}\left(\varepsilon + \frac{\alpha_i h}{2}\right) + \beta_i \quad \text{for } i = N-1$$

Since $0 < \varepsilon << 1$, the matrix M is monotone and irreducible. Then, M^{-1} exists and its entries are non-negative.

Hence using Eq. (23), we get

$$E = M^{-1} T_E(h) \tag{24}$$

and
$$\|E\| \le \left\|M^{-1}\right\| . \left\|T_E(h)\right\| \tag{25}$$

Let $\bar{p}_{k,i}$ be the $\left(k, i\right)^{th}$ element of M^{-1}. Since $\bar{p}_{k,i} \ge 0$, using the matrix theory, we have

$$\sum_{i=1}^{N-1} \bar{p}_{k,i}\, S_i = 1 \quad , \quad k = 1, 2, ..., N\text{-}1 \tag{26}$$

Therefore,

$$\sum_{i=1}^{N\text{-}1} \bar{p}_{k,i} \le \frac{1}{\min\limits_{1 \le i \le N-1} S_i} = \frac{1}{\tilde{\beta}_{i_o}} \le \frac{1}{\left|\tilde{\beta}_{i_o}\right|} \tag{27}$$

for some $1 < i_o < N$ and $\tilde{\beta}_{i_o} = 2\beta_i$.

We define $\left\|M^{-1}\right\| = \max_{1\le k\le N-1} \sum_{i=1}^{N-1} \left|\bar{p}_{ki}\right|$ and $\left\|T_E\left(h\right)\right\| = \max_{1\le i\le N-1} \left|T_i\left(h\right)\right|$.

Using Eq. (19), Eq. (24) and Eq. (27), we get

$$e_j = \sum_{i=1}^{N-1} \bar{p}_{ki} \ T_{\mathrm{i}}(h), \quad j = 1,\ 2,\ 3,\ \ldots,\ N\text{-}1$$

implies
$$e_j \le \frac{Kh}{\left|\beta_i\right|} \ , \ j = 1,2,3,\ldots,N\text{-}1 \tag{28}$$

where $K = \frac{\tau\alpha_i\theta_i''}{4}$ is a constant.

Hence, using Eq. (28), we have $\quad \|E\| = O(h)$.

Therefore, the proposed scheme has first order convergent on uniform mesh.

5. NUMERICAL EXPERIMENTS

In order to check the efficiency of proposed scheme computationally, six problems are considered, chosen from the literature. Since, the exact solution of the considered problems is given, so the maximum absolute errors are estimated by using $E_{N,\varepsilon} = \max_{0\le i\le N} \left|\theta\left(s_i\right) - \theta_i\right|$ where the exact solution is $\theta\left(s_i\right)$ and the computed solution is θ_i.

Example 1. $\varepsilon\theta''(s) + \theta'(s) + 2\theta(s-\delta) - 3\theta\left(s\right) = 0$ with $\theta(0) = 1$, $-\delta \le s \le 0$ and $\theta(1) = 1$, $1 \le s \le 1+\eta$.

Example 2. $\varepsilon\theta''(s) + \theta'(s) - 3\theta\left(s\right) + 2\theta\left(s+\eta\right) = 0$ with $\theta(0) = 1$, $-\delta \le s \le 0$ and $\theta(1) = 1$, $1 \le s \le 1+\eta$.

Example 3. $\varepsilon\theta''(s) + \theta'(s) - 2\theta(s-\delta) - 5\theta\left(s\right) + \theta\left(s+\eta\right) = 0$ with $\theta(0) = 1$, $-\delta \le s \le 0$ and $\theta(1) = 1, \quad 1 \le z \le 1+\eta$.

Example 4. $\varepsilon\theta''(s) - \theta'(s) - 2\theta(s-\delta) + \theta\left(s\right) = 0$ with $\theta(0) = 1$, $-\delta \le s \le 0$ and $\theta(1) = -1$,

$1 \le s \le 1+\eta$.

Example 5. $\varepsilon\theta''(s)-\theta'(s)+\theta(s)-2\theta(s+\eta)=0$ with $\theta(0)=1$, $-\delta \le s \le 0$ and $\theta(1)=-1$, $1 \le s \le 1+\eta$.

Example 6. $\varepsilon\theta''(s)-\theta'(s)-2\theta(s-\delta)+\theta(s)-2\theta(s+\eta)=0$ with $\theta(0)=1$, $-\delta \le s \le 0$ and $\theta(1)=-1$, $1 \le s \le 1+\eta$.

6. DISCUSSIONS AND CONCLUSION

To solve differential-difference equation having layer behaviour, a difference scheme with modified finite differences and multiple fitting parameters is introduced. Initially, the expansion of Taylor series used to minimize the given problem to differential equation with layer structure. Using modified finite differences of the first order derivatives, the numerical scheme is derived. Then introduced the fitting parameters at the convective and diffusion terms to handle the small values of the perturbation and to get accurate solution of the problem. The method is used with various examples of left layer and right layer, with distinct values of the delay parameter δ, advanced parameter η and the perturbation ε. The outcomes of the computations were compared and tabulated. The effects of the delay and the advanced parameters have been examined via graphs on the problem solutions. When the solution exhibits the layer on the left-end, the effect of delay or advanced parameters in the layer domain is observed to be negligible, whereas in the outer region it is significant. The variation of the advanced parameter influences the solution in the same way that the change in delay has an influence but reverse effect (see the Figures 1-4). In layer region as well as external region, there is an impact when the problem shows right-end layer on the region with respect to the delay or advanced variations. We also observed that the layer thickness decreases as the delay parameter increases while the advanced parameter increases the layer thickness (Figures 5-8). Results show that the proposed scheme is very well suited to the exact solution.

Table 1. Maximum absolute errors in the solution of Example 1 for $\varepsilon = 0.1$ and for different values of δ.

$N \rightarrow$	8	32	128	512
$\delta \downarrow$		Present method		
0.00	7.6406e-13	6.7724e-15	9.9920e-16	3.3307e-16
0.05	7.5151e-13	2.5535e-15	3.3307e-15	2.2204e-16
0.09	7.4929e-13	2.6645e-15	3.5527e-15	3.3307e-16
		Results in [17]		
0.00	0.03998462	2.3211e-003	1.4207e-004	8.8822e-006
0.05	0.04117834	2.3918e-003	1.4572e-004	9.0930e-006
0.09	0.04193952	2.4339e-003	1.4773e-004	9.2252e-006
		Results in [8]		
0.00	0.09907804	0.03700736	0.00954678	0.00214501
0.05	0.09659609	0.03640566	0.00924661	0.00202998
0.09	0.09277401	0.03556652	0.00895172	0.00192488

Table 2. Maximum absolute errors in the solution of Example 2 for $\varepsilon = 0.1$ and for different values of η.

$N \rightarrow$	8	32	128	512
$\eta \downarrow$		Present Method		
0.00	7.6406e-13	6.7724e-15	9.9920e-16	3.3307e-16
0.05	3.0642e-14	4.2188e-14	2.3315e-15	3.3307e-16
0.09	1.6986e-14	3.3862e-14	1.6653e-15	3.3307e-16
		Results in [17]		
0.00	3.5917e-003	1.9114e-004	1.1686e-005	7.2967e-007
0.05	3.3119e-003	1.8015e-004	1.0902e-005	6.8007e-007
0.09	3.0919e-003	1.7198e-004	1.0756e-005	6.7236e-007
		Results [8]		
0.00	0.09907804	0.03700736	0.00954678	0.00214501
0.05	0.09977501	0.03727087	0.00979659	0.00224472
0.09	0.10031348	0.03723863	0.00996284	0.00458698

Table 3. Maximum absolute errors in the solution of Example 3 for $\varepsilon=0.1$ and for different values of η and δ.

	N=8	N = 32	N =128	N =512
$\delta \downarrow \eta = 0.5\varepsilon$		Present method		
0.00	1.1213e-14	6.3283e-15	3.3307e-16	2.2204e-16
0.05	1.2490e-14	4.1078e-15	3.3307e-16	2.2204e-16
0.09	1.8785e-13	1.0603e-14	4.4409e-16	1.1102e-16
$\eta \downarrow \delta = 0.5\varepsilon$				
0.00	1.7564e-13	1.2546e-14	2.2204e-16	1.1102e-16
0.05	1.2490e-14	4.1078e-15	3.3307e-16	2.2204e-16
0.09	1.8030e-13	1.8874e-15	2.2204e-16	2.2204e-16
$\delta \downarrow \eta = 0.5\varepsilon$		Results in [17]		
0.00	0.03998462	0.00232117	0.00014207	8.8822e-006
0.05	0.04117834	0.00239180	0.00014572	9.0930e-006
0.09	0.04193952	0.00243399	0.00014773	9.2252e-006
$\eta \downarrow \delta = 0.5\varepsilon$				
0.00	0.04061578	0.00235898	0.00014404	8.9940e-006
0.05	0.04117834	0.00239180	0.00014572	9.0930e-006
0.09	0.04157997	0.00241448	0.00014683	9.1629e-006
$\delta \downarrow \eta = 0.5\varepsilon$		Results in [8]		
0.00	0.09190267	0.03453494	0.01164358	0.00300463
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.11018870	0.04110846	0.01400144	0.00362925
$\eta \downarrow \delta = 0.5\varepsilon$				
0.00	0.09720079	0.03640446	0.01229476	0.0031786
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.10632014	0.03965833	0.01348348	0.00349050

Table 4. Maximum absolute errors in the solution of Example 4 for ε =0.1 and for different values of δ .

$N \rightarrow$	8	32	128	512
$\delta \downarrow$		Present method		
0.00	7.4529e-13	2.3315e-15	8.3267e-16	3.3307e-16
0.05	7.5462e-13	1.5543e-15	2.8866e-15	1.1102e-16
0.09	7.5784e-13	.6645e-15	3.3307e-15	2.7756e-16
		Results in [17]		
0.00	0.01729728	8.9760e-004	5.5488e-005	3.4650e-006
0.05	0.01614989	8.5195e-004	5.3014e-005	3.3311e-006
0.09	0.01511535	8.1843e-004	5.0710e-005	3.1680e-006
		Results in [8]		
0.00	0.07847490	0.04678972	0.01727912	0.00443086
0.05	0.09222560	0.03828329	0.01487799	0.00380679
0.09	0.10509460	0.03149275	0.01299340	0.00331935

Table 5. Maximum absolute errors in the solution of Example 5 for ε =0.1 and for different values of η.

$N \rightarrow$	8	32	128	512
$\eta \downarrow$		Present method		
0.00	7.4529e-13	2.3315e-15	8.3267e-16	3.3307e-16
0.05	3.9857e-14	4.5963e-14	1.7764e-15	3.3307e-16
0.09	1.3545e-14	4.5852e-14	1.5543e-15	2.2204e-16
		Results in [17]		
0.00	0.01729728	9.3663e-004	5.7581e-005	3.5951e-006
0.05	0.01829655	8.5195e-004	5.3014e-005	3.3311e-006
0.09	0.01900051	9.6037e-004	5.9020e-005	3.6850e-006
		Results in [8]		
0.00	0.07847490	0.04678972	0.01727912	0.00443086
0.05	0.06834579	0.05516436	0.01972508	0.00506769
0.09	0.08328237	0.06168267	0.02169662	0.00558451

Table 6. Maximum absolute errors in the solution of Example 6 for ε=0.1 and for different values of η and δ .

$N \rightarrow$	8	32	128	512
$\delta \downarrow \eta = 0.5\varepsilon$		Present method		
0.00	2.2204e-16	3.6082e-16	7.3830e-15	2.2649e-14
0.05	1.9429e-16	3.7470e-16	8.8818e-16	3.5127e-13
0.09	1.3878e-16	1.6653e-15	2.2538e-14	1.1935e-14
$\eta \downarrow \delta = 0.5\varepsilon$				
0.00	2.2204e-16	1.9429e-15	2.9976e-15	1.5987e-14
0.05	1.9429e-16	3.7470e-16	8.8818e-16	3.5127e-13
0.09	3.3307e-16	5.5511e-16	2.2593e-14	3.5461e-13
$\delta \downarrow \eta = 0.5\varepsilon$		Results in [8]		
0.00	0.09930002	0.03685072	0.01331683	0.00342882
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.10044578	0.02850398	0.01038902	0.00266379
$\eta \downarrow \delta = 0.5\varepsilon$				
0.00	0.10055269	0.02759534	0.01007834	0.00258299
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.09944067	0.03591410	0.0129736	0.00334044

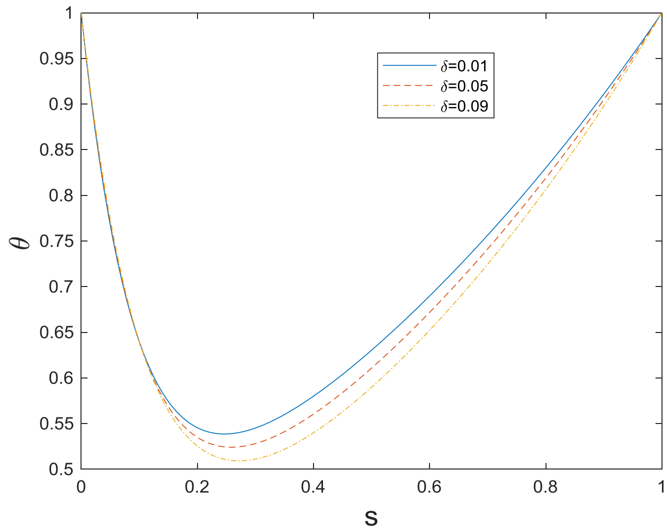

Figure 1. Numerical solution in Example 1 for $\varepsilon = 0.1$with different values of δ.

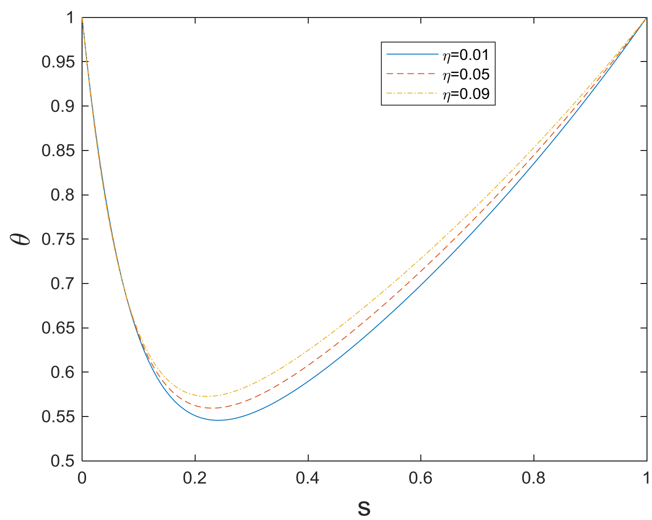

Figure 2. Numerical solution in Example 2 for$\varepsilon = 0.1$ with different values of η.

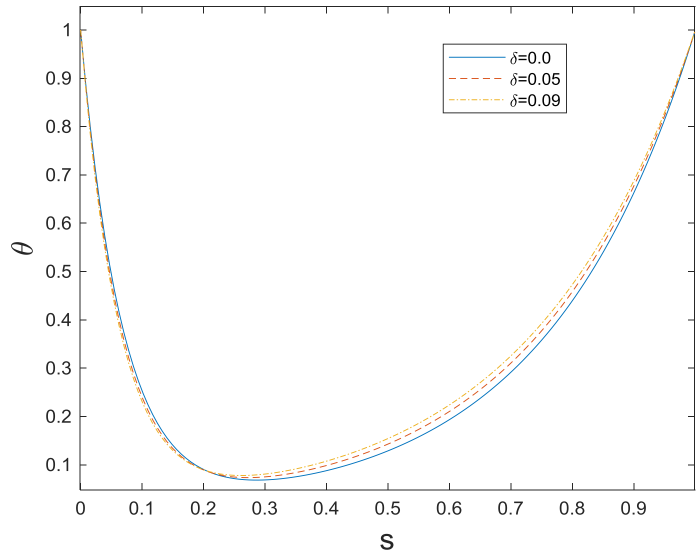

Figure 3. Numerical solution in Example 3 for$\varepsilon = 0.1$with different values of δ.

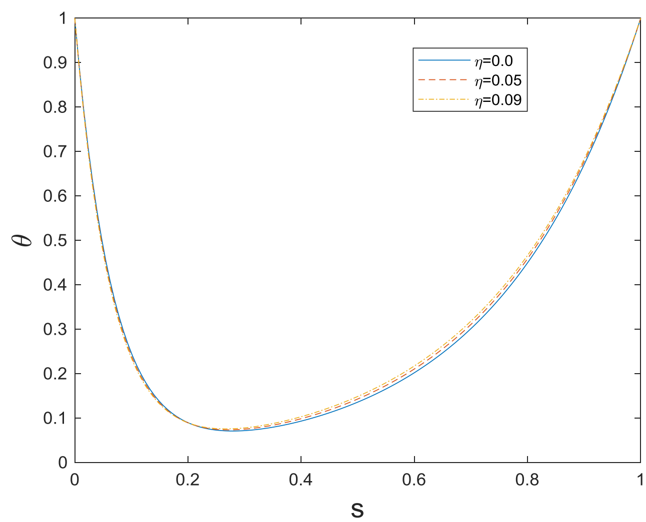

Figure 4. Numerical solution in Example 3 for$\varepsilon = 0.1$with different values of η.

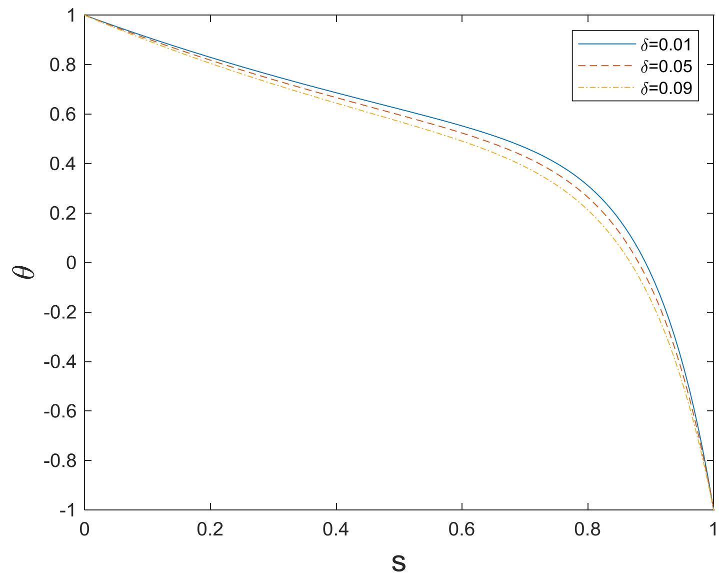

Figure 5. Numerical solution in Example 4 for$\varepsilon = 0.1$with different values of δ.

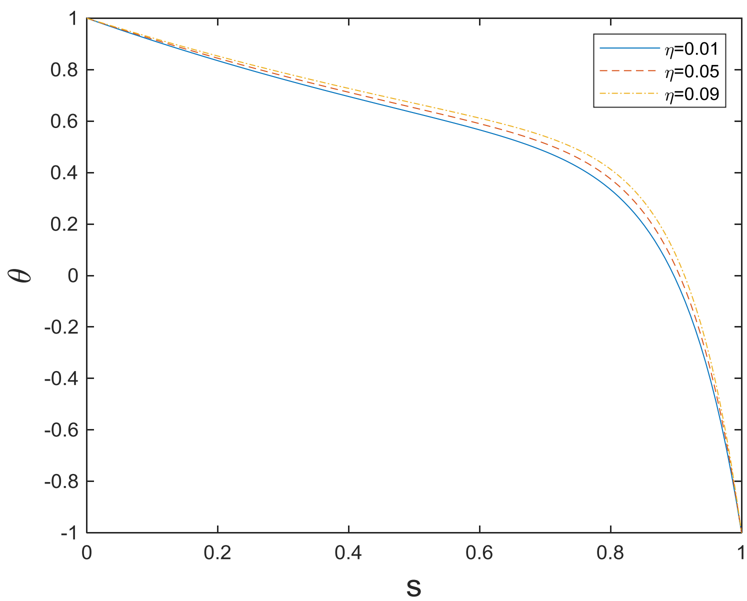

Figure 6. Numerical solution in Example 5 for $\varepsilon = 0.1$with different values of η.

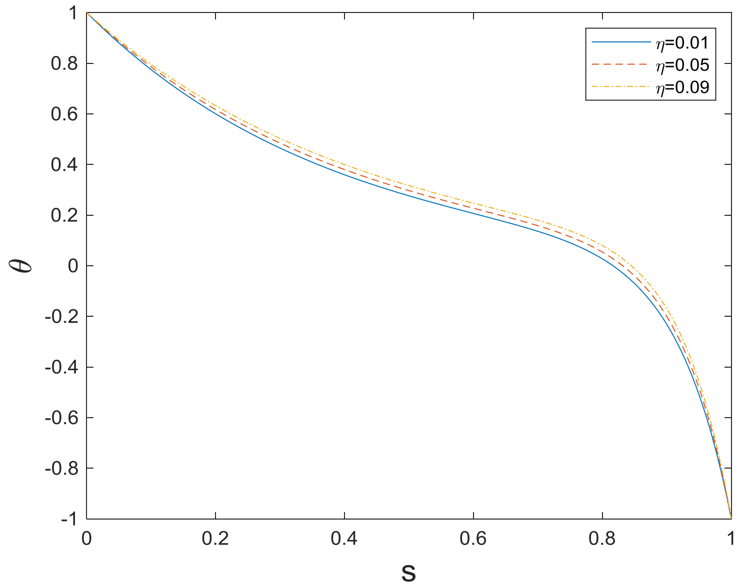

Figure 7. Numerical solution in Example 6 with $\varepsilon = 0.1$ and for different values of η.

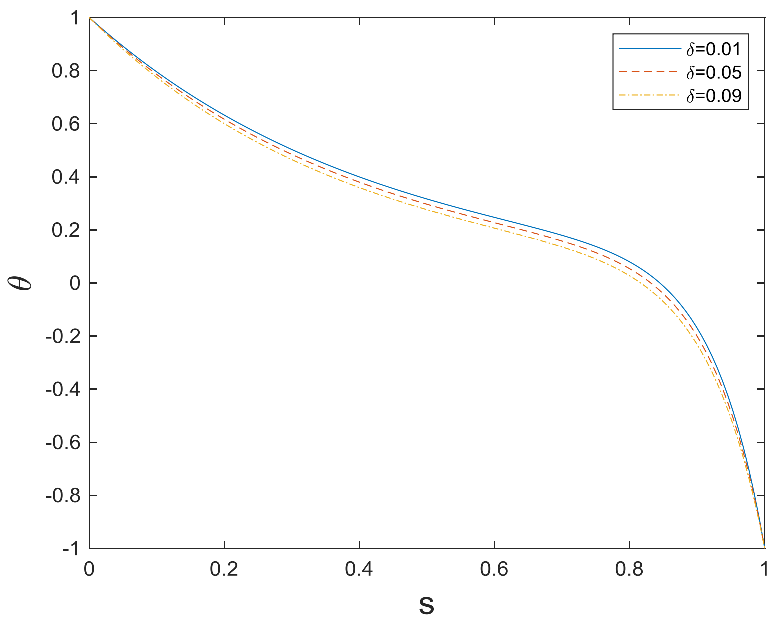

Figure 8. Numerical solution in Example 6 with $\varepsilon = 0.1$ and for different values of δ.

Conflict of Interests

The authors declare that there is no conflict of interests.

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