COSET CAYLEY DIGRAPH STRUCTURES

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Abstract. In this paper, we generalize the results in [9] to produce a new classes of Cayley digraph structures induced by groups.

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1. Introduction

A binary relation on a set \( V \) is a subset \( E \) of \( V \times V \). A digraph is a pair \((V,E)\) where \( V \) is a non empty set (called vertex set) and \( E \) is a binary relation on \( V \). The elements of \( E \) are called edges. Let \( V \) be a non empty set and let \( E_1, E_2, \ldots, E_n \) be mutually disjoint binary relations on \( V \). Then the \((n+1)\)-tuple \( G = (V; E_1, E_2, \ldots, E_n) \) is called a digraph structure[9]. The elements of \( V \) are called vertices and the elements of \( E_i \) are called \( E_i \)-edges. The following definition were introduced in [9].

A digraph structure \((V; E_1, E_2, \ldots, E_n)\) is called (i) \( E_1 E_2 \cdots E_n \)-trivial if \( E_i = \emptyset \) for all \( i \), and \( E_i \)-trivial if \( E_i = \emptyset \) (ii) \( E_1 E_2 \cdots E_n \)-reflexive if for all \( x \in G, (x,x) \in E_i \) for some \( i \), and \( E_i \)-reflexive if for all \( x \in V, (x,x) \in E_i \) (iii) \( E_1 E_2 \cdots E_n \)-symmetric if \( E_i = E_i^{-1} \) for all \( i \), and \( E_i \)-symmetric if \( E_i = E_i^{-1} \) (iv) \( E_1 E_2 \cdots E_n \)-anti symmetric, if \((x,y) \in E_i \) and

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(y, x) ∈ E_i implies x = y for all i, and E_i- anti symmetric if (x, y) ∈ E_i and (y, x) ∈ E_i implies x = y (v) E_1E_2⋯E_n- transitive if for every i and j, E_i ∩ E_j ⊆ E_k for some k, and E_i transitive if E_i ∩ E_i ⊆ E_i (vi) an E_1E_2⋯E_n- hasse diagram if for every positive integer n ≥ 2 and every v_0, v_1, ..., v_n of V, (v_i, v_{i+1}) ∈ ∪E_i for all i = 0, 1, 2, ..., n − 1, implies (v_0, v_n) ∉ E_i for all i, and E_i- hasse diagram if for every positive integer n ≥ 2 and every v_0, v_1, ..., v_n of V, (v_i, v_{i+1}) ∈ E_i for all i = 0, 1, 2, ..., n − 1, implies (v_0, v_n) ∉ E_i, (viii)E_1E_2⋯E_n- complete if ∪E_i = V × V, and E_i complete if E_i = V × V.

A digraph structure (V; E_1, E_2, ..., E_n) is called (i) an E_1E_2⋯E_n- quasi ordered set if it is both E_1E_2⋯E_n- reflexive and E_1E_2⋯E_n- transitive (ii)an E_1E_2⋯E_n - partially ordered set if it is E_1E_2⋯E_n- anti symmetric and E_1E_2⋯E_n- quasi ordered set. Similarly, we can define E_i quasi ordered set and E_i partially ordered set as in the case of ordinary relations.

An E_1E_2⋯E_n- walk of length k in a digraph structure is an alternating sequence W = v_0, e_0, v_1, ..., e_{k−1}, v_k, where e_i = (v_i, v_{i+1}) ∈ ∪E_i. An E_1E_2⋯E_n -walk W is called a E_1E_2⋯E_n- path if all the internal vertices are distinct. We use notation (v_0, v_1, v_2, ..., v_n) for the E_1E_2⋯E_n - path W. As in digraphs, we define E_i- walk and E_i- path. For example, an E_i- path between two vertices u and v consists of only E_i- edges.

A digraph structure (V; E_1, E_2, ..., E_n) is called (i) E_1E_2⋯E_n- connected if there exits at least one E_1E_2⋯E_n- path from v to u for all u, v ∈ V, (ii)E_1E_2⋯E_n- quasi connected if for every pair of vertices x, y there is a vertex z such that there is an E_1E_2⋯E_n-path from z to x and an E_1E_2⋯E_n-path from z to y, (iii) E_1E_2⋯E_n- locally connected iff for every pair of vertices u, v ∈ V there is an E_1E_2⋯E_n - path from v to u whenever there is an E_1E_2⋯E_n - path from u to v and (iv) E_1E_2⋯E_n- semi connected for every pair of vertices u, v, there is an E_1E_2⋯E_n- path from u to v or an E_1E_2⋯E_n- path from v to u.

A digraph structure (V; E_1, E_2, ..., E_n) is called E_i -connected if there exits at least one E_i path from v to u for all u, v ∈ V. Similarly we can define E_i quasi connected, E_i-locally connected and E_i - semi connected digraph structures.
The $E_1E_2 \cdots E_n$- distance between two vertices $x$ and $y$ in a digraph structure $G$ is the length of the shortest $E_1E_2 \cdots E_n$-path between $x$ and $y$, denoted by $d_{1,2,3,\ldots,n}(x,y)$. Let $G = (V; E_1, E_2, \ldots, E_n)$ be a finite $E_1E_2 \cdots E_n$-connected digraph structure. Then the $E_1E_2 \cdots E_n$ diameter of $G$ is defined as $d(G) = \max_{x,y \in G}\{d_{1,2,3,\ldots,n}(x,y)\}$. Similarly we can define $E_i$ distance and $E_i$ diameter as in digraphs.

Two digraph structures $(V_1; E_1, E_2, \ldots, E_n)$ and $(V_2; R_1, R_2, \ldots, R_m)$ are said to be isomorphic if (i) $m = n$ and (ii) there exits a bijective function $f : V_1 \mapsto V_2$ such that $(x, y) \in E_i \iff (f(x), f(y)) \in R_i$. This concept of isomorphism is a generalization of isomorphism between two digraphs. An isomorphism of a digraph structure onto itself is called an automorphism. A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is said to be vertex-transitive if, given any two vertices $a$ and $b$ of $V$, there is some digraph automorphism $f : V \to V$ such that $f(a) = b$. Let $(V; E_1, E_2, \ldots, E_n)$ be a digraph structure and let $v \in V$. Then the $E_1E_2 \cdots E_n$-out-degree of $u$ is $|\{v \in V : (u,v) \in \cup E_i\}|$ and $E_1E_2 \cdots E_n$-in-degree of $u$ is $|\{v \in V : (v,u) \in \cup E_i\}|$. Similarly we can define the $E_i$-out-degree and $E_i$-in-degree as in the case of digraphs.

Let $(V_1; E_1, E_2, \ldots, E_n)$ be a digraph structure. A vertex $v \in G$ is called an $E_1E_2 \cdots E_n$-source if for every vertex $x \in G$, there is an $E_1E_2 \cdots E_n$-path from $v$ to $x$. Similarly a vertex $u \in G$ is called an $E_1E_2 \cdots E_n$-sink if for every vertex $y \in G$ there is an $E_1E_2 \cdots E_n$-path from $y$ to $u$. As in digraphs, we define $E_i$-source and $E_i$-sink. Let $(V_1; E_1, E_2, \ldots, E_n)$ be a digraph structure and let $v \in G$. Then the $E_1E_2 \cdots E_n$-reachable set $R_{1,2,3,\ldots,n}(u)$ is $\{x \in G : \text{there is an } E_1E_2 \cdots E_n\text{-path from } u \text{ to } x\}$. Similarly, the $E_1E_2 \cdots E_n$-antecedent set $Q_{1,2,\ldots,n}(u)$ is defined as

$$Q_{1,2,\ldots,n}(u) = \{x \in G : \text{there is an } E_1E_2 \cdots E_n\text{-path from } x \text{ to } u\}.$$  

As in the case of digraphs, we can define the $E_i$-reachable set and $E_i$-antecedent set of a vertex.

2. Coset Cayley Digraph Structures

In [9] the authors introduced a class of Cayley digraph structures induced by groups. In this paper, we introduce a class of coset Cayley digraph structures induced by groups
and prove that every vertex transitive digraph structure is isomorphic to the coset Cayley digraph structure. These class of Cayley digraphs structures can be viewed as a generalization of those obtained in [9].

We start with the following definition:

**Definition 2.1.** Let $G$ be a group and $S_1, S_2, \ldots, S_n$ be mutually disjoint subsets of $G$ and $H$ be a subgroup of $G$. Then coset Cayley digraph structure of $G$ with respect to $S_1, S_2, \ldots, S_n$ is defined as the digraph structure $(G/H; E_1, E_2, \ldots, E_n)$, where

$$E_i = \{(xH, yH) : x^{-1}y \in HS_i H\}.$$

The sets $S_1, S_2, \ldots, S_n$ are called connection sets of $(G/H; E_1, E_2, \ldots, E_n)$. We denote the coset Cayley digraph structure of $G$ with respect to $S_1, S_2, \ldots, S_n$ by

$$\mathcal{C} = \text{Cay}(G/H; HS_1 H, HS_2 H, \ldots, HS_n H).$$

In this paper, we may use the following notations: Let $\mathcal{C}$ be a coset Cayley digraph structure induced by the group $G$ with respect to the connection sets $S_1, S_2, \ldots, S_n$.

(1) Let $A_k$ be the union of set of all $k$ products of the form $(HS_{i1} H)(HS_{i2} H) \cdots (HS_{ik} H)$ from the set $\{HS_1 H, HS_2 H, \ldots, HS_n H\}$. Then $\bigcup_k A_k$ is denoted by $[HSH]$.

(2) Let $A_k^{-1}$ be the union of set of all $k$ products of the form:

$$(HS_{i1}^{-1} H)(HS_{i2}^{-1} H) \cdots (HS_{ik}^{-1} H).$$

Then $\bigcup_k A_k^{-1}$ is denoted by $[HS^{-1} H]$.

(3) Let $A$ be a subset of a group $G$, then the semigroup generated by $A$ is denoted by $< A >$.

### 2.1 Main Theorems

**Theorem 2.1.1** If $G$ is a group and let $S_1, S_2, \ldots, S_n$ are mutually disjoint subsets of $G$ and $H$ is a subgroup of $G$, then the coset Cayley digraph structure $\mathcal{C}$ is vertex transitive.

**Proof.** To see that $\text{Cay}(G/H; HS_1 H, HS_2 H, \ldots, HS_n H)$ is a vertex transitive digraph structure, we first need only show that $E_i$’s are well defined. Let $x, y, x', y'$ be any four elements of $G$ with $xH = x'H$ and $yH = y'H$. Then $x = x'h_1$ and $y = y'h_2$ for some
$h_1, h_2 \in H$. Observe that

$$(xH, yH) \in E_i \iff x^{-1}y \in HS_iH$$

$$\iff (x'h_1)^{-1}(y'h_2) \in HS_iH$$

$$\iff h_1^{-1}(x')^{-1}y'h_2 \in HS_iH$$

$$\iff (x')^{-1}y' \in HS_iH$$

$$\iff (x'H, y'H) \in HS_iH.$$  

Hence each $E_i$'s are well defined and hence Cay$(G/H; HS_1H, HS_2H, \ldots, HS_nH)$ is a digraph structure. Let $aH$ and $bH$ be any two arbitrary elements in $G/H$. Define a mapping $\varphi : G \mapsto G$ by

$$\varphi(xH) = ba^{-1}xH \text{ for all } xH \in G/H.$$  

This mapping defines a permutation of the vertices of Cay$(G/H; HS_1H, HS_2H, \ldots, HS_nH)$. It is also an automorphism. Note that

$$(xH, yH) \in E_i \iff x^{-1}y \in HS_iH$$

$$\iff (ba^{-1}x)^{-1}(ba^{-1}y) \in HS_iH$$

$$\iff (ba^{-1}xH, ba^{-1}yH) \in E_i$$

$$\iff (\varphi(xH), \varphi(yH)) \in E_i.$$  

Also we note that

$$\varphi(aH) = ba^{-1}aH = bH.$$  

Hence Cay$(G/H; HS_1H, HS_2H, \ldots, HS_nH)$ is vertex transitive digraph structure.

**Theorem 2.1.2**

Let $(V; W_1, W_2, \cdots, W_n)$ be any vertex transitive digraph structure such that $|V| \geq n$. Then $(V; W_1, W_2, \cdots, W_n)$ is isomorphic to Cay$(G/H; HS_1H, HS_2H, \ldots, HS_nH)$.  

Proof. Let $G$ be the automorphism group of the digraph structure $(V; W_1, W_2, \ldots, W_n)$. Let $q_1, q_2, \ldots, q_n$ be fixed elements in $V$. For $i = 1, 2, \ldots, n$, define the following:

$$H_i := \{ \theta \in G : \theta(q_i) = q_i \},$$
$$S_i := \{ \theta \in G : (q_i, \theta(q_i)) \in W_i \}.$$

Note that $H = \cap_{i=1}^n H_i$ is a subgroup of $G$. Construct the Cayley digraph structure $\text{Cay}(G/H; HS_1H, HS_2H, \ldots, HS_nH)$ as in theorem 2.2.1.

Define a map $\varphi : G/H \mapsto V$ by

$$(xH)\varphi = x(q_i) \text{ for all } xH \in G/H.$$  

where $q_i$ is a fixed element in the set $\{q_1, q_2, \ldots, q_n\}$.

(i) $\varphi$ is well defined:
Let $xH = yH$. Then $y = xh_1$, for some $h_1 \in H$. Observe that

$$\varphi(yH) = y(q_i)$$
$$= (xh_1)(q_i)$$
$$= x[h_1(q_i)]$$
$$= x(q_i)$$
$$= \varphi(xH)$$

(ii) $\varphi$ is one to one:

$$\varphi(xH) = \varphi(yH) \Leftrightarrow x(q_i) = y(q_i)$$
$$\Leftrightarrow y^{-1}x(q_i) = q_i$$
$$\Leftrightarrow y^{-1}x \in H$$
$$\Leftrightarrow xH = yH.$$

(iii) $\varphi$ is onto:
Let $v$ be any element in $V$. Since $(V; W_1, W_2, \ldots, W_n)$ is vertex transitive, there exists an
automorphism $\theta$ such that $\theta(v) = q_i$. This implies that $v = \theta^{-1}(q_i)$. That is, $v = \varphi(\theta^{-1}H)$.

(iv) $\varphi$ preserves adjacency relation:

Observe that

$$(xH, yH) \in E_i \iff x^{-1}y \in HS_iH$$

$\iff x^{-1}y = h_1s_ih_2$

$\iff h_1^{-1}x^{-1}y h_2^{-1} = s_i \in S_i$

$\iff (q_i, (h_1^{-1}x^{-1}y h_2^{-1})(q_i)) \in W_i$

$\iff (h_1(q_i), x^{-1}y(q_i)) \in W_i$

$\iff (x(q_i), y(q_i)) \in W_i$

$\iff (\varphi(xH), \varphi(yH)) \in W_i$.

2.2 Corollaries

In this section we can prove many graph theoretic properties in terms of algebraic properties. Moreover, these results can be considered as the generalization of those obtained in [9].

Proposition 2.3 The coset Cayley graph structure $C$ is an $E_1E_2\cdots E_n$-trivial digraph structure $\iff S_i = \emptyset$ for all $i$.

Proof. By definition, $C$ is $E_1E_2\cdots E_n$- trivial $\iff E_i = \emptyset$ for all $i$. This implies that $S_i = \emptyset$ for all $i$.

Proposition 2.4 The coset Cayley graph structure $C$ is an $E_i$-trivial digraph structure $\iff S_i = \emptyset$.

Proposition 2.5 The coset Cayley graph structure $C$ is $E_1E_2\cdots E_n$- reflexive $\iff 1 \in S_i$ for some $i$. 
Proof. Assume that $C$ is an $E_1E_2 \cdots E_n$-reflexive digraph structure. Then for every $xH \in G/H$, $(xH,xH) \in E_i$ for some $i$. This implies that $1 \in HS_iH$ for some $i$. Conversely, assume that $1 \in S_i$ for some $i$. This implies for each $xH \in G/H$, $(xH,xH) \in E_i$ for some $i$. That is, $(xH,xH) \in \cup E_i$ for all $x \in G$.

**Proposition 2.6** The coset Cayley graph structure $C$ is $E_i$-reflexive $\iff 1 \in HS_iH$.

**Proposition 2.7** The coset cayley graph structure $C$ is $E_1E_2 \cdots E_n$-symmetric if and only if $HS_iH = HS_i^{-1}H$ for all $i$.

**Proof.** First, assume that $C$ is an $E_1E_2 \cdots E_n$-symmetric digraph structure. Let $a \in HS_iH$. Then $(H,aH) \in E_i$. Since $C$ is symmetric $(a,1) \in E_i$. This implies that $a^{-1} \in HS_iH$. That is $a \in HS_i^{-1}H$. Hence $HS_iH \subseteq HS_i^{-1}H$. Similarly, we can prove that $HS_i^{-1}H \subseteq HS_iH$.

Conversely, if $HS_iH = HS_i^{-1}H$, we can prove that $C$ is an $E_1E_2 \cdots E_n$-symmetric digraph structure.

**Proposition 2.8** $C$ is $E_i$ symmetric if and only if $HS_iH = HS_i^{-1}H$.

**Proposition 2.9** $C$ is an $E_1E_2 \cdots E_n$-transitive if and only if for every $i,j$, $HS_iHS_jH = HS_kH$ for all $k$.

**Proof.** First, assume that $C$ is $E_1E_2 \cdots E_n$-transitive. We will show that for all $(i,j)$, $HS_iHS_jH \subseteq HS_kH$ for some $k$. Let $x \in HS_iHS_jH = HS_iHHS_jH$. Then

$$x = z_1z_2 \text{ for some } z_1 \in HS_iH, z_2 \in HS_jH$$

This implies that $(H,z_1H) \in E_i$ and $(z_1H,z_1z_2H) \in E_j$. Since $C$ is $E_1E_2 \cdots E_n$-transitive, $(H,z_1z_2H) \in HS_kH$ for some $k$. That is $z_1z_2 \in HS_kH$. Hence $HS_iHS_jH \subseteq HS_kH$.

Conversely, assume that all $(i,j)$, $HS_iHS_jH \subseteq HS_kH$ for some $k$. We will show that $C$ is $E_1E_2 \cdots E_n$-transitive. Let $(H,xH) \in E_i$, $(xH,yH) \in E_j$. Then $x \in HS_iH$ and $x^{-1}y \in HS_jH$. This implies that $y = xx^{-1}y \in HS_iHS_jH$. Since $HS_iHS_jH \subseteq HS_kH$, we have $y \in HS_kH$. It follows that $(H,yH) \in E_k$. 
Proposition 2.10 $\mathcal{C}$ is an $E_1E_2 \cdots E_n$-transitive if and only if for every $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, k\}$, we have

$$(HS_{i_1}H)(HS_{i_2}H) \cdots (HS_{i_k}H) \subseteq (HS_{j_1}H)$$

for some $j_1$;

$$(HS_{i_1}H)(HS_{i_2}H) \cdots (HS_{i_{k-1}}H) \subseteq (HS_{j_2}H)$$

for some $j_2$;

$$\vdots$$

$$(HS_{i_1}H)(HS_{i_2}H) \subseteq (HS_{j_{k-1}}H)$$

for some $j_1$.

Proof. First, assume that $\mathcal{C}$ is an $E_1E_2 \cdots E_n$-transitive. Let $x \in (HS_{i_1}H)(HS_{i_2}H) \cdots (HS_{i_k}H)$. Then there exits $z_j \in (HS_{i_j}H)$, $j = 1, 2, \ldots, k$ such that $x = z_1z_2 \cdots z_k$. This implies that

$$(H, z_1H, z_1z_2H, z_1z_2z_3H, \ldots, z_1z_2z_3 \cdots z_kH)$$

is a path from 1 to $x$. Since $\mathcal{C}$ is an $E_1E_2 \cdots E_n$-transitive, we have

$$(H, z_1z_2z_3 \cdots z_kH) \in E_{j_1}$$

for some $j_1$,

$$(H, z_1z_2z_3 \cdots z_{k-1}H) \in E_{j_1}$$

for some $j_2$,

$$\vdots$$

$$(H, z_1z_2H) \in E_{j_{k-1}}$$

for some $j_{k-1}$.

The above statements tells us that

$$(HS_{i_1}H)(HS_{i_2}H) \cdots (HS_{i_k}H) \subseteq (HS_{j_1}H)$$

for some $j_1$;

$$(HS_{i_1}H)(HS_{i_2}H) \cdots (HS_{i_{k-1}}H) \subseteq (HS_{j_2}H)$$

for some $j_2$;

$$\vdots$$

$$(HS_{i_1}H)(HS_{i_2}H) \subseteq (HS_{j_{k-1}}H)$$

for some $j_{k-1}$.

Conversely, assume that the above conditions holds. Let $x_1H, x_2H, \ldots, x_nH \in G/H$ such that $(x_1H, x_2H) \in E_{i_1}, (x_2H, x_3H) \in E_{i_2}, \ldots, (x_{k-1}H, x_nH) \in E_{i_k}$. Then

$$x_2 = x_1t_1, x_3 = x_2t_2, \ldots, x_k = x_{k-1}t_{k-1}$$
for some \( t_i \in HS_i H \).

The above equations can be written as:

\[
x_3 = x_1(t_1t_2) \\
x_4 = x_1(t_1t_2t_3) \\
\vdots \\
x_{k-1} = x_1(t_1t_2 \cdots t_n)
\]

The above equations tell us that \((x_1 H, x_3 H) \in E_{i_1}, (x_1 H, x_4 H) \in E_{i_2}, \ldots, (x_1 H, x_{k-1} H) \in E_{i_{k-1}}\). This completes the proof.

**Proposition 2.11** \(\mathcal{C}\) is an \(E_i\)-transitive if and only if \((HS_i H)^n \subseteq (HS_i H)\) for \(n = 2, 3, \ldots, k\).

**Proposition 2.12** \(\mathcal{C}\) is \(E_1E_2 \cdots E_n\)-complete if and only if \(G = \cup HS_i H\).

**Proof.** Suppose \(\mathcal{C}\) is \(E_1E_2 \cdots E_n\) complete. Then for every \(xH \in G/H\), we have \((H, xH) \in \cup E_i\). This implies that \(x \in HS_i H\) for some \(i\). This implies that \(G = \cup HS_i H\).

Conversely, assume that \(G = \cup HS_i H\). Let \(xH\) and \(yH\) be two arbitrary elements in \(G/H\) such that \(y = xz\). Then \(z \in G\). This implies that \(z \in HS_i H\) for some \(i\). That is, \((H, zH) \in \cup E_i\). That is \((xH, xzH) = (xH, yH) \in \cup E_i\). This shows that \(\mathcal{C}\) is complete.

**Proposition 2.13** \(\mathcal{C}\) is \(E_i\) complete if and only if \(G = HS_i H\).

**Proposition 2.14** \(\mathcal{C}\) is \(E_1E_2 \cdots E_n\) connected if and only if \(G = [HS H]\).

**Proof.** Suppose \(\mathcal{C}\) is \(E_1E_2 \cdots E_n\) connected and let \(xH \in G/H\).

Let \((H, y_1 H, y_2 H, \ldots, y_n H, xH)\) be an \(E_1E_2 \cdots E_n\)-path leading from \(H\) to \(xH\). Then

\[
y_1 \in HS_{i_1} H \quad \text{for some } i_1; \\
y_1^{-1}y_2 \in HS_{i_2} H \quad \text{for some } i_2; \\
y_2^{-1}y_3 \in HS_{i_3} H \quad \text{for some } i_3; \\
\vdots \\
y_n^{-1}x \in HS_{i_{n+1}} H \quad \text{for some } i_{n+1}.
\]
Note that $x = y_1 y_1^{-1} y_2 y_2^{-1} y_3 \cdots y_n^{-1} x$. Hence from the above equations, we have:

$x \in (HS_{i_1} H)(HS_{i_2} H)(HS_{i_3} H) \cdots (HS_{i_n} H) \subseteq [HSI]$. Since $x$ is arbitrary, $G = [HSI]$.

Conversely, assume that $G = [HSI]$. Let $x$ and $y$ be any arbitrary elements in $G$. Let $y = xz$. Then $z \in G$. That is, $z \in (HS_{i_1} H)(HS_{j_1} H) \cdots (HS_{k_1} H)$ for some $i, j, \ldots$ and $k$. Then clearly, $(H, s_i H, s_i s_j H, \ldots, s_i s_j \cdots s_k H)$ is an $E_1 E_2 \cdots E_n$-path from $H$ to $zH$. That is $(xH, xH, s_i s_j H, \ldots, s_i s_j \cdots s_k H)$ is a $E_1 E_2 \cdots E_n$-path from $xH$ to $yH$. Hence $\mathcal{C}$ is connected.

**Proposition 2.15** $\mathcal{C}$ is $E_i$ connected if and only if $G = \langle HS_{i_1} H \rangle$, where $\langle HS_{i_1} H \rangle$ is the semigroup generated by $HS_{i_1} H$.

**Proposition 2.16** $\mathcal{C}$ is $E_1 E_2 \cdots E_n$ quasi connected if and only if $G = [HSI]^{-1}[HSI]$.

**Proof.** First, assume that $\mathcal{C}$ is quasi strongly connected. Let $xH$ be any arbitrary element in $G/H$. Then there exists a vertex $yH \in G$ such that there is a path from $yH$ to $xH$, say: $(yH, y_1 H, y_2 H, \cdots, y_n H, H)$ and a path from $yH$ to $H$, say: $(yH, x_1 H, x_2 H, \cdots, x_m H, xH)$. Then we have the following system of equations:

\begin{align}
y^{-1} y_1 & \in HS_{i_1} H; \\
y_1^{-1} y_2 & \in HS_{i_2} H; \\
y_2^{-1} y_3 & \in HS_{i_3} H; \\
& \vdots \\
y_n^{-1} & \in HS_{i_n+1} H. \\
\end{align}

and

\begin{align}
y^{-1} x_1 & \in HS_{i_1} H; \\
x_1^{-1} x_2 & \in HS_{i_2} H; \\
x_2^{-1} x_3 & \in HS_{i_3} H; \\
& \vdots \\
x_m^{-1} x & \in HS_{i_m+1} H. \\
\end{align}
From equation (1) we obtain the following:

\[ y^{-1} = (y^{-1}y_1)(y_1^{-1}y_2)(y_2^{-1}y_3) \cdots (y_n^{-1}) \in S_{i_2} \in (HS_{i_1}H)(HS_{i_2}H) \cdots (HS_{i_{n+1}}H). \]

This implies that

\[ y \in (HS_{i_1}^{-1}H)(HS_{i_2}^{-1}H) \cdots (HS_{i_{n+1}}^{-1}H) \in [HS^{-1}H]. \]

Similarly, from equation (2) we obtain the following:

\[ y^{-1}x = (y^{-1}x_1)(x_1^{-1}x_2) \cdots (x_m^{-1}x) \in (HS_{i_1}H)(HS_{i_2}H) \cdots (HS_{i_{m+1}}H). \]

That is

\[ y^{-1}x \in [HSH]. \]

That is

\[ x \in y[HSH] \subseteq [HS^{-1}H][HSH]. \]

Since \( x \) is arbitrary, we have

\[ G = [HS^{-1}H][HSH]. \]

Conversely, assume that \( G = [HS^{-1}H][HSH] \). Let \( x \) and \( y \) be two arbitrary vertices in \( G \). Let \( y = xz \). Then \( z \in G \). This implies that \( z \in [HS^{-1}H][HSH] \). Then there exits \( z_1 \in [HS^{-1}H] \) and \( z_2 \in [HSH] \) such that \( z = z_1z_2 \). \( z_1 \in [HS^{-1}H] \) implies that there exits \( t_k \in HS_{i_k}H \) such that

\[ z_1 = t_1t_2 \ldots t_n \text{ for some } t_k \in HS_{i_k}^{-1}H, k = 1, 2, \ldots, n. \]

This implies that

\[ (z_1H, t_1t_2H \ldots t_{n-1}, \ldots, H) \]

is a path from \( z_1H \) to \( H \). That is

\[ (yz_1H, yt_1t_2H \ldots t_{n-1}H, \ldots, yH) \]

is a path from \( yz_1H \) to \( yH \).

Similarly, \( z_2 \in [HSH] \) implies that there exits \( a_k \in S_{i_k} \) such that

\[ z_2 = a_1a_2 \ldots a_m. \]
Observe that

\[(z_2 H, a_1 a_2 H, a_1 a_2 a_3 H, \ldots, H)\]

is a path from \(z_2 H\) to \(H\). That is,

\[(z_1 z_2 H, z_1 a_1 a_2 H, a_1 a_2 a_3 H, \ldots, z_1 H)\]

is a path from \(zH\) to \(z_1 H\). That is

\[(yz H, yz_1 a_1 a_2 H, ya_1 a_2 a_3 H, \ldots, z_1 H)\]

is a path from \(xH\) to \(z_1 H\).

**Proposition 2.17** \(\mathcal{C}\) is \(E_i\)-quasi connected if and only if \(G =< HS_i^{-1} H > < HS_i H >\).

**Proposition 2.18** \(\mathcal{C}\) is \(E_1 E_2 \cdots E_n\) - locally connected if and only if \([HSH] = [HS^{-1}H]\).

**Proof.**

Assume that \(\mathcal{C}\) is \(E_1 E_2 \cdots E_n\) - locally connected. Let \(x \in [HSH]\). Then \(x \in A_m\) for some \(m\). Then \(x = s_1 s_2 \ldots s_m\). Let \(x_0 = 1, x_1 = s_i, x_2 = s_i s_j, \ldots, x_m = s_i s_j \ldots s_m\). Then

\[(x_0 H, x_1 H, x_2 H, \ldots, x_m H)\]

is a path leading from 1 to \(x\). Since \(\mathcal{C}\) is locally connected, there exits a path from \(xH\) to \(H\), say:

\[(xH, y_1 H, y_2 H, \ldots, y_m H, H)\]

This implies that

\[x^{-1} y_1 \in S_{i_1}\]
\[y_1^{-1} y_2 \in S_{i_2}\]
\[
\vdots
\]
\[y_m^{-1} \in S_{i_n}\]

The above equations tells us that \(x^{-1} \in [HSH]\). That is \(x \in [HS^{-1}H]\). Hence \([HSH] = [HS^{-1}H]\). Conversely, if \([HSH] = [HS^{-1}H]\), one can easily verify that \(\mathcal{C}\) is \(E_1 E_2 \cdots E_n\) - locally connected.
Proposition 2.19 \( \mathcal{C} \) is \( E_i \)-locally connected if and only if \( <HS_i^{-1}H> = <HS_iH> \).

Proposition 2.20 \( \mathcal{C} \) is \( E_1E_2\cdots E_n \)-semi connected if and only if \( G = [HSH] \cup [HS^{-1}H] \).

Proof. Assume that \( \mathcal{C} \) is \( E_1E_2\cdots E_n \)-semi connected and let \( xH \in G/H \). Then there is a path from \( H \) to \( xH \), say

\[
(H, x_1H, x_2H, \cdots, x_nH, xH)
\]
or a path from \( xH \) to \( H \), say

\[
(xH, y_1H, y_2H, \cdots, y_mH, H)
\]

This implies that \( x \in [HSH] \) or \( x \in [HS^{-1}H] \). This implies that \( G = [HSH] \cup [HS^{-1}H] \). Similarly, if \( G = [HSH] \cup [HS^{-1}H] \), then one can prove that \( \mathcal{C} \) is \( E_1E_2\cdots E_n \)-semi connected.

Proposition 2.21 \( \mathcal{C} \) is \( E_i \)-semi connected if and only if \( G = <HS_iH> \cup <HS_i^{-1}H> \).

Proposition 2.22 \( \mathcal{C} \) is an \( E_1E_2\cdots E_n \)-quasi ordered set if and only if

\[
(i) 1 \in (HS_1H) \cup (HS_2H) \cdots \cup (HS_nH),
\]

\[
(ii) \text{for every}(i,j), HS_iHS_jH \subseteq HS_kH \text{ for some } k.
\]

Proposition 2.23 \( \mathcal{C} \) is an \( E_i \) quasi ordered set if and only if

\[
(i) 1 \in HS_iH,
\]

\[
(ii) (HS_iH)^2 \subseteq HS_iH.
\]

Proposition 2.24 \( \mathcal{C} \) is an \( E_1E_2\cdots E_n \)-partially ordered set if and only if

\[
(i) 1 \in (HS_1H) \cup (HS_2H) \cdots \cup (HS_nH),
\]

\[
(ii) \text{for every}(i,j), (HS_iH)(HS_jH) \subseteq (HS_kH) \text{ for some } k,
\]

\[
(iii) \cup (HS_iH) \cap (HS_i^{-1}H) = \{1\}.
\]

Proof. Observe that

\[
x \in \cup (HS_iH) \cap H(S_i)^{-1}H \iff x \in (HS_iH) \cap (HS_i^{-1}H) \text{ for some } i
\]
\[ \iff x \in HS_i H \text{ and } x \in HS_i^{-1} H \]
\[ \iff (H, xH) \in E_i \text{ and } (xH, H) \in E_i \]
\[ \iff x = 1. \]

**Proposition 2.25** \( \mathcal{C} \) if an \( E_i \) partially ordered set if and only if

1. \( (i)1 \in HS_i H \),
2. \( (ii)(HS_i H)^2 \subseteq HS_i H \)
3. \( (iii)(HS_i H) \cap (HS_i^{-1} H) = \{1\} \)

**Proposition 2.26** Let \( A_m (m \geq 2) \) is the set of \( m \) products of the form \( S_{i_1} S_{i_2} \cdots S_{i_m} \).

Then \( \mathcal{C} \) is an \( E_1 E_2 \cdots E_n \) - hasse diagram if and only if \( C \cap S_i = \emptyset \) for all \( i \) and for all \( C \in A_m \).

**Proof.** Suppose the condition holds. Let \( x_0 H, x_1 H, \ldots, x_m H \) be \( (m + 1) \) elements in \( G/H \) such that \( (x_i H, x_{i+1} H) \in \cup E_i \) for \( i = 0, 1, \ldots, m - 1 \). This implies that

\[
x_0^{-1} x_1 \in S_{i_1}; \\
x_1^{-1} x_2 \in S_{i_2}; \\
x_2^{-1} x_3 \in S_{i_3}; \\
\vdots \\
x_{m-1}^{-1} x_m \in S_{i_m}.
\]

The above equation tells us that \( x_0^{-1} x_m \in A_m \). Since \( C \cap S_i = \emptyset \) for all \( i \) and for all \( C \in A_m \), \( (x_0, x_m) \notin \cup E_i \).

Conversely assume that \( \mathcal{C} \) is an \( E_1 E_2 \cdots E_n \) hasse diagram. We will show that \( C \cap S_i = \emptyset \) for all \( i \) and for all \( C \in A_m \). Let \( S_{i_1} S_{i_2} S_{i_3} \cdots S_{i_m} \) be any element in \( A_m \). Let \( x \in S_{i_1} S_{i_2} S_{i_3} \cdots S_{i_m} \). Then \( x = s_{i_1} s_{i_2} s_{i_3} \ldots s_{i_m} \) for some \( s_{i_k} \in S_{i_k} \). This implies that

\[ (H, s_{i_1} H, s_{i_2} s_{i_3} H, \ldots, x H) \]
is a path from $H$ to $xH$. Since $C$ is an $E_1 E_2 \cdots E_n$ hasse-diagram, $x \notin S_i$ for any $i$. That is, $A_m \cap S_i = \emptyset$ for all $i$.

**Proposition 2.27** The $E_1 E_2 \cdots E_n$ out-degree of $C$ is the cardinal number $|S_1 \cup S_2 \cup \cdots \cup S_n/H|$.

**Proof.** Since $C$ is vertex-transitive it suffices to consider the out degree of the vertex $H \in G/H$. Observe that

$$\rho(H) = \{uH : (H, uH) \in E\}$$

$$= \{uH : u \in HS_i H \text{ for some } i\}$$

$$= (HS_1 H) \cup (HS_2 H) \cup \cdots \cup (HS_n H)/H$$

Hence $|\rho(H)| = |(HS_1 H) \cup (HS_2 H) \cup \cdots \cup (HS_n H)/H|$.

**Proposition 2.28** The $E_i$ out-degree of $C$ is the cardinal number $|HS_i H/H|$.

**Proposition 2.29** The $E_1 E_2 \cdots E_n$ in-degree of $C$ is the cardinal number $|(HS_1^{-1} H) \cup (HS_2^{-1} H) \cup \cdots \cup (HS_n^{-1} H)/H|$. 

**Proof.** Since $C$ is vertex-transitive it suffices to consider the in degree of the vertex $H \in G/H$. Observe that

$$\sigma(H) = \{uH : (uH, H) \in E\}$$

$$= \{uH : (uH, H) \in E_i\}$$

$$= \{uH : u^{-1} \in HS_i H\}$$

$$= \{uH : u \in HS_i^{-1} H\}$$

Hence $|\sigma(H)| = |(HS_1^{-1} H) \cup (HS_2^{-1} H) \cup \cdots \cup (HS_n^{-1} H)/H|$.

**Proposition 2.30** The $E_i$ in-degree of $C$ is the cardinal number $|HS_i^{-1} H/H|$.

**Proposition 2.31** For $k = 1, 2, 3, \ldots$ let $A_k$ be the set of all $k$ products of the form $(HS_{i_1} H)(HS_{i_2} H) \cdots (HS_{i_k} H)$. If $C$ has finite diameter, then the diameter of $C$ is the least positive integer $m$ such that

$$G = A_m$$
Proof. Let \( m \) be the smallest positive integer such that \( G = A_m \). We will show that the diameter of \( \mathcal{C} \) is \( m \). Let \( xH \) and \( yH \) be any two arbitrary elements in \( G \) such that \( y = xz \). Then \( z \in G \). This implies that \( x \in A_m \). But then \( z \) has a representation of the form \( x = s_{i_1}s_{i_2} \cdots s_{i_m} \). This implies that

\[
(H, s_{i_1}H, s_{i_1}s_{i_2}H, \ldots, zH)
\]

is a path of \( m \) edges from \( H \) to \( zH \). That is

\[
(xH, xs_{i_1}H, xs_{i_1}s_{i_2}H, \ldots, yH)
\]

is a path of length \( m \) from \( xH \) to \( yH \). This shows that \( d(xH, yH) \leq n \). Since \( xH \) and \( yH \) are arbitrary,

\[
\max_{xH, yH \in G} \{d_{1,2,\ldots,n}(xH, yH)\} \leq m
\]

Therefore the diameter of \( \mathcal{C} \) is less than or equal to \( n \). On the other hand let the diameter of \( \mathcal{C} \) be \( k \). Let \( x \in G \) and \( d_{1,2,\ldots,n}(H, xH) = k \). Then we have \( x \in B \) for some \( B \in A_k \). That is

\[
G = A_k
\]

Now by the minimality of \( k \), we have \( m \leq k \). Hence \( k = m \).

**Proposition 2.32** The vertex \( H \) is an \( E_1E_2\cdots E_n \)-source of \( \mathcal{C} \) if and only if \( G = [HSH] \).

Proof. First, assume that \( H \) is an \( E_1E_2\cdots E_n \)-source of \( \mathcal{C} \). Then for any vertex \( xH \in G/H \), there is an \( E_1E_2\cdots E_n \)-path from \( H \) to \( xH \). This implies that \( G = [HSH] \). Conversely, if \( G = [HSH] \), one can prove that \( H \) is an \( E_1E_2\cdots E_n \)-source.

**Proposition 2.33** The vertex \( H \) is an \( E_i \)-source of \( \mathcal{C} \) if and only if \( G =< H S_i H > \).

**Proposition 2.34** The vertex \( H \) is an \( E_1E_2\cdots E_n \)-sink of \( \mathcal{C} \) if and only if \( G = [HS^{-1}H] \).

Proof. First, assume that \( H \) is an \( E_1E_2\cdots E_n \)-sink of \( \mathcal{C} \). Then for each \( xH \in G/H \), there is an \( E_1E_2\cdots E_n \)-path from \( xH \) to \( H \). This implies that \( x \in [HS^{-1}H] \). Hence \( G = [HS^{-1}H] \). Conversely, if \( G = [HS^{-1}H] \), one can easily prove that \( H \) is an \( E_1E_2\cdots E_n \)-sink of \( \mathcal{C} \).

**Proposition 2.35** The vertex \( H \) is an \( E_i \) sink of \( \mathcal{C} \) if and only if \( G =< H S_i^{-1}H > \).
**Proposition 2.36** The $E_1E_2\cdots E_n$-reachable set $R_{1,2,\ldots,n}(H)$ of the vertex $H$ is the set $[HS'H]$.  

**Proof.** By definition,

$$R(H) = \{xH : \text{there exists an } E_1E_2\cdots E_n \text{-path from } H \text{ to } xH\}$$

Observe that

$$xH \in R_{1,2,\ldots,n}(H) \iff \text{there exists an } E_1E_2\cdots E_n \text{-path from } H \text{ to } xH, \text{ say}$$

$$(H, x_1H, x_2H, \ldots, x_nH, xH)$$

$$\iff x \in [HS'H]$$

Therefore, $R_{1,2,3,\ldots,n}(H) = [HS'H]$.  

**Proposition 2.37** The $E_i$ reachable set $R_i(H)$ of the vertex $H$ is the set $<S_i>$.  

**Proposition 2.38** The $E_1E_2\cdots E_n$-antecedent set $Q_{1,2,\ldots,n}(H)$ of the vertex $H$ is the set $[HS^{-1}H]$.  

**Proof.** Observe that

$$x \in Q_{1,2,\ldots,n}(H) \iff \text{there exists an } E_1E_2\cdots E_n \text{-path from } xH \text{ to } H, \text{ say}$$

$$(xH, x_1H, x_2H, \ldots, x_nH, H)$$

$$\iff x \in [HS^{-1}H].$$

$$\therefore Q_{1,2,\ldots,n}(H) = [HS^{-1}H].$$  

**Proposition 2.39** The $E_i$ antecedent set $Q_i(H)$ of the vertex $H$ is the set $<HS_i^{-1}H>$.  

**References**


