Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 3, 535-543
https://doi.org/10.28919/jmcs/4460
ISSN: 1927-5307

ON THE PATH ENERGY OF SOME GRAPHS

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Abstract. Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. We define a matrix whose $(i, j)^{th}$ entry is the maximum number of vertex disjoint paths between the corresponding vertices if they are adjacent and is zero otherwise. We call this matrix as path matrix of G and its eigenvalues as path eigenvalues of G. In this paper, we investigate path eigenvalues and path energy of some graphs.

Keywords: symmetric matrix; eigenvalues; path eigenvalues of a graph; path energy of a graph.

2010 AMS Subject Classification: 05C50.

1. Introduction

The eigenvalues of a graph G are the eigenvalues of its adjacency matrix. The spectrum of a matrix is the list of its eigenvalues together with their multiplicities. The eigenvalues of graphs have several useful properties. For undefined terminology and notations, we refer to West [5] and Varga [4]. For an extensive survey on graph spectra we refer to Brouwer A. E. [3] and Beineke L. W. [6].

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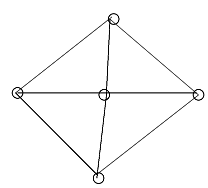
Received January 7, 2020

We define a new matrix, called the path matrix ([1], [2]) of a graph in the following way. Let G be a graph without loops and let $V(G) = \{v_1, v_2, ..., v_n\}$ be the vertex set of G. Define the matrix $P = (p_{ij})$ of size $n \times n$ such that

$$p_{ij} = \begin{cases} \text{maximum number of vertex disjoint paths from } v_i \text{ to } v_j & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

We call P as Path Matrix of G. The matrix P is real and symmetric. Therefore, its eigenvalues are real. We call eigenvalues of P as path eigenvalues of G.

Consider the graph as shown in the following figure



G

Then the path matrix of G is

$$\mathbf{P} = \begin{bmatrix} 0 & 3 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 \end{bmatrix}.$$

The characteristic polynomial of the matrix P is
$C_P(x) = |P - xI| = (x-12)(x+3)^4$. The path eigenvalues of G are $12, -3, -3, -3$ and -3. The eigenvalues of G are $3.236, -2, -1.236, 0$ and 0.

2. Path Energy Obtained from Some Operations on Graphs

The ordinary energy ([7], [8]), $E(G)$, of a graph G is defined to be the sum of the absolute values of the ordinary eigenvalues of G. Recently much work on ordinary graph energy appeared in the mathematical literature. In analogy, the path energy [2], $PE(G)$ is defined as the sum of the absolute values of the path eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of G, i.e.,

$$PE = PE(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{2.1}$$

We know if G is a r-regular, r-connected graph with n vertices, then its path matrix has row sum $(n-1)r$ and this row sum $(n-1)r$ is one of the path eigenvalues of G and the other path eigenvalues are -1 with multiplicity $n-1$.

In the following Theorem, we investigate the path eigenvalues and path energy of a graph which is obtained by joining a vertex of r_1-regular, r_1-connected graph with a vertex of r_2-regular, r_2-connected graph by an edge.

Theorem 2.1. *Let G_1 be r_1-regular, r_1-connected graph with m vertices and G_2 be r_2-regular r_2-connected graph with n vertices. If $\lambda_1, \lambda_2, ..., \lambda_m$ and $\mu_1, \mu_2, ..., \mu_n$ are path eigenvalues of G_1 and G_2 respectively. Let G be a graph obtained by joining a vertex of G_1 to a vertex of G_2 by an edge. Then the path eigenvalues of G are $-r_1$ with multiplicity $m-1$, $-r_2$ with multiplicity $n-1$, $\frac{(m-1)r_1+(n-1)r_2+\sqrt{[(m-1)r_1+(n-1)r_2]^2+4[mn-(m-1)r_1(n-1)r_2]}}{2}$ with multiplicity 1 and $\frac{(m-1)r_1+(n-1)r_2-\sqrt{[(m-1)r_1+(n-1)r_2]^2+4[mn-(m-1)r_1(n-1)r_2]}}{2}$ with multiplicity 1. $PE(G) = 2[(m-1)r_1 + (n-1)r_2] = PE(G_1) + PE(G_2)$.*

Proof. Let P, Q, and R be the path matrices of G, G_1, and G_2 respectively. As G_1 is r_1-regular, r_1-connected with m vertices, the path eigenvalues of G_1 are $\lambda_1 = (m-1)r_1$ with multiplicity 1, $-r_1$ with multiplicity $m-1$ and as G_2 is r_2-regular, r_2-connected on n vertices, the path eigenvalues of G_2 are $\mu_1 = (n-1)r_2$ with multiplicity 1, $-r_2$ with multiplicity $n-1$. The path matrix P can be written as

$$P = \begin{bmatrix} Q & J_{m\times n} \\ J_{n\times m} & R \end{bmatrix}$$

where $J_{m\times n}$ is m by n matrix with all entries 1. We know that $\mathbf{1}$ is an eigenvector of Q corresponding to $(m-1)r_1$, so we assume $\mathbf{1}X = 0$, where $X = [x_1, ..., x_m]'$ is an eigenvector of of Q corresponding to $\lambda_i \neq (m-1)r_1$. Now

$$P\begin{bmatrix} X \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} = \begin{bmatrix} Q & J_{m\times n} \\ J_{n\times m} & R \end{bmatrix} \begin{bmatrix} X \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} = \lambda_i \begin{bmatrix} X \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

implies that $\lambda_i = -r_1$ is a path eigenvalue of P for $i = 2,...,m$. Similarly $\mu_i = -r_2$ is a path eigenvalue of P for $i = 2,...,n$. Thus

$tr(P) = 0 = tr(Q) + tr(R)$, where $tr(Q) = 0, tr(R) = 0$ and $\lambda_1 + \mu_1 = (m-1)r_1 + (n-1)r_2$. Let A be a square matrix, then the sum of all 2×2 principal minors of A is equal to $s_2(A)$, where $s_2(A)$ is the second elementary symmetric function of the eigenvalues of A. Thus $s_2(P) = \lambda_1\mu_1 + \lambda_1(-r_1)(m-1) + \lambda_1(-r_2)(n-1) + \mu_1(-r_1)(m-1) + \mu_1(-r_2)(n-1) + \sum_{2\le i<j}\lambda_i\lambda_j + \sum_{2\le i<j}\mu_i\mu_j + \sum_{2\le i,j}\lambda_i\mu_j$. We can write this as $s_2(P) = \lambda_1\mu_1 + (\lambda_1+\mu_1)[-(m-1)r_1 - (n-1)r_2] + \sum_{2\le i<j}\lambda_i\lambda_j + \sum_{2\le i<j}\mu_i\mu_j + [-(n-1)r_2][-(m-1)r_1]$ (i)

Now for the path matrices Q and R, we get

$s_2(Q) = (m-1)r_1(-r_1)(m-1) + \sum_{2\le i<j}\lambda_i\lambda_j$

$= (m-1)r_1(-(m-1)r_1) + \sum_{2\le i<j}\lambda_i\lambda_j$ and

$s_2(R) = (n-1)r_2(-r_2)(n-1) + \sum_{2\le i<j}\mu_i\mu_j$

$= (n-1)r_2(-(n-1)r_2) + \sum_{2\le i<j}\mu_i\mu_j$.

Again every principal minor of size 2×2 of P is either a 2×2 principal minor of Q or R, or it has the form $\begin{vmatrix} q_{ii} & 1 \\ 1 & r_{jj} \end{vmatrix} = q_{ii}\, r_{jj} - 1,\ i = 1,2,...,m,\ j = 1,2,...,n$, where $Q = (q_{ij})$ and $R = (r_{ij})$.

Using this, we write

$s_2(P) = s_2(Q) + s_2(R) + \sum_{i=1}^{m}\sum_{j=1}^{n}(q_{ii}\, r_{jj} - 1) = s_2(Q) + s_2(R) - mn.$ (ii)

From (i) and (ii), we get

$\lambda_1\mu_1 + (\lambda_1+\mu_1)[-(m-1)r_1 - (n-1)r_2] + [-(n-1)r_2][-(m-1)r_1] = (m-1)r_1[-(m-1)r_1] + (n-1)r_2[-(n-1)r_2] - mn.$

This and $\lambda_1 + \mu_1 = (m-1)r_1 + (n-1)r_2$ gives $\lambda_1\mu_1 = (m-1)r_1(n-1)r_2 - mn$. Solving this equations for λ_1 and μ_1, we get

$\lambda_1 = \frac{(m-1)r_1+(n-1)r_2+\sqrt{[(m-1)r_1+(n-1)r_2]^2+4[mn-(m-1)r_1(n-1)r_2]}}{2}$ and
$\mu_1 = \frac{(m-1)r_1+(n-1)r_2-\sqrt{[(m-1)r_1+(n-1)r_2]^2+4[mn-(m-1)r_1(n-1)r_2]}}{2}$.
Hence $PE(G) = 2[(m-1)r_1 + (n-1)r_2] = PE(G_1) + PE(G_2)$.

□

We know for a tree T with m vertices, its path matrix has row sum $m-1$ and this row sum $m-1$ is one of the path eigenvalue of T and the other path eigenvalues are -1 with multiplicity $m-1$.

In the following Proposition, we investigate the path eigenvalues and path energy of a graph which is obtained by joining a vertex of a tree with a vertex of r-regular, r-connected graph by an edge.

Proposition 2.2. *Let G_1 be a tree with m vertices and G_2 be r-regular, r-connected graph with n vertices. Let G be a graph obtained by joining a vertex of G_1 to a vertex of G_2 by an edge. Then the path eigenvalues of G are -1 with multiplicity $m-1$, $-r$ with multiplicity $n-1$,*
$\frac{(m-1)+(n-1)r+\sqrt{[(m-1)+(n-1)r]^2+4[mn-(m-1)(n-1)r]}}{2}$ *with multiplicity* 1 *and*
$\frac{(m-1)+(n-1)r-\sqrt{[(m-1)+(n-1)r]^2+4[mn-(m-1)(n-1)r]}}{2}$ *with multiplicity* 1.
$PE(G) = 2[(m-1)+(n-1)r] = PE(G_1) + PE(G_2)$.

Proof. Let P, Q $(= J_m - I_m)$, and R be the path matrices of G, G_1, and G_2 respectively. Here $\mu_1 = (n-1)r$. The path matrix P can be written as

$$P = \begin{bmatrix} \text{Q} & J_{m\times n} \\ J_{n\times m} & \text{R} \end{bmatrix}$$

where $J_{m\times n}$ is m by n matrix with all entries 1. We know that $\mathbf{1}$ is an eigenvector of Q corresponding to $\lambda_1 = (m-1)$, so we assume $\mathbf{1}X = 0$, where $X = [x_1, \ldots, x_m]'$ is an eigenvector of Q corresponding to $\lambda_i \neq (m-1)$. Now

$$P\begin{bmatrix} X \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} = \begin{bmatrix} Q & J_{m\times n} \\ J_{n\times m} & R \end{bmatrix} \begin{bmatrix} X \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} = \lambda_i \begin{bmatrix} X \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

implies that $\lambda_i = -1$ is a path eigenvalue of P for $i = 2,...,m$. Similarly $\mu_i = -r$ is a path eigenvalue of P for $i = 2,...,n$. Thus $tr(P) = 0 = tr(Q)+tr(R)$, where $0 = tr(Q) = (m-1)+\lambda_2+...+\lambda_m,\ 0 = tr(R) = (n-1)r+\mu_2+...+\mu_n$ and $\lambda_1+\lambda_2 = (m-1)+(n-1)r$. Let A be a square matrix, then the sum of all 2×2 principal minors of A is equal to $s_2(A)$, where $s_2(A)$ is the second elementary symmetric function of the eigenvalues of A. Thus $s_2(P) = \lambda_1\mu_1+\lambda_1(-1)(m-1)+\lambda_1(-r)(n-1)+\mu_1(-1)(m-1)+\mu_1(-r)(n-1)+\sum_{2\le i<j}\lambda_i\lambda_j+\sum_{2\le i<j}\mu_i\mu_j+\sum_{2\le i,j}\lambda_i\mu_j$. We can write this as

$s_2(P) = \lambda_1\mu_1 + (\lambda_1+\mu_1)(-(m-1)-(n-1)r) + \sum_{2\le i<j}\lambda_i\lambda_j + \sum_{2\le i<j}\mu_i\mu_j + (-(n-1)r)(-(m-1)r) = \lambda_1\lambda_2 + (\lambda_1-\lambda_2)[(m-1)+(n-1)r)] + \sum_{2\le i<j}\lambda_i\lambda_j + \sum_{2\le i<j}\mu_i\mu_j + (n-1)r(m-1).$ (i)

Now for the path matrices Q and R, we get

$s_2(Q) = (m-1)(-1)(m-1)+\sum_{2\le i<j}\lambda_i\lambda_j$

$= (m-1)(-(m-1))+\sum_{2\le i<j}\lambda_i\lambda_j$ and

$s_2(R) = (n-1)r(-r)(n-1)+\sum_{2\le i<j}\mu_i\mu_j$

$= (n-1)r(-(n-1)r)+\sum_{2\le i<j}\mu_i\mu_j.$

Again every principal minor of size 2×2 of P is either a 2×2 principal minor of Q or R, or it has the form

$$\begin{vmatrix} q_{ii} & 1 \\ 1 & r_{jj} \end{vmatrix}$$

$= q_{ii}\, r_{jj}-1,\ i = 1,2,...,m,\ j = 1,2,...,n$, where $Q = (q_{ij})$ and $R = (r_{ij})$.

Using this, we write $s_2(P) = s_2(Q)+s_2(R)+\sum_{i=1}^{m}\sum_{j=1}^{n}(q_{ii}\ \ r_{jj}-1) = s_2(Q)+s_2(R)-mn$ (ii)

From (i) and (ii), we get

$\lambda_1\mu_1 + (\lambda_1+\mu_1)(-(m-1)r_1-(n-1)r_2)+(-(n-1)r_2)(-(m-1)r_1) = (m-1)r_1(-(m-1)r_1)+(n-1)r_2(-(n-1)r_2)-mn.$

This and $\lambda_1+\mu_1 = (m-1)+(n-1)r$ gives $\lambda_1\mu_1 = (m-1)(n-1)r-mn$. Solving this equations for λ_1 and μ_1, we get

$\lambda_1 = \frac{(m-1)+(n-1)r+\sqrt{[(m-1)+(n-1)r]^2+4[mn-(m-1)(n-1)r]}}{2}$ and

$\mu_1 = \frac{(m-1)+(n-1)r-\sqrt{[(m-1)+(n-1)r]^2+4[mn-(m-1)(n-1)r]}}{2}$.

Hence $PE(G) = 2[(m-1)+(n-1)r] = PE(G_1)+PE(G_2)$. □

We investigate the path eigenvalues and path energy of a graph which is obtained by taking k copies of r-regular, r-connected graph and joining a vertex of one graph with a vertex of other graph.

Theorem 2.3. *Let $G_1,\ G_2\ ,...,G_k$ be the k copies of some r-regular r-connected graph on n vertices and let G be a graph obtained by joining a vertex of G_i with a vertex of G_{i+1} $(1\le i\le k-1)$ by an edge. Then the path eigenvalues of G are $n(k-1)+r(n-1)$ with multiplicity 1, $-r$ with multiplicity $k(n-1)$ and $n(r-1)-r$ with multiplicity $k-1$. $PE(G) = \sum_{i=1}^{k} PE(G_i)$.*

Proof. Let P be the path matrix of G and Q be the path matrix of G_i, for $i = 1,2,...,k$. Let J_n be the $n\times n$ matrix with all entries 1. The path matrix P can be written as

$$\mathbf{P} = \begin{bmatrix} Q & J_n & \ldots & J_n \\ J_n & Q & \ldots & J_n \\ \vdots & \vdots & \ddots & \vdots \\ J_n & J_n & \ldots & Q \end{bmatrix}$$

Adding $2^{nd},\ 3^{rd},...,k^{th}$ columns to the first column, we get

$$\begin{bmatrix} Q+(k-1)J_n & J_n & \ldots & J_n \\ Q+(k-1)J_n & Q & \ldots & J_n \\ \vdots & \vdots & \ddots & \vdots \\ Q+(k-1)J_n & J_n & \ldots & Q \end{bmatrix}$$

Now subtracting the first row from 2^{nd}, $3^{rd}, \ldots, k^{th}$ rows, we get

$$\begin{bmatrix} Q+(k-1)J_n & J_n & \ldots & J_n \\ 0 & Q-J_n & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & Q-J_n \end{bmatrix}.$$

This is a triangular block matrix. Hence the characteristic polynomial of P is $C_P(x) = |Q+(k-1)J_n - xI_n||Q-J_n-xI_n|\ldots|Q-J_n-xI_n|$ (here $|Q-J_n-xI_n|$ appears $k-1$ times). This implies that the path eigenvalues of G are the path eigenvalues of $Q+(k-1)J_n$ and the path eigenvalues of $Q-J_n$, $k-1$ times. Now, the path eigenvalues of $Q+(k-1)J_n$ are $n(k-1)+r(n-1)$ with multiplicity 1 and $-r$ with multiplicity $n-1$ whereas the path eigenvalues of $Q-J_n$ are $n(r-1)-r$ with multiplicity 1 and $-r$ with multiplicity $n-1$. Hence the path eigenvalues of G are $n(k-1)+r(n-1)$ with multiplicity 1, $-r$ with multiplicity $k(n-1)$ and $n(r-1)-r$ with multiplicity $k-1$. Hence $PE(G) = n(k-1)+r(n-1)+rk(n-1)+[n(r-1)-r](k-1) = 2kr(n-1) = \sum_{i=1}^{k} PE(G_i)$. □

3. CONCLUSION

In the present paper, path eigenvalues and path energy of graphs which are obtained by joining a vertices of some specific classes of graphs are obtained and studied.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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