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SOME COUPLED FIXED POINT THEOREMS IN CONE S_b-METRIC SPACE

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Abstract: In this paper, we extend the definition of coupled fixed point to mappings on cone S_b-metric space and prove some coupled fixed point theorems. Our results extend the coupled fixed point results of F. Sabetghadam et al. [Some coupled fixed point theorems in cone metric spaces, Fixed point theory and applications, Volume 2009, Article ID 125426, 8 pages] to cone S_b-metric space. An example is also given to illustrate the validity of our result.

Keywords: coupled fixed point; cone metric space; cone S-metric space; cone S_b-metric space.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

The Banach contraction principle is the most celebrated fixed point theorem. This result has been generalised in various directions. As a generalisation of metric space, Huang and Zhang [5] introduced the concept of cone metric space by replacing the set of real numbers by a general Banach space E which is partially ordered with respect to a cone $P \subset E$.

On the other hand, Sedghi et al. [14] generalised metric space to S-metric space and Bakhtin generalised it to b-metric space. Nizar and Nabil [11] introduced the concept of S_b-metric space. Dhamodharan and Krishnakumar [2] also further extended S-metric space to cone

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S-metric space. K. Anthony Singh and M.R. Singh [6] generalized the concept of cone *S*-metric space to cone S_b-metric space and prove some fixed point theorems. In the meantime, Bhaskar and Lakshmikantham [17] introduced the concept of coupled fixed point of a mapping $F: X \times X \to X$. Lakshmikantham and Ciric [18] investigated some more coupled fixed point theorems in partially ordered sets. Then, F. Sabetghadam et al. [3] considered the corresponding definition of coupled fixed point for mappings on cone metric spaces and proved some coupled fixed point theorems.

The aim of this paper is to further extend the definition of coupled fixed point to mappings on cone S_b-metric space and prove some coupled fixed point theorems. Our results extend the coupled fixed point results of F. Sabetghadam et al. [3] to cone S_b-metric space. We also give an example to illustrate the validity of our result.

2. PRELIMINARIES

Following definitions, properties will be needed in the sequel.

Definition 2.1. [5] Let E be a Banach space. A subset P of E is called a cone if and only if :

1. P is closed, nonempty and $P \neq \{0\}$,
2. $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b,
3. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ in E with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of *P*. The cone *P* is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant of *P*.

The cone *P* is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone *P* is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Definition 2.2. [5] Let X be a non-empty set. Suppose the mapping $d: X \times X \to E$ satisfies

1. $d(x,y)\geq 0$, and $d(x,y)=0$ if and only if $x=y,\forall x,y\in X$,
2. $d(x,y)=d(y,x),\forall x,y\in X$,
3. $d(x,y)\leq d(x,z)+d(z,y),\forall x,y,z\in X$.

Then (X,d) is called a cone metric space or simply CMS.

Example 2.3. [5] Let $E=\mathbb{R}^2, P=\{(x,y):x,y\geq 0\}, X=\mathbb{R}$ and $d:X\times X\to E$ such that $d(x,y)=(|x-y|,\alpha|x-y|)$, where $\alpha\geq 0$ is a constant. Then (X,d) is a cone metric space.

Definition 2.4. [13] Let $X\neq\phi$ be any set and $S:X\times X\times X\to[0,\infty)$ be a function satisfying the following conditions for all $u,v,z,a\in X$:

1. $S(u,v,z)\geq 0$,
2. $S(u,v,z)=0$ if and only if $u=v=z$,
3. $S(u,v,z)\leq S(u,u,a)+S(v,v,a)+S(z,z,a)$.

Then the function *S* is called an *S*-metric on *X* and the pair (X,S) is called an *S*-metric space simply SMS.

Example 2.5. [12] Let *X* be a non-empty set and *d* be ordinary metric on *X*. Then $S(x,y,z)=d(x,z)+d(y,z)$ is an *S*-metric on *X*.

Definition 2.6. [11] Let *X* be a nonempty set and $b\geq 1$ be a given real number. A function $S_b:X\times X\times X\to[0,\infty)$ is said to be S_b-metric on *X* if and only if for all $x,y,z,a\in X$ the following conditions are satisfied:

(S_b1) $S_b(x,y,z)=0$ if and only if $x=y=z$,

(S_b2) $S_b(x,y,z)\leq b[S_b(x,x,a)+S_b(y,y,a)+S_b(z,z,a)]$.

The pair (X,S_b) is called an S_b-metric space.

It is quite obvious that S_b-metric spaces are the generalizations of *S*-metric spaces since every *S*-metric is an S_b-metric with b=1.

Example 2.7. [15] Let (X,S) be an *S*-metric space, and $S_*(x,y,z)=\{S(x,y,z)\}^p$, where p >1 is a real number. Then, S_* is an S_b-metric on *X* with $b=2^{2(p-1)}$.

Example 2.8. [6] Let $X=\mathbb{R}$ and let the function $S:X\times X\times X\to\mathbb{R}$ be defined as $S(x,y,z)=|x-z|+|y-z|$. Then S is an S-metric on X. Therefore, the function $S_b(x,y,z)=\{S(x,y,z)\}^2=\{|x-z|+|y-z|\}^2$ is an S_b-metric on X with $b=2^{2(2-1)}=4$.

Definition 2.9. [2] Suppose that E is a real Banach space, P is a cone in E with $\operatorname{int}P\neq\phi$ and $\leq$ is partial ordering with respect to P. Let X be a non-empty set, and let the function $S:X\times X\times X\to E$ satisfy the following conditions:

1. $S(u,v,z)\geq 0$,
2. $S(u,v,z)=0$ if and only if $u=v=z$,
3. $S(u,v,z)\leq S(u,u,a)+S(v,v,a)+S(z,z,a),\ \forall u,v,z,a\in X$.

Then the function S is called a cone S-metric on X and the pair (X,S) is called a cone S-metric space or simply CSMS.

Example 2.10. [2] Let $E=\mathbb{R}^2$, $P=\{(x,y):x,y\geq 0\}$, $X=\mathbb{R}$ and d be the ordinary metric on X. Then, the function $S:X\times X\times X\to E$ defined by

$$S(x,y,z)=\big(d(x,z)+d(y,z),\alpha(d(x,z)+d(y,z))\big),\ \alpha>0$$

is a cone S-metric on X.

Lemma 2.11. [2] *Let* (X,S) *be a cone S-metric space. Then we have* $S(u,u,v)=S(v,v,u)$.

Definition 2.12. [6] Suppose that E is a real Banach space, P is a cone in E with $\operatorname{int}P\neq\phi$ and $\leq$ is partial ordering in E with respect to P. Let X be a non-empty set, and let the function $S:X\times X\times X\to E$ satisfy the following conditions:

1. $S(u,v,z)\geq 0$,
2. $S(u,v,z)=0$ if and only if $u=v=z$,
3. $S(u,v,z)\leq b\big[S(u,u,a)+S(v,v,a)+S(z,z,a)\big],\ \forall u,v,z,a\in X$, where $b\geq 1$ is a constant.

Then, the function S is called a cone S_b-metric on X and the pair (X, S) is called a cone S_b-metric space or simply CS_bMS.

We note that cone S_b-metric spaces are generalizations of cone S-metric spaces since every cone S-metric is a cone S_b-metric with b=1.

Example 2.13. [6] Let $E=\mathbb{R}^2$, the Euclidean plane, and $P=\{(x,y)\in E:x,y\geq 0\}$, a normal cone in E. Let $X=\mathbb{R}$ and $S:X\times X\times X\to E$ be such that

$$S(x,y,z)=\left(\alpha S_*(x,y,z),\beta S_*(x,y,z)\right),$$

where $\alpha,\beta>0$ are constants and S_* is an S_b-metric on X. Then S is a cone S_b-metric on X. In particular, we have, the function $S_*(x,y,z)=\{|x-z|+|y-z|\}^2$, $x,y,z\in X$ is an S_b-metric on X with b=4. Therefore, the function

$$S(x,y,z)=\left(\{|x-z|+|y-z|\}^2,\frac{1}{4}\{|x-z|+|y-z|\}^2\right),\ x,y,z\in X$$

is a cone S_b-metric on X with b=4.

Definition 2.14. [6] Let (X, S) be a cone S_b-metric space.

1. A sequence $\{u_n\}$ in X is said to converge to u if for each $c\in E$, $0\ll c$ there exists $n_0\in N$ such that for all $n\geq n_0$, $S(u_n,u_n,u)\ll c$. We denote this by $\lim_{n\to\infty}u_n=u$ or $u_n\to u$ as $n\to\infty$.
2. A sequence $\{u_n\}$ in X is called a Cauchy sequence if for each $c\in E$, $0\ll c$ there exists $n_0\in N$ such that for all $n,m\geq n_0$, $S(u_n,u_n,u_m)\ll c$.
3. The cone S_b-metric space (X, S) is called complete if every Cauchy sequence is convergent.

Lemma 2.15. [6] *Let (X, S) be a cone S_b-metric space, P be a normal cone with normal constant K. Then a sequence $\{u_n\}$ in X converges to u if and only if $S(u_n,u_n,u)\to 0$ as $n\to\infty$.*

Lemma 2.16. [6] *Let (X, S) be a cone S_b-metric space, P be a normal cone with normal constant K. Let $\{u_n\}$ be a sequence in X. If $\{u_n\}$ converges to w_1 and $\{u_n\}$ converges to w_2, then $w_1=w_2$. That is the limit of a convergent sequence is unique.*

Lemma 2.17. [6] *Let (X, S) be a cone S_b-metric space, P be a normal cone with normal constant K. Then a sequence $\{u_n\}$ in X is a Cauchy sequence if and only if $S(u_n,u_n,u_m)\to 0$ as $n,m\to\infty$.*

Lemma 2.18. [6] *Let (X, S) be a cone S_b-metric space, P be a normal cone with normal constant K. Let $\{u_n\}$ be a sequence in X. If $\{u_n\}$ converges to w, then $\{u_n\}$ is a Cauchy sequence. That is every convergent sequence is Cauchy.*

Lemma 2.19. *Let (X,S) be a cone S_b-metric space. Then we have*

(1) $S(u,u,v) \le bS(v,v,u)$,

(2) $S(u,u,v) \le 2bS(u,u,a) + bS(v,v,a) \le 2bS(u,u,a) + b^2 S(a,a,v)$.

3. MAIN RESULTS

We now state and prove our main results.

First we give the corresponding definition of coupled fixed point in cone S_b-metric space.

Definition 3.1. Let (X,S) be a cone S_b-metric space. An element $(x,y) \in X \times X$ is said to be a coupled fixed point of $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

Theorem 3.2. *Let (X, S) be a complete cone S_b-metric space and P be a normal cone with normal constant K. Suppose the mapping $F : X \times X \to X$ satisfies the following contractive condition :*

$$S\big(F(x,y), F(x,y), F(u,v)\big) \le k\, S(x, x, u) + l\, S(y, y, v), \tag{3.1}$$

for all $x,y,u,v \in X$, where k,l are nonnegative constants with $k + l \in \left[0, \frac{1}{b^2}\right)$. Then, F has a unique coupled fixed point.

Proof : Let us choose $x_0, y_0 \in X$ and set

$x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), x_2 = F(x_1, y_1), y_2 = F(y_1, x_1), \ldots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n)$.
Then using (3.1), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S\big(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) \\ &\le kS(x_{n-1}, x_{n-1}, x_n) + lS(y_{n-1}, y_{n-1}, y_n). \end{aligned} \tag{3.2}$$

Also, we have

$$\begin{aligned} S(y_n, y_n, y_{n+1}) &= S\big(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big) \\ &\le kS(y_{n-1}, y_{n-1}, y_n) + lS(x_{n-1}, x_{n-1}, x_n). \end{aligned} \tag{3.3}$$

If we let $S_n = S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1})$, then we have

$$\begin{aligned} S_n &= S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) \\ &\le (k+l)\big(S(x_{n-1}, x_{n-1}, x_n) + S(y_{n-1}, y_{n-1}, y_n)\big) \\ &= (k+l) S_{n-1}. \end{aligned} \tag{3.4}$$

Thus if we take $\alpha = k + l < 1$, then for each $n \in \mathbb{N}$, we have

$$0 \le S_n \le \alpha S_{n-1} \le \alpha^2 S_{n-2} \le \ldots \le \alpha^n S_0. \tag{3.5}$$

If $S_0 = 0$, then (x_0, y_0) is a coupled fixed point of F. Let us therefore suppose that $S_0 > 0$.

Then for $m > n$, we have

$$\begin{aligned}
S(x_n, x_n, x_m) &\le b\left[2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1})\right] \\
&\le 2bS(x_n, x_n, x_{n+1}) + b^2 S(x_{n+1}, x_{n+1}, x_m) \\
&\le 2bS(x_n, x_n, x_{n+1}) + 2b^3 S(x_{n+1}, x_{n+1}, x_{n+2}) + b^4 S(x_{n+2}, x_{n+2}, x_m) \\
&\le 2bS(x_n, x_n, x_{n+1}) + 2b^3 S(x_{n+1}, x_{n+1}, x_{n+2}) + 2b^5 S(x_{n+2}, x_{n+2}, x_{n+3}) \\
&\quad + \ldots\ldots + b^{2(m-n-1)} S(x_{m-1}, x_{m-1}, x_m) \\
&\le 2b\left\{S(x_n, x_n, x_{n+1}) + b^2 S(x_{n+1}, x_{n+1}, x_{n+2}) + b^4 S(x_{n+2}, x_{n+2}, x_{n+3})\right. \\
&\quad \left. + \ldots\ldots + b^{2(m-n-1)} S(x_{m-1}, x_{m-1}, x_m)\right\}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
S(y_n, y_n, y_m) \le 2b&\left\{S(y_n, y_n, y_{n+1}) + b^2 S(y_{n+1}, y_{n+1}, y_{n+2}) + b^4 S(y_{n+2}, y_{n+2}, y_{n+3})\right. \\
&\left. + \ldots\ldots + b^{2(m-n-1)} S(y_{m-1}, y_{m-1}, y_m)\right\}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
S(x_n, x_n, x_m) + S(y_n, y_n, y_m) &\le 2b\left\{S_n + b^2 S_{n+1} + b^4 S_{n+2} + \ldots\ldots + b^{2(m-n-1)} S_{m-1}\right\} \\
&\le 2b\left\{\alpha^n + b^2 \alpha^{n+1} + b^4 \alpha^{n+2} + \ldots\ldots + b^{2(m-n-1)} \alpha^{m-1}\right\} S_0 \\
&= 2b\alpha^n \left\{1 + b^2 \alpha + b^4 \alpha^2 + \ldots\ldots + b^{2(m-n-1)} \alpha^{m-n-1}\right\} S_0 \\
&= 2b\alpha^n \left\{1 + b^2 \alpha + (b^2\alpha)^2 + \ldots\ldots + (b^2\alpha)^{m-n-1}\right\} S_0 \\
&\le \frac{2b\alpha^n}{1 - b^2\alpha} S_0.
\end{aligned}$$

$$\Rightarrow \left\|S(x_n, x_n, x_m) + S(y_n, y_n, y_m)\right\| \le \frac{2b\alpha^n K}{1 - b^2\alpha}\left\|S_0\right\| \to 0 \text{ as } n, m \to \infty.$$

$$\Rightarrow S(x_n, x_n, x_m) \to 0 \text{ and } S(y_n, y_n, y_m) \to 0 \text{ as } n, m \to \infty.$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in *X*. Using the completeness hypothesis, there exist $x^*, y^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. We show that (x^*, y^*) is a coupled fixed point of *F*.

Using condition 3(2.12.) and (3.1), we have

$$S\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x^*\right) \le 2bS\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x_{n+1}\right)+bS\left(x^*,x^*,x_{n+1}\right)$$

$$= 2bS\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),F\left(x_n,y_n\right)\right)+bS\left(x^*,x^*,x_{n+1}\right)$$

$$\le 2bkS\left(x^*,x^*,x_n\right)+2blS\left(y^*,y^*,y_n\right)+bS\left(x^*,x^*,x_{n+1}\right)$$

$$\Rightarrow \left\|S\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x^*\right)\right\| \le K\left[2bk\left\|S\left(x^*,x^*,x_n\right)\right\|+2bl\left\|S\left(y^*,y^*,y_n\right)\right\|+b\left\|S\left(x^*,x^*,x_{n+1}\right)\right\|\right]$$

$$\to 0 \text{ as } n\to\infty$$

$$\Rightarrow \left\|S\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x^*\right)\right\| = 0$$

$$\Rightarrow S\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x^*\right) = 0$$

$$\Rightarrow F\left(x^*,y^*\right) = x^*.$$

Similarly, we have $F\left(y^*,x^*\right)=y^*$. Thus $\left(x^*,y^*\right)$ is a coupled fixed point of *F*.

Now if (x',y') is another coupled fixed point of *F*, then we have

$$S\left(x',x',x^*\right)=S\left(F\left(x',y'\right),F\left(x',y'\right),F\left(x^*,y^*\right)\right)\le kS\left(x',x',x^*\right)+lS\left(y',y',y^*\right)$$

and $$S\left(y',y',y^*\right)=S\left(F\left(y',x'\right),F\left(y',x'\right),F\left(y^*,x^*\right)\right)\le kS\left(y',y',y^*\right)+lS\left(x',x',x^*\right).$$

Therefore we have

$$S\left(x',x',x^*\right)+S\left(y',y',y^*\right)\le (k+l)\left(S\left(x',x',x^*\right)+S\left(y',y',y^*\right)\right). \tag{3.6}$$

Since $k+l<1$, (3.6) implies that $S\left(x',x',x^*\right)+S\left(y',y',y^*\right)=0$. This means that $S\left(x',x',x^*\right)=0$ and $S\left(y',y',y^*\right)=0$. Hence we have $(x',y')=\left(x^*,y^*\right)$. Therefore, the coupled fixed point of *F* is unique.

Theorem 3.3. *Let (X, S) be a complete cone S_b-metric space and P be a normal cone with normal constant K. Suppose the mapping $F: X\times X\to X$ satisfies the following contractive condition :*

$$S\left(F(x,y),F(x,y),\ F(u,v)\right)\le k\,S\left(F(x,y),F(x,y),\ x\right)+l\,S\left(F(u,v),\ F(u,v),\ u\right),\qquad (3.7)$$

for all $x,y,u,v\in X$ *, where k,l are nonnegative constants with* $k+l\in\left[0,\frac{1}{2b^3}\right)$*. Then, F has a unique coupled fixed point.*

Proof : Let us choose $x_0,y_0\in X$ and set

$$x_1=F(x_0,y_0),y_1=F(y_0,x_0),x_2=F(x_1,y_1),y_2=F(y_1,x_1),\ldots,x_{n+1}=F(x_n,y_n),y_{n+1}=F(y_n,x_n).$$

Using the condition (3.7) and 3(2.12.) , we get

$$\begin{aligned}S(x_n,x_n,x_{n+1})&=S\left(F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1}),F(x_n,y_n)\right)\\&\le kS\left(F(x_{n-1},y_{n-1}),F(x_{n-1},y_{n-1}),x_{n-1}\right)+lS\left(F(x_n,y_n),F(x_n,y_n),x_n\right)\\&\le kS(x_n,x_n,x_{n-1})+lS(x_{n+1},x_{n+1},x_n)\\&\le kbS(x_{n-1},x_{n-1},x_n)+lbS(x_n,x_n,x_{n+1})\end{aligned}$$

$$\Rightarrow S(x_n,x_n,x_{n+1})\le\frac{kb}{1-lb}S(x_{n-1},x_{n-1},x_n)$$

$$\Rightarrow S(x_n,x_n,x_{n+1})\le\beta S(x_{n-1},x_{n-1},x_n)\ \text{where}\ \beta=\frac{kb}{1-lb}<1.\qquad (3.8)$$

Similarly, we have

$$S(y_n,y_n,y_{n+1})\le\beta S(y_{n-1},y_{n-1},y_n).\qquad (3.9)$$

Therefore for $m>n,$ we get

$$S(x_n,x_n,x_m)\le\frac{2b\beta^n}{1-b^2\beta}S(x_0,x_0,x_1)\ \text{and}\ S(y_n,y_n,y_m)\le\frac{2b\beta^n}{1-b^2\beta}S(y_0,y_0,y_1).$$

$$\Rightarrow S(x_n,x_n,x_m)\to 0\ \text{and}\ S(y_n,y_n,y_m)\to 0\ \text{as}\ n,m\to\infty.$$

Here we note that $b^2\beta=\frac{kb^3}{1-lb}<1.$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in *X*. Using the completeness hypothesis, there exist $x^*,y^*\in X$ such that $\lim_{n\to\infty}x_n=x^*$ and $\lim_{n\to\infty}y_n=y^*$. We show that (x^*,y^*) is a coupled fixed point of *F*.

Using condition 3(2.12.) and (3.7), we have

$$S\left(F(x^*,y^*),F(x^*,y^*),x^*\right)\le 2bS\left(F(x^*,y^*),F(x^*,y^*),x_{n+1}\right)+bS\left(x^*,x^*,x_{n+1}\right)$$

$$= 2bS\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),F\left(x_n,y_n\right)\right)+bS\left(x^*,x^*,x_{n+1}\right)$$

$$\leq 2bkS\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x^*\right)+2blS\left(F\left(x_n,y_n\right),F\left(x_n,y_n\right),x_n\right)+bS\left(x^*,x^*,x_{n+1}\right)$$

$$\Rightarrow S\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x^*\right)\leq \frac{2bl}{1-2bk}S\left(F\left(x_n,y_n\right),F\left(x_n,y_n\right),x_n\right)+\frac{b}{1-2bk}S\left(x^*,x^*,x_{n+1}\right)$$

$$\Rightarrow S\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x^*\right)\leq \frac{2bl}{1-2bk}S\left(x_{n+1},x_{n+1},x_n\right)+\frac{b^2}{1-2bk}S\left(x_{n+1},x_{n+1},x^*\right)$$

$$\Rightarrow \left\|S\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x^*\right)\right\|\leq K\left[\frac{2bl}{1-2bk}\left\|S\left(x_{n+1},x_{n+1},x_n\right)\right\|+\frac{b^2}{1-2bk}\left\|S\left(x_{n+1},x_{n+1},x^*\right)\right\|\right]$$

$$\to 0 \text{ as } n\to\infty$$

$$\Rightarrow \left\|S\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x^*\right)\right\|=0$$

$$\Rightarrow S\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x^*\right)=0$$

$$\Rightarrow F\left(x^*,y^*\right)=x^*.$$

Similarly, we can get $F\left(y^*,x^*\right)=y^*$. Thus $\left(x^*,y^*\right)$ is a coupled fixed point of *F*.

Now if $\left(x',y'\right)$ is another coupled fixed point of *F*, then we have

$$S\left(x',x',x^*\right)=S\left(F\left(x',y'\right),F\left(x',y'\right),F\left(x^*,y^*\right)\right)$$

$$\leq kS\left(F\left(x',y'\right),F\left(x',y'\right),x'\right)+lS\left(F\left(x^*,y^*\right),F\left(x^*,y^*\right),x^*\right)=0.$$

Therefore, $S\left(x',x',x^*\right)=0$ and so $x'=x^*$. Similarly, we can get $y'=y^*$. Hence we have $\left(x',y'\right)=\left(x^*,y^*\right)$ which shows that the coupled fixed point of *F* is unique.

Theorem 3.4. *Let* $(X,\ S)$ *be a complete cone* S_b*-metric space and P be a normal cone with normal constant K. Suppose the mapping* $F: X\times X\to X$ *satisfies the following contractive condition :*

$$S\left(F\left(x,y\right),F\left(x,y\right),\ F\left(u,v\right)\right)\leq k\,S\left(F\left(x,y\right),F\left(x,y\right),\ u\right)+l\,S\left(F\left(u,v\right),\ F\left(u,v\right),\ x\right), \quad (3.10)$$

for all $x,y,u,v\in X$ *, where k,l are nonnegative constants with* $k+l\in\left[0,\frac{1}{4b^3}\right)$*. Then, F has a unique coupled fixed point.*

Proof : Let us choose $x_0, y_0 \in X$ and set

$x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), x_2 = F(x_1, y_1), y_2 = F(y_1, x_1), \ldots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n).$

Using the condition (3.10) and 3(2.12.) , we get

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\le kS(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), x_n) + lS(F(x_n, y_n), F(x_n, y_n), x_{n-1}) \\ &\le kS(x_n, x_n, x_n) + lS(x_{n+1}, x_{n+1}, x_{n-1}) \\ &\le lS(x_{n+1}, x_{n+1}, x_{n-1}) \\ &\le l(2bS(x_{n+1}, x_{n+1}, x_n) + bS(x_{n-1}, x_{n-1}, x_n)) \\ &\le l(2b^2 S(x_n, x_n, x_{n+1}) + bS(x_{n-1}, x_{n-1}, x_n)) \end{aligned}$$

$$\Rightarrow S(x_n, x_n, x_{n+1}) \le \frac{lb}{1-2lb^2} S(x_{n-1}, x_{n-1}, x_n)$$

$$\Rightarrow S(x_n, x_n, x_{n+1}) \le \delta S(x_{n-1}, x_{n-1}, x_n), \text{ where } \delta = \frac{lb}{1-2lb^2} < 1. \quad (3.11)$$

Similarly, we can get

$$S(y_n, y_n, y_{n+1}) \le \delta S(y_{n-1}, y_{n-1}, y_n). \quad (3.12)$$

Therefore for $m > n,$ we get

$$S(x_n, x_n, x_m) \le \frac{2b\delta^n}{1-b^2\delta} S(x_0, x_0, x_1) \text{ and } S(y_n, y_n, y_m) \le \frac{2b\delta^n}{1-b^2\delta} S(y_0, y_0, y_1).$$

$$\Rightarrow S(x_n, x_n, x_m) \to 0 \text{ and } S(y_n, y_n, y_m) \to 0 \text{ as } n, m \to \infty.$$

Here we note that $b^2\delta = \dfrac{lb^3}{1-2lb^2} < 1.$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in *X*. Using the completeness hypothesis, there exist $x^*, y^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. We show that (x^*, y^*) is a coupled fixed point of *F*.

Using condition 3 (2.12.) and (3.10), we have

$$S(F(x^*, y^*), F(x^*, y^*), x^*) \le 2bS(F(x^*, y^*), F(x^*, y^*), x_{n+1}) + bS(x^*, x^*, x_{n+1})$$

$$= 2bS\left(F\left(x^*, y^*\right), F\left(x^*, y^*\right), F\left(x_n, y_n\right)\right) + bS\left(x^*, x^*, x_{n+1}\right)$$

$$\leq 2bkS\left(F\left(x^*, y^*\right), F\left(x^*, y^*\right), x_n\right) + 2blS\left(F\left(x_n, y_n\right), F\left(x_n, y_n\right), x^*\right) + bS\left(x^*, x^*, x_{n+1}\right)$$

$$\leq 2bkS\left(F\left(x^*, y^*\right), F\left(x^*, y^*\right), x_n\right) + 2blS\left(x_{n+1}, x_{n+1}, x^*\right) + b^2S\left(x_{n+1}, x_{n+1}, x^*\right)$$

$$\leq 4b^2kS\left(F\left(x^*, y^*\right), F\left(x^*, y^*\right), x^*\right) + 2b^2kS\left(x_n, x_n, x^*\right) + \left(2bl + b^2\right)S\left(x_{n+1}, x_{n+1}, x^*\right)$$

$$\Rightarrow S\left(F\left(x^*, y^*\right), F\left(x^*, y^*\right), x^*\right) \leq \frac{2b^2k}{1-4b^2k}S\left(x_n, x_n, x^*\right) + \frac{2bl+b^2}{1-4b^2k}S\left(x_{n+1}, x_{n+1}, x^*\right)$$

$$\Rightarrow \left\|S\left(F\left(x^*, y^*\right), F\left(x^*, y^*\right), x^*\right)\right\| \leq K\left[\frac{2b^2k}{1-4b^2k}\left\|S\left(x_n, x_n, x^*\right)\right\| + \frac{2bl+b^2}{1-4b^2k}\left\|S\left(x_{n+1}, x_{n+1}, x^*\right)\right\|\right]$$

$$\to 0 \text{ as } n \to \infty$$

$$\Rightarrow \left\|S\left(F\left(x^*, y^*\right), F\left(x^*, y^*\right), x^*\right)\right\| = 0$$

$$\Rightarrow S\left(F\left(x^*, y^*\right), F\left(x^*, y^*\right), x^*\right) = 0$$

$$\Rightarrow F\left(x^*, y^*\right) = x^*.$$

Similarly, we can get $F\left(y^*, x^*\right) = y^*$. Thus $\left(x^*, y^*\right)$ is a coupled fixed point of *F*.

Now if $\left(x', y'\right)$ is another coupled fixed point of *F*, then we have

$$S\left(x', x', x^*\right) = S\left(F\left(x', y'\right), F\left(x', y'\right), F\left(x^*, y^*\right)\right)$$

$$\leq kS\left(F\left(x', y'\right), F\left(x', y'\right), x^*\right) + lS\left(F\left(x^*, y^*\right), F\left(x^*, y^*\right), x'\right)$$

$$= kS\left(x', x', x^*\right) + lS\left(x^*, x^*, x'\right)$$

$$\leq \left(k + lb\right)S\left(x', x', x^*\right). \tag{3.13}$$

Since $k + lb < 1$, (3.13) implies $S\left(x', x', x^*\right) = 0$ and so $x' = x^*$. Similarly, we can get $y' = y^*$. Hence we have $\left(x', y'\right) = \left(x^*, y^*\right)$, which shows that the coupled fixed point of *F* is unique.

When $k = l$ in Theorems 3.2., 3.3. and 3.4., we get the following corollaries.

Corollary 3.5. *Let* (X, S) *be a complete cone* S_b*-metric space and P be a normal cone with normal constant K. Suppose the mapping* $F : X \times X \to X$ *satisfies the following contractive condition :*

$$S\left(F(x,y),F(x,y),\ F(u,v)\right)\le k\left(S(x,\ x,\ u)+S(y,\ y,\ v)\right), \tag{3.14}$$

for all $x,y,u,v\in X$ *, where* $k\in\left[0,\frac{1}{2b^2}\right)$ *is a constant. Then, F has a unique coupled fixed point.*

Corollary 3.6. *Let* $(X,\ S)$ *be a complete cone* S_b*-metric space and P be a normal cone with normal constant K. Suppose the mapping* $F: X\times X\to X$ *satisfies the following contractive condition :*

$$S\left(F(x,y),F(x,y),\ F(u,v)\right)\le k\left(S\left(F(x,y),F(x,y),\ x\right)+S\left(F(u,v),\ F(u,v),\ u\right)\right), \tag{3.15}$$

for all $x,y,u,v\in X$ *, where* $k\in\left[0,\frac{1}{4b^3}\right)$ *is a constant. Then, F has a unique coupled fixed point.*

Corollary 3.7. *Let* $(X,\ S)$ *be a complete cone* S_b*-metric space and P be a normal cone with normal constant K. Suppose the mapping* $F: X\times X\to X$ *satisfies the following contractive condition :*

$$S\left(F(x,y),F(x,y),\ F(u,v)\right)\le k\left(S\left(F(x,y),F(x,y),\ u\right)+S\left(F(u,v),\ F(u,v),\ x\right)\right), \tag{3.16}$$

for all $x,y,u,v\in X$ *, where* $k\in\left[0,\frac{1}{8b^3}\right)$ *is a constant. Then, F has a unique coupled fixed point.*

Here we give an example to illustrate Corollary 3.5.

Example 3.8. Let $E=\mathbb{R}^2$, the Euclidean plane, and $P=\{(x,y)\in E: x,y\ge 0\}$, a normal cone in E. Let $X=\mathbb{R}$ and $S: X\times X\times X\to E$ be such that $S(x,y,z)=\left(|x-z|+|y-z|,|x-z|+|y-z|\right)$, $x,y,z\in X$. Then S is a cone S_b-metric on X with $b=1$ and (X, S) is a complete cone S_b-metric space. Let us consider the mapping $F: X\times X\to X$ defined by $F(x,y)=\frac{x+y}{3}$.

Then we have $S\left(F(x,y),F(x,y),F(u,v)\right)=S\left(\frac{x+y}{3},\frac{x+y}{3},\frac{u+v}{3}\right)$

$$=\left(\left|\frac{x+y}{3}-\frac{u+v}{3}\right|+\left|\frac{x+y}{3}-\frac{u+v}{3}\right|,\left|\frac{x+y}{3}-\frac{u+v}{3}\right|+\left|\frac{x+y}{3}-\frac{u+v}{3}\right|\right)$$

$$=\frac{2}{3}\left(|x+y-u-v|,|x+y-u-v|\right)$$

$$\leq \frac{2}{3}\left(\left(|x-u|,|x-u|\right)+\left(|y-v|,|y-v|\right)\right)$$

And $S\left(x,x,u\right)+S\left(y,y,v\right)=\left(|x-u|+|x-u|,|x-u|+|x-u|\right)+\left(|y-v|+|y-v|,|y-v|+|y-v|\right)$

$$=2\left(\left(|x-u|,|x-u|\right)+\left(|y-v|,|y-v|\right)\right)$$

Therefore we have $S\left(F\left(x,y\right),F\left(x,y\right),F\left(u,v\right)\right)\leq k\left(S\left(x,x,u\right)+S\left(y,y,v\right)\right)$, for $k=\frac{1}{3}\in\left[0,\frac{1}{2}\right)$.

Thus the condition of Corollary 3.5. is satisfied and F has a unique coupled fixed point $\left(0,0\right)$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Gupta, Cyclic contraction on cone *S*-metric space, Int. J. Anal. Appl. 3 (2) (2013), 119-130.

[2] D. Dhamodharan and R. Krishnakumar, Cone *S*-Metric Space and Fixed Point Theorems of Contractive Mappings, Ann. Pure Appl. Math. 14 (2) (2017), 237-243.

[3] F. Sabetghadam, H.P. Masiha, A.H. Sanatpour, Some coupled fixed point theorems in cone metric spaces, Fixed Point Theory Appl. 2009 (2009), Article ID 125426.

[4] H. K. Nashine, Yumnam Rohen and Th. Chhatrajit, Common coupled fixed point theorems of two mappings satisfying generalized contractive condition in cone metric space, Int. J. Pure Appl. Math. 106 (3) (2016), 791-799.

[5] H. Long-Guang and Z. Xian, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332 (2007), 1468-1476.

[6] K. Anthony Singh, M. R. Singh, Some fixed point theorems of cone S_b-metric space, J. Indian Acad. Math. 40 (2) (2018), 255-272.

[7] L. Shanjit, Yumnam Rohen, Tripled Fixed point in Ordered Multiplicative Metric Spaces, J. Nonlinear Anal. Appl. 2017 (1) (2017), 56-65.

[8] M.U. Rahman and M. Sarwar, Fixed point results of Altman integral type mappings in *S*-metric spaces, Int. J. Anal. Appl. 10 (1) (2016), 58-63.

[9] N. Mlaiki, Yumnam Rohen, Some Coupled fixed point theorems in partially ordered A_b-metric space, J. Nonlinear Sci. Appl. 10 (2017), 1731-1743.

[10] N. Priyobarta, Yumnam Rohen and N. Mlaiki, Complex valued S_b-metric spaces, J. Math. Anal. 8 (3) (2017), 13-24.

[11] N. Souayah and N. Mlaiki, A fixed point theorem in S_b-metric spaces, J. Math. Computer Sci., 16 (2016), 131-139.

[12] N. Tas and N. Yilmaz Ozgur, New generalized fixed point results on S_b -metric spaces, arxiv: 1703.01868v2 [math.GN], 2017.

[13] N.Y. Ozgur and N. Tas, Some fixed point theorems on *S*-metric spaces, Mat. Vesnik, 69 (1) (2017), 39-52.

[14] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in *S*-metric spaces, Mat. Vesnik, 64 (3) (2012), 258-266.

[15] S. Sedghi, A. Gholidahneh and K.P.R. Rao, Common Fixed Point of Two R-Weakly Commuting Mappings in S_b-Metric Spaces, Math. Sci. Lett. 6 (3) (2017), 249-253.

[16] Y. Rohen, T. Dosenovic and S. Radenovic, A note on the paper "A fixed point theorem in S_b-metric spaces", Filomat, 31 (11) (2017), 3335-3346.

[17] T.G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal., Theory Methods Appl. 65 (7) (2006), 1379-1393.

[18] V. Lakshmikantham and L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal., Theory Methods Appl. 70 (12) (2009), 4341-4349.