SOME NEW RESULTS ON PROPER COLOURING OF EDGE-SET GRAPHS

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Abstract. In this paper, we present a foundation study for proper colouring of edge-set graphs. The authors consider that a detailed study of the colouring of edge-set graphs corresponding to the family of paths is best suitable for such foundation study. The main result is deriving the chromatic number of the edge-set graph of a path, $P_{n+1}$, $n \geq 1$. It is also shown that edge-set graphs for paths are perfect graphs.

Keywords: chromatic colouring; rainbow neighbourhood; rainbow neighbourhood number; edge-set graph.

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1. INTRODUCTION

For general notation and concepts in graphs and digraphs see [1, 2, 12]. Unless mentioned otherwise, all graphs we consider in this paper are finite, simple, connected and undirected graphs.

For a set of distinct colours $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\ell\}$, a vertex colouring of a graph $G$ is an assignment $\varphi : V(G) \mapsto \mathcal{C}$. A vertex colouring is said to be a proper vertex colouring of a graph $G$ if no two distinct adjacent vertices have the same colour. The cardinality of a minimum set of colours in a proper vertex colouring of $G$ is called the chromatic number of $G$ and is denoted $\chi(G)$.
A colouring of $G$ with exactly $\chi(G)$ colours may be called a $\chi$-colouring or a chromatic colouring of $G$.

A minimum parameter colouring of a graph $G$ is a proper colouring of $G$ which consists of the colours $c_i; 1 \leq i \leq \ell$, with minimum possible values for the subscripts $i$. Unless stated otherwise, we consider minimum parameter colouring throughout this paper.

The set of vertices of $G$ having the colour $c_i$ is said to be the colour class of $c_i$ in $G$ and is denoted by $C_i$. The cardinality of the colour class $C_i$ is said to be the weight of the colour $c_i$, denoted by $\theta(c_i)$. Note that $\sum_{i=1}^{\ell} \theta(c_i) = \nu(G)$.

Unless mentioned otherwise, we colour the vertices of a graph $G$ in such a way that $C_1 = I_1$, the maximal independent set in $G$, $C_2 = I_2$, the maximal independent set in $G_1 = G - C_1$ and proceed like this until all vertices are coloured. This convention is called rainbow neighbourhood convention (see [5]). The number of vertices in $G$ which yield rainbow neighbourhoods, denoted by $r_{\chi}(G)$, is called the rainbow neighbourhood number of $G$.

In [5], the bounds on $r_{\chi}(G)$ corresponding to of minimum proper colouring, denoted by $r_{\chi}^{-}(G)$ and $r_{\chi}^{+}(G)$, have been defined as the minimum value and maximum value of $r_{\chi}(G)$ over all permissible colour allocations. If we relax connectedness, it follows that the null graph $\emptyset_n$ of order $n \geq 1$ has $r^{-}(\emptyset_n) = r^{+}(\emptyset_n) = n$. For bipartite graphs and complete graphs, $K_n$ it follows that, $r^{-}(G) = r^{+}(G) = n$ and $r^{-}(K_n) = r^{+}(K_n) = n$.

We observe that if it is possible to permit a chromatic colouring of any graph $G$ of order $n$ such that the star subgraph obtained from vertex $v$ as center and its open neighbourhood $N(v)$ the pendant vertices, has at least one coloured vertex from each colour for all $v \in V(G)$ then $r_{\chi}(G) = n$. Certainly, examining this property for any given graph is complex.

**Lemma 1.1.** [5] For any graph $G$ the graph $G' = K_1 + G$ has $r_{\chi}(G') = 1 + r_{\chi}(G)$.

### 2. Rainbow Neighbourhood Number of Edge-set Graphs

Edge-set graphs were introduced in [4]. As the notion of an edge-set graph seems to be largely unknown. Therefore, the main definition and some important observations from [4] will be presented in this section.
Let $A$ be a non-empty finite set. Let the set of all $s$-element subsets of $A$ (arranged in some order), where $1 \leq s \leq |A|$, be denoted by $S$ and the $i$-th element of $S$ by $A_{i,s}$.

**Definition 2.1.** [4] Let $G(V,E)$ be a non-empty finite graph with $|E| = \varepsilon \geq 1$ and $E = \mathcal{P}(E) - \{\emptyset\}$, where $\mathcal{P}(E)$ is the power set of the edge set $E(G)$. For $1 \leq s \leq \varepsilon$, let $S$ be the collection of all $s$-element subsets of $E(G)$ and $E_{s,i}$ be the $i$-th element of $S$. Then, the edge-set graph corresponding to $G$, denoted by $G_G$, is the graph with the following properties.

(i) $|V(G_G)| = 2\varepsilon - 1$ so that there exists a one to one correspondence between $V(G_G)$ and $E$;

(ii) Two vertices, say $v_{s,i}$ and $v_{t,j}$, in $G_G$ are adjacent if some elements (edges of $G$) in $E_{s,i}$ is adjacent to some elements of $E_{t,j}$ in $G$.

From the above definition, it can be seen that the edge-set graph $G_G$ of a given graph $G$ is dependent not only on the number of edges $\varepsilon$, but the structure of $G$ also. Note that it was erroneously remarked in [4] that non-isomorphic graphs of the same size have distinct edge-set graphs. Figure 2 illustrates one contradictory case.

Note that an edge-set graph $G_G$ has an odd number of vertices. If $G$ is a trivial graph, then $G_G$ is an empty graph (since $\varepsilon = 0$). Also, $G_{P_2} = K_1$ and $G_{P_3} = C_3$. In [4] the following conventions were used.

(i) If an edge $e_j$ is incident with vertex $v_k$, then we write it as $(e_j \rightarrow v_k)$.

(ii) If the edges $e_i$ and $e_j$ of a graph $G$ are adjacent, then we write it as $e_i \sim e_j$.

(iii) The $n$ vertices of the path $P_n$ are positioned horizontally and the vertices and edges are labeled from left to right as $v_1, v_2, v_3, \ldots, v_n$ and $e_1, e_2, e_3, \ldots, e_{n-1}$, respectively.

(iv) The $n$ vertices of the cycle $C_n$ are seated on the circumference of a circle and the vertices and edges are labeled clockwise as $v_1, v_2, v_3, \ldots, v_n$ and $e_1, e_2, e_3, \ldots, e_n$, respectively such that $e_i = v_iv_{i+1}$, in the sense that $v_{n+1} = v_1$.

Invoking the definition and observations given above, it is noticed that both $d_{G_{(e)}}(G)$ and $d_{G_{(e)}}(v_i)$ are single values, while $d_{G_{(v)}}(e_j) \leq d_{G_{(v)}}(e_j), (e_j \rightarrow v_k), (e_j \rightarrow v_m)$. The graphs having three edges $e_1, e_2, e_3$ are graphs $P_4, C_3$, and $K_{1,3}$. The corresponding edge-set graphs
on the vertices $v_{1,1} = \{e_1\}, v_{1,2} = \{e_2\}, v_{1,3} = \{e_3\}, v_{2,1} = \{e_1, e_2\}, v_{2,2} = \{e_1, e_3\}, v_{2,3} = \{e_2, e_3\}, v_{3,1} = \{e_1, e_2, e_3\}$ are depicted below.

Figure 1 depicts the edge-set graph $\mathcal{G}_P$.

![Figure 1](image1.png)

Figure 1. Edge-set graph $\mathcal{G}_P$.

Figure 2 depicts the edge-set graph $\mathcal{G}_C = \mathcal{G}_{K_{1,3}} = K_7$.

![Figure 2](image2.png)

Figure 2. Edge-set graph $\mathcal{G}_C = \mathcal{G}_{K_{1,3}} = K_7$.

Notice that both $\mathcal{G}_C$ and $\mathcal{G}_{K_{1,3}}$ are complete graphs.

3. **Proper Colouring of the Edge-set Graphs of Paths**

It is known that for a given size $\varepsilon \geq 1$ a graph of maximum order $\nu$, is a tree. Hence, for a given size the graphs with maximum structor index $si(G)$ are the corresponding trees,
It easily follows that for $\varepsilon(T) \geq 3$ only the star graphs have $\mathcal{G}_{S_{\varepsilon+1}}$, complete. Put another way, a tree $T$ has $\mathcal{G}_T$ complete if and only if $diam(T) \leq 2$. From the family of trees, a path corresponding to a given $\varepsilon$, denoted by $P_\varepsilon$, has largest diameter. These observations motivate a detailed study of the proper colouring and associated colour parameters of edge-set graphs of paths to lay the foundation for studying more complex graph classes.

For this section paths of the form $P_{n+1} = v_1e_1v_2e_2v_3\cdots e_nv_{n+1}$, will be considered. Such graph will be abbreviated to $P_{n+1} = v_1e_1v_2 \succ v_{n+1}$. To easily relate the results with Definition 2.1, note that $\varepsilon(P_{n+1}) = n$. It can be easily verified that $\mathcal{G}_{P_2} = K_1$. Hence, $\chi(\mathcal{G}_{P_2}) = 1$. Also, $\mathcal{G}_{P_3} = K_3$ and hence, $\chi(\mathcal{G}_{P_3}) = 3$. These observations bring the main results. First, we state an important lemma.

**Lemma 3.1.** Let $G(V,E)$ be a non-empty finite graph with $|E| = \varepsilon \geq 1$ and $\mathcal{E} = \mathcal{P}(E) - \{\emptyset\}$, where $\mathcal{P}(E)$ is the power set of the edge set $E(G)$. Then each edge $e_i$ is in exactly $2^{\varepsilon-1}$ subsets of $\mathcal{E}$.

**Proof.** The result follows directly from the well-definedness and well-ordering of the power set, $\mathcal{P}(E)$. □

It is observed that if the number of subsets which has say, $e_i$ as element is $t$, then within the corresponding $t$ subsets the edge $e_j$, $j \neq i$ will be in $\frac{t}{2} = 2^{\varepsilon-2}$ of those subsets.

**Theorem 3.2.** The edge-set graph $\mathcal{G}_{P_{n+1}}$, $n \geq 1$ has

$$
\chi(\mathcal{G}_{P_{n+1}}) = \begin{cases} 
1 \text{ or } 3, & \text{if } P_2 \text{ or } P_3 \text{ respectively,} \\
5, & \text{if } P_4, \\
2^{n-1} + 2^{n-2} - 2, & \text{for } P_{n+1}, \ n \geq 4.
\end{cases}
$$

**Proof. Part 1:** Trivial is the observation that $\mathcal{G}_{P_2} = K_1$ and that result in equality. It has been observed that $\mathcal{G}_{P_3} = K_3$ and hence $\chi(\mathcal{G}_{P_3}) = 3$.

**Part 2:** In constructing $\mathcal{G}_{P_4}$ begin with $\mathcal{G}_{P_3}$ which has vertices $\{e_1\}, \{e_2\}, \{e_1,e_2\}$. Add a disjoint copy of $\mathcal{G}_{P_3}$ and relabel the vertices of this copy to be $\{e_1,e_3\}, \{e_2,e_3\}, \{e_1,e_2,e_3\}$ to obtain, $\mathcal{G}_{P_4}$. Clearly, $\mathcal{G}_{P_4}$ complies with Definition 2.1.
Consider $H = \mathcal{G}_P \cup \mathcal{G}_F$ and add the cut edges, \{e_2\}\{e_1, e_3\}, \{e_2\}\{e_2, e_3\}, \{e_2\}\{e_1, e_2, e_3\}, \{e_1, e_2\}\{e_1, e_3\}, \{e_1, e_2\}\{e_1, e_2, e_3\}, \{e_1, e_2\}\{e_2, e_3\}, \{e_1, e_2\}\{e_1, e_2, e_3\}. Clearly, the induced subgraph, \{\{e_2\}, \{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}\} = K_5. Now add all additional bridges in accordance with Definition 2.1 to obtain graph $H'$. Due to symmetry considerations between edges $e_1$ and $e_2$ in $P_3$, exactly two maximum cliques $K_5$ come into existence hence, $w(H') = 5$. Finally, by adding vertex \{e_3\} and the corresponding edges in accordance with Definition 2.1 and by symmetry considerations between edges $e_1$ and $e_3$ in $P_4$, the edge-set graph $\mathcal{G}_P$ has exactly four maximum cliques $K_5$. Therefore, $\chi(\mathcal{G}_P) \geq 5$.

Invoking Definition 2.1, consider the following colouring of $\mathcal{G}_P$. Let $c(v_{1,1}) = c_1$, $c(v_{1,3}) = c_1$, $c(v_{2,2}) = c_1$, $c(v_{1,2}) = c_2$, $c(v_{2,1}) = c_3$, $c(v_{2,3}) = c_4$, $c(v_{3,1}) = c_5$. Clearly, the colouring is proper and hence $\chi(\mathcal{G}_P) \leq 5$. Hence we have $\chi(\mathcal{G}_P) = 5$.

Part 3: For $n \geq 4$, and the path path $P_{n+1}$ the edge-set graph $\mathcal{G}_{P(n-1)+1}$ of the preceding path hence, the $(n - 1)$-edge path $P_{(n-1)+1}$, is incomplete. In accordance with the procedure described in Part 2, consider $\mathcal{G}_{P_{(n-1)+1}}$ and $\mathcal{G}_{P_{(n-1)+1}}$. Since in $\mathcal{G}_{P_{(n-1)+1}}$ the edge $e_n$ has been added to each vertex corresponding to the vertices $v_{i,j} \in V(\mathcal{G}_{P_{(n-1)+1}})$, the new edges in accordance with Definition 2.1 are those between all pairs of vertices for which at least one vertex has $e_{n-1} \in v'_{i,j}$. From Lemma 3.1, it follows that at least one complete induced subgraph, $K_{2n-2}$ exists in $\mathcal{G}_{P_{(n-1)+1}}$. All pairs of vertices which has both $e_{n-2}, e_{n-1} \in v'_{i,j}$ is an edge in $\mathcal{G}_{P_{(n-1)+1}}$, so least one complete induced subgraph, $K_{2n-3}$ exists in $\mathcal{G}_{P_{(n-1)+1}}$. Proceeding to vertices for which edge $e_{n-3} \in v'_{i,j}$ and so on until the edge $e_1$ has been accounted for results in $\mathcal{G}_{P_{(n-1)+1}}$ being complete. Hence, $\chi(\mathcal{G}_{P_{(n-1)+1}}) = 2^{n-1} - 1$.

Finally, by adding the bridges between $\mathcal{G}_{P_{(n-1)+1}}$ and $\mathcal{G}_{P_{(n-1)+1}}$ and through similar arguments in respect of edges $e_{n-2}, e_{n-1} \in v_{i,j} \in V(\mathcal{G}_{P_{(n-1)+1}})$ and so on, it follows that at least one maximum induced clique, of order $2^{n-2} - 1 + \chi(\mathcal{G}_{P_{(n-1)+1}})$, exists in $\mathcal{G}_{P_{n+1}}$. Therefore, $\chi(\mathcal{G}_{P_{n+1}}) \geq 2^{n-1} + 2^{n-2} - 2$. By allocating colours similar to the procedure described in Part-2, it follows that $2^{n-1} + 2^{n-2} - 2 \leq \chi(\mathcal{G}_{P_{n+1}}) \leq 2^{n-1} + 2^{n-2} - 2 \Leftrightarrow \chi(\mathcal{G}_{P_{n+1}}) = 2^{n-1} + 2^{n-2} - 2$. Therefore, by immediate induction, the result follows for all $n \geq 4$.

**Corollary 3.3.** (a) Each vertex in an edge-set graph $\mathcal{G}_{P_{n+1}}$, $n \geq 2$ belongs to some maximum clique in $\mathcal{G}_{P_{n+1}}$. 

(b) The edge-set graphs $G_{P_{n+1}}, n \geq 1$ has clique number, $\omega(G_{P_{n+1}}) = 2^{n-1} + 2^{n-2} - 2$.

(c) The edge-set graphs $G_{P_{n+1}}, n \geq 1$ are perfect graphs.

(d) The edge-set graph $G_{P_{n+1}}$ has, $r^- c_{hi}(G_{P_{n+1}}) = r^+ c_{hi}(G_{P_{n+1}}) = 2^n - 1$.

Proof. The results are a direct consequence from the proof of Theorem 3.2. □

Theorem 3.4. An edge-set graph $G_{P_{n+1}}, n \geq 1$ is a perfect graph.

Proof. For $P_1, P_2$ the result is trivial. From Theorem 3.2 and Corollary 3.3(b) we have, $n \geq 2$ and hence it follows that $\omega(G_{P_{n+1}}) = 2^{n-1} + 2^{n-2} - 2 = \chi(G_{P_{n+1}})$. Hence, an edge-set graph is weakly perfect. From Definition 2.1, it follows that an edge-set graph has a unique maximum independent set $X$. Furthermore, $\langle X \rangle$ is a null graph hence, any subgraph thereof is perfect.

Also, from Corollary 3.3(a), each vertex in $V(G_{P_{n+1}})$ is in some induced maximum clique. It then follows that $\omega(H) = \chi(H), \forall H \subseteq G_{P_{n+1}}, n \geq 1$. Hence the result. □

Conjecture 1. The edge-set graphs of acyclic graphs are perfect graphs.

4. CONCLUSION

Research problem: The notion of a chromatic core subgraph of a graph $G$ was introduced in [9]. We recall that, for a graph $G$ its structural size is measured by its structor index denoted and defined as, $si(G) = v(G) + e(G)$. We say that the smaller of graphs $G$ and $H$ is the graph satisfying the condition, $min\{si(G), si(H)\}$. If $si(G) = si(H)$ the graphs are of equal structural size but not necessarily isomorphic. A straight forward example is the path, $P_4$ and the star graph, $S_3$.

Definition 4.1. For a finite, undirected simple graph $G$ of order $v(G) = n \geq 1$ a chromatic core subgraph $H$ is a smallest induced subgraph $H$ (smallest in respect of $si(H)$) such that, $\chi(H) = \chi(G)$.

From the construction used in the proof of Theorem 3.2 it follows that a finite number of distinct maximum cliques can be associated with a given edge-set graph $G_{P_{n+1}}$. As an application, the largest number of vertices common to the maximum number of chromatic core subgraphs can be considered the most strategic vertices for protection from a disaster management and
recovery plan in the event of graph destruction. The aforesaid observation motivates us to introduce a new graph parameter called the \textit{chromatic cluster number} of a graph $G$. It is denoted by $\mathcal{C}(G)$. From Theorem 3.2 it follows that $\mathcal{C}(G_{P_2}) = \mathcal{C}(G_{P_3}) = 1$ and $\mathcal{C}(G_{P_4}) = 4$. Note that the vertices $v_{1,1} = \{e_1\}$, $v_{1,3} = \{e_3\}$, $v_{2,2} = \{e_1, e_3\}$ and $v_{1,3} = \{e_1, e_2, e_3\}$ corresponds to $\mathcal{C}(G_{P_4})$.

\textbf{Problem 1.} For the edge-set graph $G_{P_{n+1}}, n \geq 4$, determine $\mathcal{C}(G_{P_{n+1}})$.

The research on set-graphs (see [3]) and edge-set graphs naturally leads to new concepts such as vertex degree sequence set-graphs and colour set-graphs and colour-string set-graphs. Preliminary definitions are provided below.

(1) If the degree sequence of a graph $G$ of order $n \geq 1$ is $(d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_n)$, then for a subsequence $(d_{t+1} = d_{t+2} = \cdots = d_{t+\ell} = m_i)$, $t \geq 0$, $1 \leq \ell \leq n$, label the corresponding vertices to be $m_{i,1}, m_{i,2}, m_{i,3}, \ldots, m_{i,\ell}$. Consider the set $\mathcal{V}(G) = \mathcal{P}(V(G)) - \emptyset$ where, $\mathcal{P}(V)$ is the power set of $V(G)$.

\textbf{Definition 4.2.} The degree sequence set-graph corresponding to $G$, denoted by $\mathcal{G}_{\mathcal{V}(G)}$, is the graph with the following properties.

(i) $|\mathcal{G}_{\mathcal{V}(G)}| = 2^\nu - 1$ so that there exists a one to one correspondence between $V(\mathcal{G}_{\mathcal{V}(G)})$ and $\mathcal{V}(G)$.

(ii) Two vertices, say $v_{s,i}$ and $v_{t,j}$, in $\mathcal{G}_{\mathcal{V}(G)}$ are adjacent if some element(s) (specific vertex degree(s) of $G$) in $v_{s,i}$ is adjacent to some element(s) of $v_{t,j}$ in $G$.

It follows easily that for a complete graph $K_n$, $n \geq 1$ has its corresponding degree sequence set-graph, a complete graph.

\textbf{Problem 2.} Discuss the properties of the degree sequence set-graph corresponding to graph $G$.

(2) Let the minimum colour set $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\chi\}$ permit a chromatic colouring of $G$ in accordance with the rainbow neighbourhood convention. Let $\mathcal{C}^{[1]}(G) = \mathcal{P}(\mathcal{C}) - \emptyset$ where, $\mathcal{P}(\mathcal{C})$ is the power set of $\mathcal{C}$.

\textbf{Definition 4.3.} The colour set-graph corresponding to $G$, denoted by $\mathcal{G}_{\mathcal{C}^{[1]}(G)}$, is the graph with the following properties.
(i) \(|\mathcal{G}_{\mathcal{C}(G)}| = 2^\chi - 1\) so that there exists a one to one correspondence between \(V(\mathcal{G}_{\mathcal{C}(G)})\) and \(\mathcal{C}(G)\).

(ii) Two vertices, say \(v_{s,i}\) and \(v_{t,j}\), in \(\mathcal{G}_{\mathcal{C}(G)}\) are adjacent if some element(s) (specific vertex degree(s) of \(G\)) in \(v_{s,i}\) is adjacent to some element(s) of \(v_{t,j}\) in \(G\).

Clearly, for all graphs \(G\) with \(\chi(G) = 2\) the colour set-graph is \(K_3\).

**Problem 3.** Discuss the properties of the colour set-graph corresponding to a chromatic colouring of a graph \(G\).

This problem is similar to (1). For a minimum colour set \(\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\chi\}\) the corresponding colour weight sequence is \((c_1, c_1, \ldots, c_1, c_\chi, c_\chi, c_\chi, \ldots, c_\chi)\),

\[
\theta(c_1) \quad \text{entries} \quad \theta(c_\chi) \quad \text{entries}
\]

Let \(\mathcal{C}^\circ(G) = \{c_1,1, c_1,2, c_1,3, \ldots, c_1,\theta(c_1), \ldots, c_\chi,1, c_\chi,2, c_\chi,3, \ldots, c_\chi,\theta(c_\chi)\}\). We can define the colour-string set-graph, \(\mathcal{G}_{\mathcal{C}^\circ(G)}\) similar to Definition 4.2.

**Problem 4.** Research the properties of the colour-string set-graph corresponding to a chromatic colouring of a graph \(G\).

**Conflict of Interests**

The author(s) declare that there is no conflict of interests.

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