COINCIDENCE POINTS AND COMMON FIXED POINTS IN CONE BANACH SPACES

RAHUL TIWARI\(^1\),\(^*\) AND D.P. SHUKLA\(^2\)

\(^1\)Department of Mathematical Sciences, A.P.S. University Rewa (M.P.) 486001, India
\(^2\)Department of Mathematics, Govt. P.G. Science College Rewa (M.P.) 486001, India

Abstract: In this manuscript we obtain coincidence points and common fixed points in cone Banach spaces. Our result generalizes and extends the result of Thabet Ableljwal, Erdal Karapinar and KenanTas [3].

Keywords: Cone normed spaces, Coincidence points, Common fixed points.

2000 AMS Mathematics Subject Classification: 54H25, 47H10.

1. Introduction:

In 2007, Huang and Zhang [5] introduced the concept of cone metric space, replacing the set of real numbers by Banach space ordered by a cone and proved some fixed point theorems for function satisfying contractive conditions in these spaces. In this setting, Bogdan Rzepecki [11] generalized the fixed point theorems of Maia type [9] and Shy-Der Lin [8] considered some results of Khan and Imdad [7] Huang and Zhang [5] also discussed some properties of convergence of sequences and proved the fixed point theorems of contractive mapping for cone metric spaces: Any mapping \( T \) of a complete cone metric space \( X \) into itself that satisfies, for some \( 0 \leq k < 1 \), the inequality

\[
d(Tx, Ty) \leq k \, d(x, y)
\]

\(^*\)Corresponding author

E-mail Addresses: tiwari.rahul.rewa@gmail.com (R. Tiwari), shukladpmp@gmail.com (D.P. Shukla)

Received June 14, 2012
for all $x, y \in X$, has a unique fixed point.

Recently, Thabet Abdeljawad et. al. [3] proved some fixed point theorems for self maps satisfying some contraction principles on a cone Banach space. More precisely they proved that for a closed and convex subset $C$ of a cone Banach space with the norm $\|\|_p$, and letting $d: X \times X \rightarrow E$ with $d(x, y) = \|x - y\|_p$, if there exist $a, b, c, s$ and $T: C \rightarrow C$ satisfies the conditions $0 \leq \frac{s + a - 2b - c}{2(a + b)} < 1$ and $a d(Tx, Ty) + b(d(x, Tx) + d(y, Ty)) + c d(y, Tx) \leq s d(x, y)$ for all $x, y \in C$, then $T$ has at least one fixed point.

Here we will give some generalization of this theorem

2. Preliminaries:

Let $E$ be a real Banach space. A subset $P$ of $E$ is said to be a cone if and only if

i. $P$ is closed, nonempty and $P \neq \{0\}$.

ii. $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers $a, b$.

iii. $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, me can define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x < y$ will stand for $y - x \in \text{int} P$, where $\text{int} P$ denotes the interior of $P$.

The cone $P$ is called normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \implies \|x\| \leq M \|y\|.$$ 

The least positive number satisfying the above is called the normal constant of $P$.

The cone $P$ is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$, is a sequence such that $x_1 \leq x_2 \leq \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. Equivalently the cone $P$ is regular if and only if every decreasing sequence which is bounded from below is convergent.
Lemma 2.1 \([4, 10]\)  

(i) Every regular cone is normal.

(ii) For each \(k > 1\), there is a normal cone with normal constant \(K > k\)

Definition 2.2 \([5]\) Let \(X\) be a nonempty set. Then any map \(d: X \times X \to E\) is said to be cone metric on \(X\) if for all \(x, y, z \in X\), \(d\) satisfies.

i. \(d(x, y) \geq 0\) and \(d(x, y) = 0\) if and only if \(x = y\).

ii. \(d(x, y) = d(y, x)\)

iii. \(d(x, y) \leq d(x, z) + d(z, y)\).

Pair \((X, d)\) is called as cone metric space (CMS).

We denote set of all reals by \(R\)

Example 2.3 Let \(E = R^2\), \(P = \{(x, y) \in E : x, y \geq 0\}\) and \(X = R\).

Define \(d : X \times X \to E\) by \(d(x, y) = (\alpha|x-y|, \beta|x-y|)\),

where \(\alpha, \beta\) are positive constants. Then \((X, d)\) is a CMS.

It is quite natural to consider cone normed spaces (CNS).

Definition 2.4 \([1, 16]\) Let \(X\) be a linear space over \(R\) and \(\|\cdot\|_p : X \to E\) be a map which satisfies

i. \(\|x\|_p > 0\) for all \(x \in X\),

ii. \(\|x\|_p = 0\) if and only if \(x = 0\),

iii. \(\|x + y\|_p \leq \|x\|_p + \|y\|_p\) for all \(x, y \in X\),

iv. \(\|kx\|_p = |k| \|x\|_p\) for all \(k \in R\),
Then $\| \cdot \|_p$ is called cone norm on $X$, and pair $(X, \| \cdot \|_p)$ is called cone normed space (CNS).

Note that each CNS is CMS. Indeed, $d(x, y) = \|x - y\|_p$.

**Definition 2.5** Let $\{x_n\}_{n \geq 1}$ be a sequence in CNS $(X, \| \cdot \|_p)$. Then

i. It is said to be a convergent sequence if for every $c \in E$ with $c \geq 0$ there is a natural number $N$ such that for all $n \geq N$, $\|x_n - x\|_p \leq c$ for some fixed $x \in X$.

ii. It is said to be a Cauchy sequence if for every $c \in E$ with $c \geq 0$ there is a natural number $N$ such that for all $n, m \geq N$, $\|x_n - x_m\|_p \leq c$.

iii. CNS $(X, \| \cdot \|_p)$ is said to be complete if every Cauchy sequence in $X$ is convergent.

**Lemma 2.6** [6] Let $(X, \| \cdot \|_p)$ be a CNS and $P$ be a normal cone with normal constant $K$. If $\{x_n\}$ is a sequence in $X$, then

i. $\{x_n\}$ converges to $x$ if and only if $\|x_n - x\|_p \to 0$, as $n \to \infty$

ii. $\{x_n\}$ is a Cauchy sequence if and only if $\|x_n - x_m\|_p \to 0$ as $n, m \to \infty$.

iii. $\{x_n\}$ converges to $x$ and sequence $\{y_n\}$ converges to $y$, then $\|x_n - y_n\|_p \to \|x - y\|_p$.

**Lemma 2.7** [14, 15, 6] Let $(X, \| \cdot \|_p)$ be a CNS over a cone $P$ in $E$. Then

i. $\text{Int}(P) + \text{Int}(P) \subseteq \text{Int}(P)$ and $\lambda \text{Int}(P) \subseteq \text{Int}(P)$, $\lambda > 0$.

ii. If $c \gg 0$ then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.

iii. For any given $c \gg 0$ and $c_o \gg 0$ there exists a natural number $n_o$ such that $c_o/n_o \ll c$.

iv. If $a_n, b_n$ are sequences in $E$ such that $a_n \to a$, $b_n \to b$ and $a_n \leq b_n$, for all $n$, then $a \leq b$. 
**Definition 2.8** [4] Cone $P$ is called minihedral cone if $\sup\{x,y\}$ exists for all $x,y \in E$ and strongly minihedral if every subset of $E$ which is bounded from above has a supremum.

**Lemma 2.9** [2] Every strongly minihedral normal cone is regular

For $T : X \rightarrow X$, the set of fixed points of $T$ is denoted by $F(T) = \{z \in X : Tz = z\}$

**Definition 2.10** [13] Let $C$ be a closed and convex subset of a cone Banach space with the norm $\|x\|_p = d(x,0)$ and $T : C \rightarrow C$ a map. Then $T$ is called nonexpansive if

$$\|Tx - Tz\|_p \leq \|x - z\|_p \quad \text{for all } x,z \in C$$

and $T$ is called quasi-nonexpansive if

$$\|Tx - z\|_p \leq \|x - z\|_p \quad \text{for all } x \in C, \ z \in F(T)$$

3. **Main Results**:

**Theorem 3.1**

Let $C$ be a closed convex subset of a cone Banach space $X$ with norm $\|x\|_p$. Suppose $E = (E \| . \|)$ is a real Banach space and let $d : X \times X \rightarrow E$ be a mapping such that $d(x,y) = \|x - y\|_p$.

If there exist $a,b,c,e$ and $T : C \rightarrow C$ satisfying the conditions

$$0 \leq \frac{e+2a-2b+c}{2a+2b+c} < 1, \ a+b+c \neq 0, a+b+c > 0 \quad \text{and } e \geq 0 \quad (3.1)$$

$$a \ d(Tx, Ty) + b \{d(x, Tx) + d(y, Ty)\} + c \{d(y, Tx) + d(x, Ty)\} \leq e \ d(x, y) \quad (3.2)$$

hold for all $x,y \in C$. Then $T$ has at least one fixed point.

**Proof**:

Pick $x_0 \in C$ and define a sequence $\{x_n\}$ in the following way:

$$x_{n+1} = x_n + \frac{Tx_n}{2} \quad n = 0, 1, 2, \ldots \quad (3.3)$$

Notice that
\[ x_n - T x_n = 2 \left( x_n - \frac{x_n + T x_n}{2} \right) = 2 \left( x_n - x_{n+1} \right) \]  
(3.4)

which yields that

\[ d(x_n, T x_n) = \| x_n - T x_n \|_p = 2 \| x_n - x_{n+1} \|_p = 2 \ d(x_n, x_{n+1}) \]  
(3.5)

for \( n = 0, 1, 2, \ldots \). Analogously, for \( n = 0, 1, 2, 3 \ldots \), one can get

\[ d(x_{n-1}, T x_{n-1}) = 2 \ d(x_{n-1}, x_n), \]  
and

\[ d(x_n, T x_{n-1}) = \frac{1}{2} \ d(x_{n-1}, T x_{n-1}) = d(x_{n-1}, x_n), \]  
(3.6)

and by the triangle inequality

\[ d(x_n, T x_n) - d(x_n, T x_{n-1}) \leq d(T x_{n-1}, T x_n). \]  
(3.7)

We put \( x = x_{n-1} \) and \( y = x_n \) in inequality (3.2),

\[ a \ d(T x_{n-1}, T x_n) + b[d(x_{n-1}, T x_{n-1}) + d(x_n, T x_n)] + c[d(x_{n-1}, T x_{n-1}) + d(x_{n-1}, T x_n)] \leq e \ d(x_{n-1}, x_n). \]  
(3.8)

for all \( a, b, c, e \) that satisfy (3.1). Taking into account (3.5) and (3.6) one can observe.

\[ a \ d(T x_{n-1}, T x_n) + b[2d(x_{n-1}, x_n) + 2d(x_n, x_{n+1})] + c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \leq e \ d(x_{n-1}, x_n). \]  
(3.9)

which is equivalent to

\[ a \ d(T x_{n-1}, T x_n) \leq e \ d(x_{n-1}, x_n) - 2b[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] - c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \]  
(3.10)

By using (3.7), the statement (3.10) turns into

\[ a \ [d(x_n, T x_n) - d(x_n, T x_{n-1})] \leq e \ d(x_{n-1}, x_n) - 2b[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] - c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \]  
(3.11)

Regarding (3.5) and (3.6), in (3.11),

\[ 2a \ d(x_n, x_{n+1}) - a \ d(x_n, x_n) \leq e \ d(x_{n-1}, x_n) - 2b \ d(x_{n-1}, x_n) - 2b \ d(x_n, x_{n+1}) - c \ d(x_n, x_{n+1}) - c \ d(x_n, x_{n+1}). \]

\[ \Leftrightarrow (2a+2b+c) \ d(x_n, x_{n+1}) \leq (e+a-2b-c) \ d(x_{n-1}, x_n) \]
Since \(a+b+c \neq 0\), we get 
\[
\begin{align*}
\text{d}(x_n, x_{n+1}) & \leq \frac{e+a-2b-c}{2a+2b+c} \text{d}(x_{n-1}, x_n) \\
\Rightarrow \quad \text{d}(x_n, x_{n+1}) & \leq K \text{d}(x_{n-1}, x_n), \quad \text{where } K = \frac{e+a-2b-c}{2a+2b+c}
\end{align*}
\]

Thus the sequence \(\{x_n\}\) is a Cauchy sequence that converges to some element of \(C\), say \(z\). We claim that \(z\) is a fixed point of \(T\). When we substitute \(x = z\) and \(y = x_n\) in (3.2).

\[
a \text{d}(Tz, Tx_n) + b\{d(z, Tz) + d(x_n, Tx_n)\} + c\{d(x_n, Tz) + d(z, Tx_n)\} \leq e \text{d}(z, x_n)
\]

Due to the equation (3.3) and \(x_n \rightarrow z\), we have \(Tx_n \rightarrow z\)

\[
\Rightarrow \quad a \text{d}(Tz, z) + b \text{d}(z, Tz) + c \text{d}(z, Tz) \leq 0 \quad \text{as } n \rightarrow \infty
\]

\[
\Rightarrow \quad (a+b+c) \text{d}(z, Tz) \leq 0
\]

\[
\Rightarrow \quad Tz = z \quad \text{as } a+b+c > 0.
\]

**Definition 3.2** Let \((X, d)\) be a complete metric space and \(S, T\) be self maps on \(X\). A point \(z \in X\) is said to be a coincidence point of \(S, T\) if \(Sz = Tz\) and it is called common fixed point of \(S, T\) if \(Sz = Tz = z\).

Moreover a pair \((S, T)\) of self maps is called weakly compatible on \(X\) if they commute at their coincidence points i.e. \(z \in X\), \(Sz = Tz\) implies \(STz = TSz\).

**Theorem 3.3** Let \(C\) be a closed convex subset of a cone Banach space \(X\) with norm \(\| \|_p\) and let \(d : X \times X \rightarrow E\) with \(d(x, y) = \|x-y\|_p\). If \(T\) and \(S\) are self maps on \(C\) that satisfy the conditions.

\[
(3.31) \quad T(C) \subseteq S(C)
\]

\[
(3.32) \quad S(C) \text{ is a complete subspace}
\]

\[
(3.33) \quad a \text{d}(Tx, Ty) + b\{d(Sx, Tx) + d(Sy, Ty)\} + c\{d(Sy, Tx) + d(Sx, Ty)\} \leq r \text{d}(Sx, Sy)
\]

for \(a+b+c \neq 0, 0 \leq r < a+2b, r < b, a \neq r\).
hold for all \( x, y \in C \), then \( S \) and \( T \) have a common coincidence point. Moreover if \( S \) and \( T \) are weakly compatible, then they have a unique common fixed point in \( C \).

**Proof:** Pick \( x_o \in C \). By (3.31) we can find a point in \( C \), say \( x_1 \), such that \( T(x_o) = Sx_1 \). Since \( S, T \) are self maps, there exists \( y_o \in C \) such that \( y_o = Tx_o = Sx_1 \).

Inductively we can define a sequence \( \{y_n\} \) and sequence \( \{x_n\} \) in \( C \) such that

\[
y_n = Sx_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots
\]

We put \( x = x_n \) and \( y = x_{n+1} \) in inequality (3.33), it implies that

\[
a \ d(Tx_n, Tx_{n+1}) + b \{d(Sx_n, Tx_n) + d(Sx_{n+1}, Tx_{n+1})\} + c \{d(Sx_{n+1}, Tx_n) + d(Sx_n, Tx_{n+1})\} \leq r \ d(Sx_n, Sx_{n+1})
\]

\[
\Rightarrow a \ d(y_n, y_{n+1}) + b \{d(y_{n-1}, y_n) + d(y_n, y_{n+1})\} + c \{d(y_n, y_{n+1}) + d(y_{n-1}, y_{n+1})\} \leq r \ d(y_{n-1}, y_n)
\]

By using triangle inequality and suitable choices of \( a, b, c \), it implies,

\[
(a+b) \ d(y_n, y_{n+1}) + b \ d(y_{n-1}, y_n) + c \ d(y_{n-1}, y_n) + c \ d(y_{n+1}, y_{n+1}) \leq r \ d(y_{n-1}, y_n)
\]

\[
\Rightarrow d(y_n, y_{n+1}) \leq \frac{r-b-c}{a+b+c} \ d(y_{n-1}, y_n) = k \ d(y_{n-1}, y_n)
\]

where \( k = \frac{r-b-c}{a+b+c} \). Similarly \( d(y_{n-1}, y_n) \leq k \ d(y_{n-2}, y_{n-1}) \)

Since \( 0 \leq r < a+2b, \ r < b, \) then \( 0 \leq k < 1 \).

By routine calculations,

\[
d(y_n, y_{n+1}) \leq k^n \ d(y_o, y_1).
\]

We claim that \( \{y_n\} \) is a Cauchy sequence. Let \( n > m \),

Then by (3.35) and the triangle inequality,

\[
d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m+1}, y_m).
\]

\[
\leq k^{n-1}d(y_o, y_1) + k^{n-2}d(y_o, y_1) + \ldots + k^m d(y_o, y_1).
\]

\[
\leq k^m \ d(y_o, y_1)
\]
Therefore \( \{y_n\} \) is a Cauchy sequence. Since \( S(C) \) is complete, then \( \{y_n = Sx_{n+1} = Tx_n\} \) converges to some point in \( S(C) \), say \( z \).

Now by replacing \( x \) with \( p \) and \( y \) with \( x_{n+1} \) in (3.33), we get

\[
\begin{align*}
& a \{d(Tp, y_{n+1}) + b\{d(z, Tp) + d(y_n, y_{n+1})\} + c\{d(y_n, Tp) + d(z, y_{n+1})\}\} \\
& \leq r d(Sp, Sx_{n+1}).
\end{align*}
\]

\( \Rightarrow \)

\[
\begin{align*}
& a \{d(Tp, y_{n+1}) + b\{d(z, Tp) + d(y_n, y_{n+1})\} + c\{d(y_n, Tp) + d(z, y_{n+1})\}\} \leq r d(z, y_n)
\end{align*}
\]

As \( n \to \infty \), it becomes

\[
\begin{align*}
& a d(Tp, z) + b\{d(z, Tp) + d(y_n, y_{n+1})\} + c\{d(y_n, Tp) + d(z, y_{n+1})\} \leq r d(z, y_n)
\end{align*}
\]

\( \Rightarrow \)

\[
\begin{align*}
(a + 2c - r) d(z, Tz) \leq 0.
\end{align*}
\]

Since \( a + 2c - r \neq 0 \), then \( Tp = z \). Hence \( Tp = z = Sp \).

i.e. \( p \) is a coincidence point of \( S \) and \( T \).

If \( S \) and \( T \) are weakly compatible, then they commute at a coincidence point. Therefore, \( Tp = z = Sp \Rightarrow STp = TSp \) for some \( p \in C \), which implies \( Tz = Sz \). We claim that \( z \) is a common fixed point of \( S \) and \( T \).

Substitute \( x = p \) and \( y = Tp = z \) in (3.33), to give

\[
\begin{align*}
& a \{d(Tp, TTp) + b\{d(Sp, Tp) + d(STp, TTp)\} + c\{d(STp, Tp) + d(Sp, TTp)\}\} \leq r d(Sp, STp).
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
& a \{d(z, Tz) + b\{d(z, z) + d(Sz, Tz)\} + c\{d(Sz, z) + d(z, Tz)\}\} \leq r d(z, Sz).
\end{align*}
\]

\( \Rightarrow \)

\[
\begin{align*}
(a + 2c - r) d(z, Tz) \leq 0.
\end{align*}
\]

Since \( a + 2c - r \neq 0 \), then \( z = Tz = Sz \).

To prove uniqueness, suppose the contrary, that \( w \) is another common fixed point of \( S \) and \( T \). Put \( x \) by \( z \) and \( y \) by \( w \) in the inequality (3.33), one can get.

\[
\begin{align*}
& a \{d(Tz, Tw) + b\{d(Sz, Tz) + d(Sw, Tw)\} + c\{d(Sw, Tz) + d(Sz, Tw)\}\} \leq r d(Sz, Sw).
\end{align*}
\]
a \ d(z, w) + 2c \ d(z, w) \leq r \ d(z, w) \\
\iff (a + 2c - r) \ d(z, w) \leq 0.

which is a contradiction since \(a + 2c - r \neq 0\). Hence the common fixed point of \(S\) and \(T\) is unique.

**Acknowledgment:**

The authors express their gratitude to the referees for constructive and useful remarks and suggestions.

**REFERENCES**


