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A NOTE ON THE ZEROS OF POLAR DERIVATIVE OF A POLYNOMIAL WITH COMPLEX COEFFICIENTS

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Abstract. According to the Enestrom-Kakeya theorem "zeros of the polynomial whose coefficients are positive, real and increasing along with the powers of the variable are lie in the unit circle" see [6, 10]. In [1], Aziz and Mahammad, showed that zeros of $f(z)$ satisfies $|z| \geq \frac{n}{n+1}$ are simple, under the same conditions. This article shows that the result of Gulzar, Zargar and Akthar in [8] is simplified in terms of real and imaginary parts of complex coefficients of the polynomial, also it extends some generalizations by imposing conditions on hypothesis in different ways.

Keywords: zeros; polynomial; Eneström-Kakeya theorem; polar derivative.

2010 AMS Subject Classification: 30C10, 30C15.

1. Introduction

Let $f(z)$ be the n^{th} degree polynomial with real coefficients. Let $D_\alpha f(z)$ denote the polar derivative of $f(z)$ w.r.t the point α and it is defined by $D_\alpha f(z) = nf(z) + (\alpha - z)f^{'}(z)$. In this case the degree of $D_\alpha f(z)$ is at most $n-1$ and if α tends to ∞ then it generalize the ordinary

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derivative

$$i.e \lim_{\alpha \longrightarrow \infty} \frac{D_\alpha f(z)}{\alpha} = f^{'}(z)$$

Regarding the distribution of zeros of $f(z)$, Enestrom Kakeya proved the folowing result.

Theorem 1. *Let $f(z) = \sum_{j=0}^{n} k_j z^j$ be the n^{th} degree polynomial with real coefficients such that for some $0 < k_0 \leq k_1 \leq ... \leq k_{n-2} \leq k_{n-1} \leq k_n$ then all zeros of $f(z)$ lies in $|z| \leq 1$.*

Instead of taking only positive coefficients, A.Joyal, Labelle and Rahman[3] given the following result

Theorem 2. *Let $f(z) = \sum_{j=0}^{n} k_j z^j$ be the n^{th} degree polynomial with real coefficients such that for some $k_0 \leq k_1 \leq ... \leq k_{n-2} \leq k_{n-1} \leq k_n$ then all zeros of $f(z)$ lies in $|z| \leq \frac{k_n - k_0 + |k_0|}{|k_n|}$.*

Regarding the multiplicity of zeros of $f(z)$, Aziz and Mahammad [1] proved the folowing result

Theorem 3. *Let $f(z) = \sum_{j=0}^{n} k_j z^j$ be the n^{th} degree polynomial with real coefficients such that for some $0 < k_0 \leq k_1 \leq ... \leq k_n$ then all zeros of $f(z)$ of modulus greater than or equal to $\frac{n}{n+1}$ are simple.*

Gulzar, Zargar and Akthar [8] result by substituting $b_t \ with \ (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $\quad t = 2, 3, 4, .., n$

Theorem 4. *Let $f(z) = \sum_{j=0}^{n} k_j z^j$ be the n^{th} degree polynomial with real coefficients, and α be a real number, such that for some $b_n \geq b_{n-1} \geq ... \geq b_4 \geq b_3 \geq b_2$ then all zeros of $D_\alpha f(z)$ which does not lie in $|z| \leq \frac{b_n - b_2 + |b_2|}{|b_n|}$ are simple.*

Theorem 5. *If $f(z) = \sum_{j=0}^{n} k_j z^j$ is the n^{th} degree polynomial with real coefficients, and α be a real number such that for some $b_n \leq b_{n-1} \leq ... \leq b_4 \leq b_3 \leq b_2$ then all zeros of $D_\alpha f(z)$ which does not lie in $|z| \leq \frac{b_2 + |b_2| - b_n}{|b_n|}$ are simple.*

M.H.Gulzar, Zargar and Akthar [8] have extended the above results to the polar derivatives, there exist some generalizations and extentions of Enestrom and Kakeya theorem in [2, 4, 5, 7, 9]. This paper providing the region about the simple zeros of polar derivative in terms of real and imaginary parts by imposing some conditions on hypethesis in different ways by replacing b_t *with* $(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

2. MAIN RESULTS

Theorem 6. *Let* $f(z)=\sum_{j=0}^{n}k_jz^j$ *be the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$p_n\geq p_{n-1}\geq ...\geq p_4\geq p_3\geq p_2 \quad and \quad q_n\geq q_{n-1}\geq ...\geq q_4\geq q_3\geq q_2$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z|\leq\frac{p_n+q_n-(p_2+q_2)+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}$$

are simple, where $Re(b_t)=p_t$, $Im(b_t)=q_t$ *and* $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ *for* $t=2,3,4,...,n$

Corollary 1. *If* $f(z)=\sum_{j=0}^{n}k_jz^j$ *is the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$p_n\geq p_{n-1}\geq ...\geq p_4\geq p_3\geq p_2>0 \quad and \quad q_n\geq q_{n-1}\geq ...\geq q_4\geq q_3\geq q_2>0$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z|\leq\frac{p_n+q_n}{\sqrt{p_n^2+q_n^2}}$$

are simple, where $Re(b_t)=p_t$, $Im(b_t)=q_t$ *and* $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ *for* $t=2,3,4,...,n$

Corollary 2. *If* $f(z)=\sum_{j=0}^{n}k_jz^j$ *is the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$p_n\geq p_{n-1}\geq ...\geq p_4\geq p_3\geq p_2>0 \quad and \quad q_n\geq q_{n-1}\geq ...\geq q_4\geq q_3\geq q_2$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z|\leq\frac{p_n+q_n-q_2+|q_2|}{\sqrt{p_n^2+q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}] \quad for \quad t = 2,3,4,...,n$

Corollary 3. *If* $f(z) = \sum_{j=0}^{n} k_j z^j$ *is the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$p_n \geq p_{n-1} \geq ... \geq p_4 \geq p_3 \geq p_2 \quad and \quad q_n \geq q_{n-1} \geq ... \geq q_4 \geq q_3 \geq q_2 > 0$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z| \leq \frac{p_n + q_n - p_2 + |p_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}] \quad for \quad t = 2,3,4,...,n$

Remark 1. *(1) Corollary 1 follows from theorem 6 by substituting* $p_t > 0$, $q_t > 0$.

(2) Corollary 2 follows from theorem 6 by substituting $p_t > 0$.

(3) Corollary 3 follows from theorem 6 by substituting $q_t > 0$.

Theorem 7. *Let* $f(z) = \sum_{j=0}^{n} k_j z^j$ *be the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$p_n \leq p_{n-1} \leq ... \leq p_4 \leq p_3 \leq p_2 \quad and \quad q_n \leq q_{n-1} \leq ... \leq q_4 \leq q_3 \leq q_2$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z| \leq \frac{-p_n - q_n + p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}] \quad for \quad t = 2,3,4,...,n$

Corollary 4. *If* $f(z) = \sum_{j=0}^{n} k_j z^j$ *is the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$0 < p_n \leq p_{n-1} \leq ... \leq p_4 \leq p_3 \leq p_2 \quad and \quad 0 < q_n \leq q_{n-1} \leq ... \leq q_4 \leq q_3 \leq q_2$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z| \leq \frac{-p_n - q_n + 2(p_2 + q_2)}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ $for \quad t = 2,3,4,...,n$

Corollary 5. *If* $f(z) = \sum_{j=0}^{n} k_j z^j$ *is the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$0 < p_n \leq p_{n-1} \leq ... \leq p_4 \leq p_3 \leq p_2 \quad and \quad q_n \leq q_{n-1} \leq ... \leq q_4 \leq q_3 \leq q_2$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z| \leq \frac{-p_n - q_n + 2p_2 + q_2 + |q_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ $for \quad t = 2,3,4,...,n$

Corollary 6. *If* $f(z) = \sum_{j=0}^{n} k_j z^j$ *is the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$p_n \leq p_{n-1} \leq ... \leq p_4 \leq p_3 \leq p_2 \quad and \quad 0 < q_n \leq q_{n-1} \leq ... \leq q_4 \leq q_3 \leq q_2$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z| \leq \frac{-p_n - q_n + p_2 + 2q_2 + |p_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ $for \quad t = 2,3,4,...,n$

Remark 2. *(1) Corollary 4 follows from theorem 7 by substituting* $p_t > 0$, $q_t > 0$.

(2) Corollary 5 follows from theorem 7 by substituting $p_t > 0$.

(3) Corollary 6 follows from theorem 7 by substituting $q_t > 0$.

Theorem 8. *Let* $f(z) = \sum_{j=0}^{n} k_j z^j$ *be the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$p_n \geq p_{n-1} \geq ... \geq p_4 \geq p_3 \geq p_2 \quad and \quad q_n \leq q_{n-1} \leq ... \leq q_4 \leq q_3 \leq q_2$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z| \leq \frac{p_n - q_n - p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}] \quad for \quad t = 2,3,4,...,n$

Corollary 7. *If* $f(z) = \sum_{j=0}^{n} k_j z^j$ *is the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$p_n \geq p_{n-1} \geq ... \geq p_4 \geq p_3 \geq p_2 > 0 \quad and \quad 0 < q_n \leq q_{n-1} \leq ... \leq q_4 \leq q_3 \leq q_2$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z| \leq \frac{p_n - q_n + 2q_2}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}] \quad for \quad t = 2,3,4,...,n$

Corollary 8. *If* $f(z) = \sum_{j=0}^{n} k_j z^j$ *is the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$p_n \geq p_{n-1} \geq ... \geq p_4 \geq p_3 \geq p_2 > 0 \quad and \quad q_n \leq q_{n-1} \leq ... \leq q_4 \leq q_3 \leq q_2$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z| \leq \frac{p_n - q_n + q_2 + |q_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}] \quad for \quad t = 2,3,4,...,n$

Corollary 9. *If* $f(z) = \sum_{j=0}^{n} k_j z^j$ *is the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$p_n \geq p_{n-1} \geq ... \geq p_4 \geq p_3 \geq p_2 \quad and \quad 0 < q_n \leq q_{n-1} \leq ... \leq q_4 \leq q_3 \leq q_2$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z| \leq \frac{p_n - q_n - p_2 + 2q_2 + |p_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ $\quad for \quad t = 2,3,4,...,n$

Remark 3. *(1) Corollary 7 follows from theorem 8 by substituting* $p_t > 0$, $q_t > 0$.

(2) Corollary 8 follows from theorem 8 by substituting $p_t > 0$.

(3) Corollary 9 follows from theorem 8 by substituting $q_t > 0$.

Theorem 9. *Let* $f(z) = \sum_{j=0}^{n} k_j z^j$ *be the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$p_n \le p_{n-1} \le ... \le p_4 \le p_3 \le p_2 \quad and \quad q_n \ge q_{n-1} \ge ... \ge q_4 \ge q_3 \ge q_2$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z| \le \frac{-p_n + q_n + p_2 - q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ $\quad for \quad t = 2,3,4,...,n$

Corollary 10. *Let* $f(z) = \sum_{j=0}^{n} k_j z^j$ *be the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$0 < p_n \le p_{n-1} \le ... \le p_4 \le p_3 \le p_2 \quad and \quad q_n \ge q_{n-1} \ge ... \ge q_4 \ge q_3 \ge q_2 > 0$$

then all zeros of $D_\alpha f(z)$ *which does not lie in*

$$|z| \le \frac{-p_n + q_n + 2p_2}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ *and* $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ $\quad for \quad t = 2,3,4,...,n$

Corollary 11. *Let* $f(z) = \sum_{j=0}^{n} k_j z^j$ *be the* n^{th} *degree polynomial, Let* α *be the real or complex number, such that for some*

$$0 < p_n \le p_{n-1} \le ... \le p_4 \le p_3 \le p_2 \quad and \quad q_n \ge q_{n-1} \ge ... \ge q_4 \ge q_3 \ge q_2$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{-p_n + q_n + 2p_2 - q_2 + |q_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ and $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ $\quad for \quad t = 2, 3, 4, ..., n$

Corollary 12. *Let $f(z) = \sum_{j=0}^{n} k_j z^j$ be the n^{th} degree polynomial, Let α be the real or complex number, such that for some*

$$p_n \leq p_{n-1} \leq ... \leq p_4 \leq p_3 \leq p_2 \quad and \quad q_n \geq q_{n-1} \geq ... \geq q_4 \geq q_3 \geq q_2 > 0$$

then all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{-p_n + q_n + p_2 + |p_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t$, $Im(b_t) = q_t$ and $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ $\quad for \quad t = 2, 3, 4, ..., n$

Remark 4. *(1) Corollary 10 follows from theorem 9 by substituting $p_t > 0$, $q_t > 0$.*

(2) Corollary 11 follows from theorem 9 by substituting $p_t > 0$.

(3) Corollary 12 follows from theorem 9 by substituting $q_t > 0$.

3. Proof of the Theorems

Proof of the Theorem 6.

Let $f(z) = k_n z^n + k_{n-1} z^{n-1} + ... + k_1 z + k_0$ be the n^{th} degree polynomial with real coefficients. Then by the definition of polar derivative, we have $D_\alpha f(z) = nf(z) + \alpha f^{'}(z) - zf^{'}(z)$

$D_\alpha f(z) = n(k_n z^n + k_{n-1} z^{n-1} + ... + k_1 z + k_0) + \alpha(nk_n z^{n-1} + (n-1)k_{n-1} z^{n-2} + ... + k_1)$
$- z(nk_n z^{n-1} + (n-1)k_{n-1} z^{n-2} + ... + k_1)$

$D_\alpha f(z) = [n\alpha k_n + (n-(n-1))k_{n-1}]z^{n-1} + [(n-1)\alpha k_{n-1} + (n-(n-2))k_{n-2}]z^{n-2} + ...$
$+ [2\alpha k_2 + (n-1))k_1]z + [\alpha k_1 + nk_0]$

Now,

$D_\alpha^{'} f(z) = b_n z^{n-2} + b_{n-1} z^{n-3} + b_{n-2} z^{n-4} + ... + b_4 z^2 + b_3 z + b_2$

where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ $\quad for \quad t = 2, 3, 4, ..., n$

Now Consider $g(z)=(1-z)D_{\alpha}^{'}f(z)$, so that

$g(z)=(1-z)[b_nz^{n-2}+b_{n-1}z^{n-3}+b_{n-2}z^{n-4}+...+b_4z^2+b_3z+b_2]$

$g(z)=-b_nz^{n-1}+(b_n-b_{n-1})z^{n-2}+(b_{n-1}-b_{n-2})z^{n-3}+(b_{n-2}-b_{n-3})z^{n-4}+...$

$+(b_{m+1}-b_m)z^{m-1}+(b_m-b_{m-1})z^{m-2}...+(b_4-b_3)z^2+(b_3-b_2)z+b_2$

$|g(z)|\geq|b_n||z|^{n-2}\Big[|z|-\frac{1}{|b_n|}\{|p_n-p_{n-1}|+\frac{|p_{n-1}-p_{n-2}|}{|z|}+\frac{|p_{n-2}-p_{n-3}|}{|z|^2}+...+\frac{|p_4-p_3|}{|z|^{n-4}}+\frac{|p_3-p_2|}{|z|^{n-3}}+$
$\frac{|p_2|}{|z|^{n-2}}+|q_n-q_{n-1}|+\frac{|q_{n-1}-q_{n-2}|}{|z|}+\frac{|q_{n-2}-q_{n-3}|}{|z|^2}+...+\frac{|q_4-q_3|}{|z|^{n-4}}+\frac{|q_3-q_2|}{|z|^{n-3}}+\frac{|q_2|}{|z|^{n-2}}\}\Big]$

Also, if $|z|>1$ then $\frac{1}{|z|}<1$

then

$\geq|b_n||z|^{n-2}\Big[|z|-\frac{1}{|b_n|}\{|p_n-p_{n-1}|+|p_{n-1}-p_{n-2}|+...+|p_{m+1}-p_m|+|p_m-p_{m-1}|+...+$
$|p_4-p_3|+|p_3-p_2|+|p_2|+|q_n-q_{n-1}|+|q_{n-1}-q_{n-2}|+...+|q_{m+1}-q_m|+|q_m-q_{m-1}|+$
$...+|q_4-q_3|+|q_3-q_2|+|q_2|\}\Big]$

$\geq|b_n||z|^{n-2}\Big[|z|-\frac{1}{|b_n|}\{p_n-p_{n-1}+p_{n-1}-p_{n-2}+...+p_{m+1}-p_m+p_m-p_{m-1}+...+p_4-p_3+$
$p_3-p_2+|p_2|+q_n-q_{n-1}+q_{n-1}-q_{n-2}+...+q_{m+1}-q_m+q_m-q_{m-1}+...+q_4-q_3+$
$q_3-q_2+|q_2|\}\Big]$

$\geq|b_n||z|^{n-2}\Big[|z|-\frac{1}{|b_n|}\{p_n-p_2+|p_2|+q_n-q_2+|q_2|\}\Big]$

Hence $|g(z)|>0$, provided

$$|z|>\frac{p_n+q_n-(p_2+q_2)+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}.$$

This implies that all zeros of $g(z)$ whose modulus is greater than 1 are lie in

$$|z|\leq\frac{p_n+q_n-(p_2+q_2)+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}.$$

Since the zeros of $g(z)$ whose modulus is less than or equal to 1 are already lie in

$$|z|\leq\frac{p_n+q_n-(p_2+q_2)+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}},$$

it follows that all the zeros of $g(z)$ lie in

$$|z|\leq\frac{p_n+q_n-(p_2+q_2)+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}.$$

Since all the zeros of $g(z)$ are also the zeros of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{p_n+q_n-(p_2+q_2)+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}.$$

Thus all the zeros of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{p_n+q_n-(p_2+q_2)+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}.$$

In other words all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{p_n+q_n-(p_2+q_2)+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}$$

are simple, where $Re(b_t)=p_t, Im(b_t)=q_t$ and $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}] \quad for \quad t=2,3,4,...,n$

Proof of the Theorem 7.

Let $f(z)=k_nz^n+k_{n-1}z^{n-1}+...+k_1z+k_0$ be the n^{th} degree polynomial with real coefficients. Then by the definition of polar derivative, we have

$D_\alpha f(z)=nf(z)+\alpha f'(z)-zf'(z)$

$D_\alpha f(z)=n(k_nz^n+k_{n-1}z^{n-1}+...+k_1z+k_0)+\alpha(nk_nz^{n-1}+(n-1)k_{n-1}z^{n-2}+...+k_1)$
$-z(nk_nz^{n-1}+(n-1)k_{n-1}z^{n-2}+...+k_1)$

$D_\alpha f(z)=[n\alpha k_n+(n-(n-1))k_{n-1}]z^{n-1}+[(n-1)\alpha k_{n-1}+(n-(n-2))k_{n-2}]z^{n-2}+...$
$+[2\alpha k_2+(n-1))k_1]z+[\alpha k_1+nk_0]$

Now,

$D'_\alpha f(z)=b_nz^{n-2}+b_{n-1}z^{n-3}+b_{n-2}z^{n-4}+...+b_4z^2+b_3z+b_2$

where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}] \quad for \quad t=2,3,4,...,n$

Now Consider $g(z)=(1-z)D'_\alpha f(z)$, so that

$g(z)=(1-z)[b_nz^{n-2}+b_{n-1}z^{n-3}+b_{n-2}z^{n-4}+...+b_4z^2+b_3z+b_2]$

$g(z)=-b_nz^{n-1}+(b_n-b_{n-1})z^{n-2}+(b_{n-1}-b_{n-2})z^{n-3}+(b_{n-2}-b_{n-3})z^{n-4}+...$
$+(b_{m+1}-b_m)z^{m-1}+(b_m-b_{m-1})z^{m-2}+...+(b_4-b_3)z^2+(b_3-b_2)z+b_2$

$|g(z)| \geq |b_n||z|^{n-2}\Big[|z|-\frac{1}{|b_n|}\big\{|p_n-p_{n-1}|+\frac{|p_{n-1}-p_{n-2}|}{|z|}+\frac{|p_{n-2}-p_{n-3}|}{|z|^2}+...+\frac{|p_4-p_3|}{|z|^{n-4}}+\frac{|p_3-p_2|}{|z|^{n-3}}+$
$\frac{|p_2|}{|z|^{n-2}}+|q_n-q_{n-1}|+\frac{|q_{n-1}-q_{n-2}|}{|z|}+\frac{|q_{n-2}-q_{n-3}|}{|z|^2}+...+\frac{|q_4-q_3|}{|z|^{n-4}}+\frac{|q_3-q_2|}{|z|^{n-3}}+\frac{|q_2|}{|z|^{n-2}}\big\}\Big]$

Also, if $|z|>1$ then $\frac{1}{|z|}<1$

then

$\geq |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\big\{|p_n - p_{n-1}| + |p_{n-1} - p_{n-2}| + \dots + |p_{m+1} - p_m| + |p_m - p_{m-1}| + \dots + |p_4 - p_3| + |p_3 - p_2| + |p_2| + |q_n - q_{n-1}| + |q_{n-1} - q_{n-2}| + \dots + |q_{m+1} - q_m| + |q_m - q_{m-1}| + \dots + |q_4 - q_3| + |q_3 - q_2| + |q_2|\big\}\Big]$

$\geq |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\big\{p_{n-1} - p_n + p_{n-2} - p_{n-1} + \dots + p_m - p_{m+1} + p_{m+1} - p_{m+2} + \dots + p_3 - p_4 + p_2 - p_3 + |p_2| + q_{n-1} - q_n + q_{n-2} - q_{n-1} + \dots + q_m - q_{m+1} + q_{m+1} - q_{m+2} + \dots + q_3 - q_4 + q_2 - q_3 + |q_2|\big\}\Big]$

$\geq |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\big\{-p_n + p_2 + |p_2| + -q_n + q_2 + |q_2|\big\}\Big]$

Hence $|g(z)| > 0$, provided

$$|z| > \frac{-p_n - q_n + p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}.$$

This implies that all zeros of $g(z)$ whose modulus is greater than 1 are lie in

$$|z| \leq \frac{-p_n - q_n + p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}.$$

Since the zeros of $g(z)$ whose modulus is less than or equal to 1 are already lie in

$$|z| \leq \frac{-p_n - q_n + p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}},$$

it follows that all the zeros of $g(z)$ lie in

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Since all the zeros of $g(z)$ are also the zero of $D_{\alpha}^{'}f(z)$ lie in

$$|z| \leq \frac{-p_n - q_n + p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}.$$

Thus all the zeros of $D_{\alpha}^{'}f(z)$ lie in

$$|z| \leq \frac{-p_n - q_n + p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}.$$

In other words all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \le \frac{-p_n - q_n + p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple, where $Re(b_t) = p_t, Im(b_t) = q_t$ and $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}] \quad for \quad t = 2,3,4,...,n$

Proof of the Theorem 8.

Let $f(z) = k_n z^n + k_{n-1}z^{n-1} + ... + k_1 z + k_0$ be the n^{th} degree polynomial with real coefficients. Then by the definition of polar derivative, we have $D_\alpha f(z) = nf(z) + \alpha f^{'}(z) - zf^{'}(z)$

$D_\alpha f(z) = n(k_n z^n + k_{n-1}z^{n-1} + ... + k_1 z + k_0) + \alpha(nk_n z^{n-1} + (n-1)k_{n-1}z^{n-2} + ... + k_1)$
$- z(nk_n z^{n-1} + (n-1)k_{n-1}z^{n-2} + ... + k_1)$

$D_\alpha f(z) = [n\alpha k_n + (n-(n-1))k_{n-1}]z^{n-1} + [(n-1)\alpha k_{n-1} + (n-(n-2))k_{n-2}]z^{n-2} + ...$
$+ [2\alpha k_2 + (n-1))k_1]z + [\alpha k_1 + nk_0]$

Now,

$D^{'}_\alpha f(z) = b_n z^{n-2} + b_{n-1}z^{n-3} + b_{n-2}z^{n-4} + ... + b_4 z^2 + b_3 z + b_2$

where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}] \quad for \quad t = 2,3,4,...,n$

Now Consider $g(z) = (1-z)D^{'}_\alpha f(z)$, so that

$g(z) = (1-z)[b_n z^{n-2} + b_{n-1}z^{n-3} + b_{n-2}z^{n-4} + ... + b_4 z^2 + b_3 z + b_2]$

$g(z) = -b_n z^{n-1} + (b_n - b_{n-1})z^{n-2} + (b_{n-1} - b_{n-2})z^{n-3} + (b_{n-2} - b_{n-3})z^{n-4} + ...$
$+ (b_{m+1} - b_m)z^{m-1} + (b_m - b_{m-1})z^{m-2} + ... + (b_4 - b_3)z^2 + (b_3 - b_2)z + b_2$

$|g(z)| \ge |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\{|p_n - p_{n-1}| + \frac{|p_{n-1}-p_{n-2}|}{|z|} + \frac{|p_{n-2}-p_{n-3}|}{|z|^2} + ... + \frac{|p_4-p_3|}{|z|^{n-4}} + \frac{|p_3-p_2|}{|z|^{n-3}} + \frac{|p_2|}{|z|^{n-2}} + |q_n - q_{n-1}| + \frac{|q_{n-1}-q_{n-2}|}{|z|} + \frac{|q_{n-2}-q_{n-3}|}{|z|^2} + ... + \frac{|q_4-q_3|}{|z|^{n-4}} + \frac{|q_3-q_2|}{|z|^{n-3}} + \frac{|q_2|}{|z|^{n-2}}\}\Big]$

Also, if $|z|>1$ then $\frac{1}{|z|}<1$

then

$\ge |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\{|p_n - p_{n-1}| + |p_{n-1} - p_{n-2}| + ... + |p_{m+1} - p_m| + |p_m - p_{m-1}| + ... + |p_4 - p_3| + |p_3 - p_2| + |p_2| + |q_n - q_{n-1}| + |q_{n-1} - q_{n-2}| + ... + |q_{m+1} - q_m| + |q_m - q_{m-1}| + ... + |q_4 - q_3| + |q_3 - q_2| + |q_2|\}\Big]$

$\ge |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\{p_n - p_{n-1} + p_{n-1} - p_{n-2} + ... + p_{m+1} - p_m + p_m - p_{m-1} + ... + p_4 - p_3 + p_3 - p_2 + |p_2| + q_{n-1} - q_n + q_{n-2} - q_{n-1} + ... + q_m - q_{m+1} + q_{m+1} - q_{m+2} + ... + q_3 - q_4 +$

$q_2 - q_3 + |q_2|\}\Big]$

$\geq |b_n||z|^{n-2}\left[|z| - \frac{1}{|b_n|}\{p_n - p_2 + |p_2| - q_n + q_2 + |q_2|\}\right]$

Hence $|g(z)| > 0$, provided

$$|z| > \frac{p_n - q_n - p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}.$$

This implies that all zeros of $g(z)$ whose modulus is greater than 1 are lie in

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Since the zeros of $g(z)$ whose modulus is less than or equal to 1 are already lie in

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it follows that all the zeros of $g(z)$ lie in

$$|z| \leq \frac{p_n - q_n - p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}.$$

Since all the zeros of $g(z)$ are also the zeros of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{p_n - q_n - p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}.$$

Thus all the zeros of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{p_n - q_n - p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}.$$

In other words all zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{p_n - q_n - p_2 + q_2 + |p_2| + |q_2|}{\sqrt{p_n^2 + q_n^2}}$$

are simple. where $Re(b_t) = p_t, Im(b_t) = q_t$ and $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}] \quad for \quad t = 2, 3, 4, ..., n$

Proof of the Theorem 9.

Let $f(z) = k_n z^n + k_{n-1} z^{n-1} + ... + k_1 z + k_0$ be the n^{th} degree polynomial with real coefficients.

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$D_\alpha f(z) = nf(z) + \alpha f'(z) - zf'(z)$

$D_\alpha f(z) = n(k_n z^n + k_{n-1} z^{n-1} + ... + k_1 z + k_0) + \alpha(nk_n z^{n-1} + (n-1)k_{n-1} z^{n-2} + ... + k_1)$

$-z(nk_nz^{n-1}+(n-1)k_{n-1}z^{n-2}+...+k_1)$

$D_\alpha f(z)=[n\alpha k_n+(n-(n-1))k_{n-1}]z^{n-1}+[(n-1)\alpha k_{n-1}+(n-(n-2))k_{n-2}]z^{n-2}+...$

$+[2\alpha k_2+(n-1))k_1]z+[\alpha k_1+nk_0]$

Now,

$D'_\alpha f(z)=b_nz^{n-2}+b_{n-1}z^{n-3}+b_{n-2}z^{n-4}+...$

$+b_4z^2+b_3z+b_2$

where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}] \quad for \quad t=2,3,4,...,n$

Now Consider $g(z)=(1-z)D'_\alpha f(z)$, so that

$g(z)=(1-z)[b_nz^{n-2}+b_{n-1}z^{n-3}+b_{n-2}z^{n-4}+...+b_4z^2+b_3z+b_2]$

$g(z)=-b_nz^{n-1}+(b_n-b_{n-1})z^{n-2}+(b_{n-1}-b_{n-2})z^{n-3}+(b_{n-2}-b_{n-3})z^{n-4}+...$

$+(b_{m+1}-b_m)z^{m-1}+(b_m-b_{m-1})z^{m-2}+...+(b_4-b_3)z^2+(b_3-b_2)z+b_2$

$|g(z)|\geq|b_n||z|^{n-2}\left[|z|-\frac{1}{|b_n|}\left\{|p_n-p_{n-1}|+\frac{|p_{n-1}-p_{n-2}|}{|z|}+\frac{|p_{n-2}-p_{n-3}|}{|z|^2}+...+\frac{|p_4-p_3|}{|z|^{n-4}}+\frac{|p_3-p_2|}{|z|^{n-3}}+\right.\right.$
$\left.\left.\frac{|p_2|}{|z|^{n-2}}+|q_n-q_{n-1}|+\frac{|q_{n-1}-q_{n-2}|}{|z|}+\frac{|q_{n-2}-q_{n-3}|}{|z|^2}+...+\frac{|q_4-q_3|}{|z|^{n-4}}+\frac{|q_3-q_2|}{|z|^{n-3}}+\frac{|q_2|}{|z|^{n-2}}\right\}\right]$

Also, if $|z|>1$ then $\frac{1}{|z|}<1$

then

$\geq|b_n||z|^{n-2}\left[|z|-\frac{1}{|b_n|}\left\{|p_n-p_{n-1}|+|p_{n-1}-p_{n-2}|+...+|p_{m+1}-p_m|+|p_m-p_{m-1}|+...+\right.\right.$
$|p_4-p_3|+|p_3-p_2|+|p_2|+|q_n-q_{n-1}|+|q_{n-1}-q_{n-2}|+...+|q_{m+1}-q_m|+|q_m-q_{m-1}|+$
$\left.\left....+|q_4-q_3|+|q_3-q_2|+|q_2|\right\}\right]$

$\geq|b_n||z|^{n-2}\left[|z|-\frac{1}{|b_n|}\left\{p_{n-1}-p_n+p_{n-2}-p_{n-1}+...+p_m-p_{m+1}+p_{m+1}-p_{m+2}+...+\right.\right.$
$p_3-p_4+p_2-p_3+|p_2|+q_n-q_{n-1}+q_{n-1}-q_{n-2}+...+q_{m+1}-q_m+q_m-q_{m-1}+...+$
$\left.\left.q_4-q_3+q_3-q_2+|q_2|\right\}\right]$

$\geq|b_n||z|^{n-2}\left[|z|-\frac{1}{|b_n|}\left\{-p_n+p_2+|p_2|+q_n-q_2+|q_2|\right\}\right]$

Hence $|g(z)|>0$, provided

$$|z|\leq\frac{-p_n+q_n+p_2-q_2+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}.$$

This implies that all zeros of $g(z)$ whose modulus is greater than 1 are lie in

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Since the zeros of $g(z)$ whose modulus is less than or equal to 1 are already lie in

$$|z| \leq \frac{-p_n+q_n+p_2-q_2+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}},$$

it follows that all the zeros of $g(z)$ lie in

$$|z| \leq \frac{-p_n+q_n+p_2-q_2+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}.$$

Since all the zeros of $g(z)$ are also the zeros of $D'_{\alpha}f(z)$ lie in

$$|z| \leq \frac{-p_n+q_n+p_2-q_2+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}.$$

Thus all the zeros of $D'_{\alpha}f(z)$ lie in

$$|z| \leq \frac{-p_n+q_n+p_2-q_2+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}.$$

In other words all zeros of $D_{\alpha}f(z)$ which does not lie in

$$|z| \leq \frac{-p_n+q_n+p_2-q_2+|p_2|+|q_2|}{\sqrt{p_n^2+q_n^2}}$$

are simple, where $Re(b_t)=p_t, Im(b_t)=q_t$ and $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}] \quad for \quad t=2,3,4,...,n$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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