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# DISTANCE ENERGY OF PARTIAL COMPLEMENTARY GRAPH

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**Abstract.** The partial complement of a graph $G$ with respect to a set $S$ is the graph obtained from $G$ by removing the edges of induced subgraph $\langle S\rangle$ and adding edges which are not in $\langle S\rangle$ of $G$. In this paper we introduce the concept of distance energy of connected partial complements of a graph. Few properties on distance eigenvalues and bounds for distance energy of connected partial complement of a graph are achieved. Further distance energy of connected partial complement of some families of graphs are computed.

**Keywords:** distance partial complement; distance partial complement energy; distance partial complement eigenvalue.

**2010 AMS Subject Classification:** 05C15, 05C50.

## 1. Introduction

Let $G$ be a graph with $n$ vertices and $m$ edges. Let $A = (a_{ij})$ be the adjacency matrix of $G$. Then $|A - \lambda I| = 0$ is called characteristic equation of $G$. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A$, assumed in non increasing order, are called eigenvalues of $G$. As $A$ is real symmetric matrix, the eigenvalues of $G$ are real with sum equal to zero. The energy of $G$ [3] is defined to be sum

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of absolute values of the eigenvalues of $G$. i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For all terminologies we refer [1, 2, 5].

Recently Fedor V. Fomin et.al introduced partial complements of graph [4].

Let $G = (V,E)$ be a graph and $S \subseteq V$. The partial complement of a graph $G$ with respect to $S$, denoted by $G \oplus S$, is a graph $(V, E_S)$, where for any two vertices $u, v \in V$, $uv \in E_S$ if and only if one of the following conditions hold good:

(1) $u \notin S$ or $v \notin S$ and $uv \in E$.

(2) $u, v \in S$ and $uv \notin E$.

**Definition 1.1.** [12] *Let $G \oplus S$ be partial complement of a graph $G$ with respect to $S$. The partial complement adjacency matrix of $G \oplus S$ is $n \times n$ matrix defined by $A_p(G \oplus S) = (a_{ij})$, where*

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } i \neq j \\ 1, & \text{if } i = j \text{ and } v_i \in S \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 1.2.** *Let $G \oplus S$ be a connected partial complement of a graph $G$ with respect to $S$. Then partial complement distance matrix of the graph $G \oplus S$ is $n \times n$ matrix defined by $D_p(G \oplus S) = (d_{ij})$, where*

$$d_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } v_i \in S \\ d(v_i, v_j), & \text{otherwise.} \end{cases}$$

*Characteristic polynomial of distance partial complement of a graph $G$ is defined by $\phi\{D_p(G \oplus S)\} = |\lambda I - D_p(G \oplus S)|$ and distance partial complement energy of $G \oplus S$ is denoted by $DE_p(G \oplus S)$, is defined as $\sum_{i=1}^{n} |\lambda_i|$, where $\lambda_i^{\prime}s$ are distance partial complement eigenvalues of $G \oplus S$.*

The distance matrix has found a considerable use in other areas much less mathematical than applied mathematics [10], physics or chemistry such as [11] anthropology, archeology, genetics, geology, history, ornithology, philology, psychology, sociology, etc. The origin of distance matrix may be traced back to very first paper of Cayley. However, this matrix was first introduced in rudimentary form by Brunel in 1895. The distance matrix is used in chemistry in explicit and implicit forms [11]. The earliest explicit use of the distance matrix in chemistry is work by Clark and Kettle in 1975, although in biochemistry it was used in disguise somewhat earlier in 1971. Mihalić et al.[9] gave an overview of the use of the distance matrix in chemical graph theory.

$A_p(G \oplus S)$ is adjacency matrix of a partial complement of $G$ by attaching a loop of weight +1 to each of its vertices belonging to the induced set $S$. Graphs with loops are natural representations of heteroconjugated molecules, and have been much studied in chemical graph theory. Loops of weight +1 are just the graph representation of nitrogen atoms. Matrix $D_p(G \oplus S)$ defined in this article may have some applications in chemistry or other subjects in future.

For more information on energy of graph we refer [6, 7, 8, 13]. The paper is organized as follows. In section 2, properties of distance energy of connected partial complementary graphs are achieved. In section 3, bounds for distance energy of connected partial complementary graphs are established. In section 4, distance energy of some families of connected partial complementary graphs are computed. Partial complements of graphs need not be connected always. Since distance matrix is defined only for connected graph, throughout the paper we consider only connected partial complements of graph.

## 2. PROPERTIES OF DISTANCE ENERGY OF CONNECTED PARTIAL COMPLEMENTARY GRAPHS

**Theorem 2.1.** *Let $G \oplus S = (V, E_S)$ be a connected partial complement of a graph $G = (V, E)$. Let $\phi\{D_p(G \oplus S), \lambda\} = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \ldots + a_n$ be the characteristic polynomial of graph $G \oplus S$. Then,*

*(i)* $a_0 = 1$.

*(ii)* $a_1 = -|S|$.

*(iii)* $a_2 = \binom{|S|}{2} - 2n(n-1) + 3m_S$.

*Proof.* (i) Directly from the definition of $\phi\{D_p(G \oplus S), \lambda\}$, it follows that $a_0 = 1$.

(ii) Sum of diagonal elements of $D_p(G \oplus S)$ is equal to cardinality of the set $S$.

Hence $(-1)a_1$ = trace $\{D_p(G \oplus S)\} = |S|$.

(iii) We have

$$a_2 = \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

$$a_2 = \sum_{1 \le i < j \le n} (a_{ii}a_{jj} - a_{ij}^2)$$

$$a_2 = \sum_{1 \le i < j \le n} a_{ii}a_{jj} - \sum_{1 \le i < j \le n} a_{ij}^2$$

$$a_2 = \binom{|S|}{2} - 2n(n-1) + 3m_S.$$

□

**Theorem 2.2.** *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ represents distance eigenvalues of connected $G \oplus S$, then*
*(i) $\sum_{i=1}^{n} \lambda_i = |S|$*
*(ii) $\sum_{i=1}^{n} \lambda_i^2 = k + 4n^2 - 4n - 6m_S$, where $m_S$ is number of edges of $G \oplus S$.*

*Proof.* $(i)$ Sum of eigenvalues of $D_p(G \oplus S)$ is equal to trace of $D_p(G \oplus S)$,

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_{ii} = |S|.$$

$(ii)$ The sum of squares of eigenvalues of $D_p(G \oplus S)$ is the trace of $D_p^2(G \oplus S)$.

$$\begin{aligned} i.e,\ \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}d_{ji} \\ &= \sum_{i=1}^{n} d_{ii}^2 + \sum_{i \ne j} d_{ij}d_{ji} \\ &= |S| + 2\sum_{i<j} d_{ij}^2 \\ &= k + 4n^2 - 4n - 6m_s, \end{aligned}$$

where $m_s$ is number of edges of $G \oplus S$. □

**Theorem 2.3.** *Let $G \oplus S_1$ and $H \oplus S_2$ be two connected partial complements of graphs $G$ and $H$ respectively on $n$ vertices. Let $m_{S1}, m_{S2}$ denote number of edges of $G \oplus S_1$ and $H \oplus S_2$ respectively. If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\lambda'_1 \geq \lambda'_2 \geq \ldots \geq \lambda'_n$ are distance eigenvalues of $G \oplus S_1$ and $H \oplus S_2$ respectively, then*
$\sum\limits_{i=1}^{n} \lambda_i \lambda'_i \leq \sqrt{(|S_1| - 6m_{S1} + 4n^2 - 4n)(|S_2| - 6m_{S2} + 4n^2 - 4n)}$.

*Proof.* Applying Cauchy Schwarz inequality for $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $(\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ we get,

$$\left(\sum_{i=1}^{n} \lambda_i \lambda'_i\right)^2 \leq \left(\sum_{i=1}^{n} \lambda_i^2\right)\left(\sum_{i=1}^{n} \lambda_i'^2\right)$$

$$\sum_{i=1}^{n} \lambda_i \lambda'_i \leq \sqrt{(|S_1| - 6m_{S1} + 4n^2 - 4n)(|S_2| - 6m_{S2} + 4n^2 - 4n)}.$$

□

## 3. Bounds for Distance Energy of Connected Partial Complementary Graphs

**Theorem 3.1.** *Let $G \oplus S$ be connected partial complement of a graph $G$ with induced set $|S| = k$. Then*
$\sqrt{k + 2(2n^2 - 2n - 3m_S)} \leq DE_p(G \oplus S) \leq \sqrt{n[k + 2(2n^2 - 2n - 3m_S)]}$.

*Proof.* Taking $a_i = 1, b_i = |\lambda_i|$ in Cauchy Schwarz inequality we get,

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \lambda_i^2\right)$$

$$(DE_p(G \oplus S))^2 \leq n[k + 2(2n^2 - 2n - 3m_S)]$$

$$DE_p(G \oplus S) \leq \sqrt{n[k + 2(2n^2 - 2n - 3m_S)]}.$$

Also

$$\left[\sum_{i=1}^{n} |\lambda_i|\right]^2 \geq \sum_{i=1}^{n} |\lambda_i|^2$$

$$[DE_p(G \oplus S)]^2 \geq k + 2(2n^2 - 2n - 3m_S)$$

$$DE_p(G \oplus S) \geq \sqrt{k + 2(2n^2 - 2n - 3m_S)}.$$

□

**Theorem 3.2.** *Let $G \oplus S$ be a connected partial complement of a graph $G$ on $n$ vertices with induced set $S$ of order $k$. Then $DE_p(G \oplus S) \geq \sqrt{k + 2(2n^2 - 2n - 3m_S) + n(n-1)D^{2/n}}$ where $D = |D_p(G \oplus S)|$.*

*Proof.* Using arithmetic mean and geometric mean inequality,

$$\begin{aligned}
\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| &\geq \left[ \prod_{i \neq j} |\lambda_i||\lambda_j| \right]^{\frac{1}{n(n-1)}} \\
&= \left[ \prod_{i=1}^{n} |\lambda_i|^{2(n-1)} \right]^{\frac{1}{n(n-1)}} \\
&= \left[ \prod_{i=1}^{n} |\lambda_i| \right]^{\frac{2}{n}} \\
&= [det(D_p(G \oplus S))]^{2/n} \\
&= D^{2/n}.
\end{aligned}$$

Hence,

$$\sum_{i \neq j} |\lambda_i||\lambda_j| \geq n(n-1)D^{2/n}.$$

Now consider

$$\begin{aligned}
[DE_p(G \oplus S)]^2 &= (\sum_{i=1}^{n} |\lambda_i|)^2 \\
&= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i||\lambda_j|.
\end{aligned}$$

Thus,

$$DE_p(G \oplus S) \geq \sqrt{k + 2(2n^2 - 2n - 3m_S) + n(n-1)D^{2/n}}.$$

□

**Theorem 3.3.** *Let $G \oplus S = (V, E_S)$ be a connected graph with induced subgraph $< S >$. Then*

$$k \leq DE_p(G \oplus S) \leq \sqrt{2m_S[k + 2(2n^2 - 2n - 3m_S)]}.$$

*Proof.* Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distance eigenvalues of $G \oplus S$.

Since,

$$\sum_{i=1}^{n} \lambda_i = |S| = k$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = k + 2(2n^2 - 2n - 3m_s),$$

we have

$$\sum_{i<j} \lambda_i \lambda_j = \binom{k}{2} - 2n(n-1) + 3m_S. \tag{1}$$

Now consider

$$\begin{aligned}
[DE_p(G \oplus S)]^2 &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \\
&= \sum_{i=1}^{n} |\lambda_i|^2 + 2 \sum_{1 \leq i \leq j \leq n} |\lambda_i||\lambda_j| \\
&\geq \sum_{i=1}^{n} |\lambda_i|^2 + 2 \left| \sum_{i<j} \lambda_i \lambda_j \right| \\
[DE_p(G \oplus S)]^2 &\geq k + 2(2n^2 - 2n - 3m_s) + 2\left[\binom{k}{2} - 2n(n-1) + 3m_S\right]
\end{aligned}$$

by using theorem (2.2) and equation (1).

Hence,

$$DE_p(G \oplus S) \geq k.$$

From theorem (3.1), we have $DE_p(G \oplus S) \leq \sqrt{n[k + 2(2n^2 - 2n - 3m_S)]}$.

Since $n \leq 2m_S$,

$$DE_p(G \oplus S) \leq \sqrt{2m_S[k + 2(2n^2 - 2n - 3m_S)]}.$$

Thus,

$$k \leq DE_p(G \oplus S) \leq \sqrt{2m_S[k + 2(2n^2 - 2n - 3m_S)]}.$$

□

**Theorem 3.4.** *Let $G \oplus S = (V, E_S)$ be a connected partial complement of a graph $G = (V, E)$ of order $n$ and size $m_S$ and $\rho(G \oplus S) = \max_{1 \leq i \leq n} \{|\lambda_i|\}$ be the distance spectral radius of $G \oplus S$. Then*

$$\sqrt{\frac{k + 2(2n^2 - 2n - 3m_s)}{n}} \leq \rho(G \oplus S) \leq \sqrt{k + 2(2n^2 - 2n - 3m_s)}.$$

*Proof.* Consider

$$\begin{aligned}\rho^2(G \oplus S) &= \max_{1 \leq i \leq n} \{|\lambda_i|^2\} \\ &\leq \sum_{j=1}^{n} \lambda_j^2 = k + 2(2n^2 - 2n - 3m_s). \\ \rho(G \oplus S) &\leq \sqrt{k + 2(2n^2 - 2n - 3m_s)}.\end{aligned}$$

Next consider

$$\begin{aligned}n\rho^2(G \oplus S) &\geq \max_{1 \leq i \leq n} \{|\lambda_i|^2\} \\ &\geq k + 2(2n^2 - 2n - 3m_s).\end{aligned}$$

We have,

$$\rho(G \oplus S) \geq \sqrt{\frac{k + 2(2n^2 - 2n - 3m_s)}{n}}.$$

Hence,

$$\sqrt{\frac{k + 2(2n^2 - 2n - 3m_s)}{n}} \leq \rho(G \oplus S) \leq \sqrt{k + 2(2n^2 - 2n - 3m_s)}.$$

□

**Lemma 3.1.** [8] *Let $a, a_1, a_2, \ldots, a_n, A$ and $b, b_1, b_2, \ldots, b_n, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$, $\forall$ $i = 1, 2, ...n$. Then the following inequality is valid.*

$$\left| n \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A - a)(B - b) \tag{2}$$

*where $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$ and $[x]$ denote the integral part of a real number and equality holds if and only if $a_1 = a_2 = ... = a_n$ and $b_1 = b_2 = ... = b_n$.*

**Theorem 3.5.** *Let $G \oplus S$ be a connected partial complement of $G$ on $n$ vertices and $m_s$ edges. Let $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$ be a non-increasing order of distance eigenvalues of $G \oplus S$. Then*

$$DE_p(G \oplus S) \geq \sqrt{n(k + 4n^2 - 4n - 6m_S) - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}.$$

*Proof.* Taking $a_i = b_i = |\lambda_i|$, $a = b = |\lambda_n|$ and $A = B = |\lambda_1|$ in Lemma 3.1, we obtain

$$\left| n \sum_{i=1}^{n} |\lambda_i|^2 - \left( \sum_{i=1}^{n} \lambda_i \right)^2 \right| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2. \tag{3}$$

But

$$\sum_{i=1}^{n} |\lambda_i|^2 = k + 4n^2 - 4n - 6m_S.$$

Inequality (3) becomes $n(k + 4n^2 - 4n - 6m_S) - [DE_p(G \oplus S)]^2 \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2$

$$DE_p(G \oplus S) \geq \sqrt{n(k + 4n^2 - 4n - 6m_S) - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}.$$

□

**Lemma 3.2.** [8] *Let $a_i \neq 0$, $b_i, r$ and $R$ be real numbers satisfying $ra_i \leq b_i \leq Ra_i$, then the following inequality holds*

$$\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i \leq (r + R) \sum_{i=1}^{n} a_i b_i.$$

**Theorem 3.6.** *Let $G \oplus S$ be connected partial complement of $G$ on $n$ vertices and $m_S$ edges with induced subgraph $< S >$ . Let $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| > 0$ be a non-increasing order of eigenvalues of $D_p(G \oplus S)$ then*

$$DE_p(G \oplus S) \geq \frac{n(k + 4n^2 - 4n - 6m_S + |\lambda_1||\lambda_n|)}{|\lambda_1| + |\lambda_n|}.$$

*Proof.* Taking $b_i = |\lambda_i|$, $a_i = 1, r = |\lambda_n|$ and $R = |\lambda_1|$ in Lemma 3.2, we obtain

$$\sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| \sum_{i=1}^{n} 1 \leq (|\lambda_1| + |\lambda_n|) \sum_{i=1}^{n} |\lambda_i|$$

$$\sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n| n \leq (|\lambda_1| + |\lambda_n|) DE_p(G \oplus S)$$

$$DE_p(G \oplus S) \geq \frac{n(k + 4n^2 - 4n - 6m_S + |\lambda_1||\lambda_n|)}{|\lambda_1| + |\lambda_n|}.$$

□

**Theorem 3.7.** *Let $G \oplus S$ be connected partial complement of a graph $G$ on $n$ vertices and $m_S$ edges. Suppose that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the distance eigenvalues of $G \oplus S$, then*

$$DE_p(G \oplus S) \leq \lambda_1 + \sqrt{(n-1)(k + 4n^2 - 4n - 6m_S - \lambda_1^2)}.$$

*Proof.* Applying Cauchy Schwarz inequality for $(n-1)$ terms,

$$(\sum_{i=2}^{n} \lambda_i)^2 \leq (\sum_{i=2}^{n} 1)(\sum_{i=2}^{n} \lambda_i^2)$$

$$[DE_p(G \oplus S) - \lambda_1]^2 \leq (n-1)(k + 4n^2 - 4n - 6m_S - \lambda_1^2)$$

$$DE_p(G \oplus S) \leq \lambda_1 + \sqrt{(n-1)(k + 4n^2 - 4n - 6m_S - \lambda_1^2)}.$$

□

**Theorem 3.8.** *If $G \oplus S$ is a connected partial complement of a graph $G$ with $n$ vertices and $m_S$ edges and $2m_S \geq n$, then,*

$$DE_p(G \oplus S) \leq \frac{k + 4n^2 - 4n - 6m_S}{n} + \sqrt{(n-1)\left[k + 4n^2 - 4n - 6m_S - \left(\frac{k + 4n^2 - 4n - 6m_S}{n}\right)^2\right]}.$$

*Proof.* From theorem 3.7, we have,

$$DE_p(G \oplus S) \leq \lambda_1 + \sqrt{(n-1)(k + 4n^2 - 4n - 6m_S - \lambda_1^2)}.$$

Let

$$f(x) = x + \sqrt{(n-1)(k + 4n^2 - 4n - 6m_S - x^2)}.$$

For decreasing function

$$f'(x) \leq 0 \Rightarrow 1 - \frac{2x(n-1)}{2\sqrt{(n-1)(k + 4n^2 - 4n - 6m_S - x^2)}} \leq 0$$

$$\Rightarrow x \geq \sqrt{\frac{k + 4n^2 - 4n - 6m_S}{n}}.$$

Since $2m_S + |S| \geq n$, we have, $\sqrt{\frac{k + 4n^2 - 4n - 6m_S}{n}} \leq \frac{k + 4n^2 - 4n - 6m_S}{n} \leq \lambda_1$.

Thus,

$$DE_p(G \oplus S) \leq \frac{k + 4n^2 - 4n - 6m_S}{n} + \sqrt{(n-1)\left[k + 4n^2 - 4n - 6m_S - \left(\frac{k + 4n^2 - 4n - 6m_S}{n}\right)^2\right]}.$$

□

## 4. Distance Partial Complement Spectrum of Some Families of Graphs

**Theorem 4.1.** *If $K_n$ is complete graph of order n and $K_n \oplus S$ is its connected partial complement, then*

*$D_p$-complement spectrum of $K_n \oplus S$ with $|S| = k, k < n$ is*

$$\left\{ \begin{matrix} -1 & \frac{(n+k-2)+\sqrt{5k^2+n^2-2nk}}{2} & \frac{(n+k-2)-\sqrt{5k^2+n^2-2nk}}{2} \\ n-2 & 1 & 1 \end{matrix} \right\}.$$

*Proof.* $D_p(K_n \oplus S) = \begin{bmatrix} (2J-I)_{k\times k} & J_{k\times(n-k)} \\ J_{(n-k)\times k} & (J-I)_{(n-k)\times(n-k)} \end{bmatrix}_{n\times n}$ is the distance partial complement matrix of $K_n \oplus S$. The result is proved by showing $AZ = \lambda Z$ for certain vectors $Z$ and by making use of fact that the geometric multiplicity and algebraic multiplicity of each distance eigenvalue $\lambda$ is same, as $D_p(K_n \oplus S)$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $D_p(K_n \oplus S)$.

Consider

$$[D_p(K_n \oplus S) - \lambda I] \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} [2J-(\lambda+1)I]X + JY \\ JX + [J-(\lambda+1)I]Y \end{bmatrix}. \tag{4}$$

First, let $X = X_i = e_1 - e_i$, $i = 2,3,\ldots,k$ and $Y = Y_j = e_1 - e_j$, $j = 2,3,\ldots,n-k$.

From equation (4), we have $[2J-(\lambda+1)I]X_i + JY_j = -(\lambda+1)X_i$.

So, $\lambda = 1$ is the distance eigenvalue with multiplicity of at least $(n-2)$ since there are $(k-1)$ independent vectors of the form $X_i$ and $n-k-1$ independent vectors of the form $Y_j$.

Let $X = 1_k$ and $Y = \left(\frac{\lambda-k+1}{\lambda+1}\right) 1_{n-k}$ where $\lambda$ is any root of the equation

$$\lambda^2 + (2-n-k)\lambda - k^2 + nk - n - k + 1 = 0. \tag{5}$$

From equation (4),

$$\begin{aligned} [2J-(\lambda+1)I]1_k + J\left(\frac{\lambda-k+1}{\lambda+1}\right)1_{n-k} &= 2k - (\lambda+1)1_k + \left(\frac{\lambda-k+1}{\lambda+1}\right)(n-k)1_k \\ &= [\lambda^2 + (2-n-k)\lambda - k^2 + nk - n - k + 1]1_k. \end{aligned}$$

Using equation (5), we obtain $\lambda = \dfrac{(n+k-2)+\sqrt{5k^2+n^2-2nk}}{2}$ and $\lambda = \dfrac{(n+k-2)-\sqrt{5k^2+n^2-2nk}}{2}$ are distance eigenvalues both with multiplicity of at least one.

Thus distance partial complement spectrum of $K_n \oplus S$ is $\left\{\begin{matrix} -1 & \dfrac{(n+k-2)+\sqrt{5k^2+n^2-2nk}}{2} & \dfrac{(n+k-2)-\sqrt{5k^2+n^2-2nk}}{2} \\ n-2 & 1 & 1 \end{matrix}\right\}$ and distance partial complement energy of $(K_n \oplus S)$ is $DE_p(K_n \oplus S) = (n-2)+\sqrt{5k^2+n^2-2nk}$.

□

**Theorem 4.2.** *Let $S_n^0$ be the crown graph of order $2n$ and $S_n^0 \oplus S$ be its connected partial complement with $|S| = n$, then distance partial complement spectrum of $S_n^0 \oplus S$ is*
$$\left\{\begin{matrix} -1+\sqrt{2} & -1-\sqrt{2} & \dfrac{(3n-2)+\sqrt{5n^2+4n+8}}{2} & \dfrac{(3n-2)-\sqrt{5n^2+4n+8}}{2} \\ n-1 & n-1 & 1 & 1 \end{matrix}\right\}.$$

*Proof.* $D_p(S_n^0 \oplus S) = \begin{bmatrix} J_{n\times n} & (J+I)_{n\times n} \\ (J+I)_{n\times n} & 2(J-I)_{n\times n} \end{bmatrix}_{2n\times 2n}$ is the distance partial complement matrix of $S_n^0 \oplus S$. The result is proved by showing $AZ = \lambda Z$ for certain vector $Z$ and by making use of fact that the geometric multiplicity and algebraic multiplicity of each distance eigenvalue $\lambda$ is same, as $D_p(S_n^0 \oplus S)$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $D_p(S_n^0 \oplus S)$.

Consider

$$[D_p(S_n^0 \oplus S) - \lambda I]\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} [(J-\lambda I)X + (J+I)Y \\ (J+I)X + [2J-(\lambda+2)I]Y \end{bmatrix}. \tag{6}$$

First, Let $X = 1_n$ and $Y = \dfrac{\lambda - n}{n+1}1_n$ where $\lambda$ is any root of the equation

$$\lambda^2 + (2-3n)\lambda + n^2 - 4n - 1 = 0. \tag{7}$$

Then from equation (6), $(J-\lambda I)1+(J+I)\frac{\lambda-n}{n+1}1_n=(n-\lambda)1_n+(n+1)\frac{\lambda-n}{n+1}1_n=0$ and also

$$\begin{aligned}(J-\lambda I)1_n+[2J-(\lambda-2)]\frac{\lambda-n}{n+1}1_n&=(n+1)1_n+(2n-\lambda-2)\frac{\lambda-n}{n+1}1_n\\&=\left(\frac{-\lambda^2+(3n-2)\lambda-n^2+4n+1}{n+1}\right)1_n.\end{aligned}$$

From equation (7), $\lambda=\frac{(3n-2)+\sqrt{5n^2+4n+8}}{2}$ and $\lambda=\frac{(3n-2)-\sqrt{5n^2+4n+8}}{2}$ are distance eigenvalues both with multiplicity of at least one.

Let $X_i=e_1-e_i$, $i=2,3,\ldots,n$ and $Y_i=\lambda X_i$, where $\lambda$ is any root of the equation $\lambda^2+2\lambda-1$=0.

Observe that $JX_i=JY_i=0$.

From equation (6), $(J-\lambda I)X_i+(J+I)\lambda X_i=-\lambda X_i+\lambda X_i=0$

and $(J+I)X_i+[2I-(\lambda+2)I]\lambda X_i=(-\lambda^2-2\lambda+1)X_i$.

From equation (7), $\lambda=-1+\sqrt{2}$ and $\lambda=-1-\sqrt{2}$ both distance eigenvalues are of multiplicity at least $(n-1)$ as there are $(n-1)$ linearly independent vectors of the form $X_i$ .

Distance partial complement spectrum of $S_n^0\oplus S$ is

$$\left\{\begin{matrix}-1+\sqrt{2} & -1-\sqrt{2} & \frac{(3n-2)+\sqrt{5n^2+4n+8}}{2} & \frac{(3n-2)-\sqrt{5n^2+4n+8}}{2}\\ n-1 & n-1 & 1 & 1\end{matrix}\right\}$$ and distance partial complement energy is $DE_p(S_n^0\oplus S)=2\sqrt{2}(n-1)+\sqrt{5n^2+4n+8}$. □

**Theorem 4.3.** *Let $K_{n\times 2}$ be cocktail party graph with vertex set $V=\{v_1,v_2,\ldots,v_n,u_1,u_2,\ldots,u_n\}$ and $|S|=\{v_1,v_2,\ldots,v_n\}$. Let $K_{n\times 2}\oplus S$ be its connected partial complement. Then distance partial complement spectrum of $K_{n\times 2}\oplus S$ is*

$$\left\{\begin{matrix}0 & -2 & \frac{(3n-2)+\sqrt{5n^2+8n+4}}{2} & \frac{(3n-2)-\sqrt{5n^2+8n+4}}{2}\\ n-1 & n-1 & 1 & 1\end{matrix}\right\}.$$

*Proof.* $D_p(K_{n\times 2}\oplus S)=\begin{bmatrix}(2I-J)_{n\times n} & (J+I)_{n\times n}\\ (J+I)_{n\times n} & (J-I)_{n\times n}\end{bmatrix}_{2n\times 2n}$ is the distance partial complement matrix of $K_{n\times 2}\oplus S$. The result is proved by showing $AZ=\lambda Z$ for certain vectors $Z$ and by making use of fact that the geometric multiplicity and algebraic multiplicity of each distance eigenvalue $\lambda$ is same, as $D_p(K_{n\times 2}\oplus S)$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $D_p(K_{n\times 2} \oplus S)$.
Note that

$$[D(K_{n\times 2} \oplus S) - \lambda I] \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} [2J - (\lambda+1)I]X + (J+I)Y \\ (J+I)X + [J - (\lambda+1)Y \end{bmatrix}. \tag{8}$$

**Case 1:** Let $X = 1_n$ and $Y = \dfrac{\lambda - 2n + 1}{n+1} 1_n$, where $\lambda$ is any root of the equation

$$\lambda^2 + (2-3n)\lambda + n^2 - 5n = 0. \tag{9}$$

Then from equation (8),
$[2J - (\lambda+1)I]1_n + (J+I)\dfrac{\lambda - 2n+1}{n+1} 1_n = \left[ (2n - \lambda - 1) + \dfrac{(n+1)(\lambda - 2n+1)}{n+1} \right] 1_n = 0$
and

$$\begin{aligned}(J+I)1_n + [J - (\lambda+1)I]\frac{\lambda - 2n+1}{n+1} 1_n &= \left[ (n+1) + \frac{(n - \lambda - 1)(\lambda - 2n+1)}{n+1} \right] 1_n \\ &= -\frac{[\lambda^2 + (2-3n)\lambda + n^2 - 5n]}{n+1} 1_n.\end{aligned}$$

From equation (9), $\lambda = \dfrac{(3n-2) + \sqrt{5n^2+8n+4}}{2}$ and $\lambda = \dfrac{(3n-2) - \sqrt{5n^2+8n+4}}{2}$ are the distance eigenvalues with multiplicity of at least one.

**Case 2:** Let $X = X_i = e_1 - e_i$, $i = 2, 3, \ldots, n-1$ and $Y = Y_i = (\lambda+1)X_i$.

From equation (8), $[2J - (\lambda+1)I]X_i + (J+I)Y_i = -(\lambda+1)X_i + (\lambda+1)X_i = 0$

also $(J+I)X_i + [J - (\lambda+1)I]Y_i = X_i + [J - (\lambda+1)I](\lambda+1)X_i = (-\lambda^2 - 2\lambda)X_i$.

From equation(9), $\lambda = 0$ and $\lambda = -2$ are the distance eigenvalues with multiplicity of at least $n-1$.

Distance partial complement spectrum of $K_{n\times 2} \oplus S$ is

$$\begin{Bmatrix} 0 & -2 & \dfrac{(3n-2) + \sqrt{5n^2+8n+4}}{2} & \dfrac{(3n-2) - \sqrt{5n^2+8n+4}}{2} \\ n-1 & n-1 & 1 & 1 \end{Bmatrix}$$

and $DE_p(K_{n\times 2} \oplus S) = 2(n-1) + \sqrt{5n^2+8n+4}$.

□

**Theorem 4.4.** *Let $K_{1,n-1} \oplus S$ be the connected partial complement of a star graph of order $n$ with*

*$|S| = k$, $k$ is the number of pendant vertices. Then characteristic poynomial of $D_p(K_{1,n-1} \oplus S)$ is $\lambda^{k-1}(\lambda+2)^{n-k-2}\{\lambda^3 + (k-2n+4)\lambda^2 + (2k^2-2nk-n+1)\lambda + k^2 - kn - k\}$.*

*Proof.* $D_p(K_{1,n-1} \oplus S) = \begin{bmatrix} 0 & J_{1\times k} & J_{1\times(n-k-1)} \\ J_{k\times 1} & J_{k\times k} & 2J_{k\times(n-k-1)} \\ J_{(n-k-1)\times 1} & 2J_{(n-k-1)\times k} & 2B_{(n-k-1)\times(n-k-1)} \end{bmatrix}_{n\times n}$ is the distance partial complement matrix of $K_{1,n-1} \oplus S$, where $B$ is the adjacency matrix of complete subgraph. Characteristic polynomial of $K_{1,n-1} \oplus S$ is

$$|\lambda I - D_p(K_{1,n-1} \oplus S)| = \begin{vmatrix} \lambda & -J & -J \\ -J & \lambda I - J & -2J \\ -J & -2J & \lambda I - 2B \end{vmatrix}_{n\times n} \tag{10}$$

On applying row operation $R_i = R_i - R_{i+1}, i = 2,3,\ldots,k-1,k+1,\ldots,n-k-2$ and column operations $C_i = C_i + C_{i+1} + \ldots + C_k, i = 2,3,\ldots,k-1, C_j = C_j + C_{j+1},\ldots,C_{n-k-1}, j = k+1,\ldots,n-k-2$ in equation(10) and further simplifying the determinant, it reduces to order 3.

i.e., $|\lambda I - D_p(K_{1,n-1} \oplus S)| = \lambda^{k-1}(\lambda+2)^{n-k-2} \begin{vmatrix} \lambda & -k & -(n-k-1) \\ -1 & \lambda - k & -2(n-k-1) \\ -1 & -2k & \lambda - 2(n-k-2) \end{vmatrix}$

The result is obtained on further expansion of determinant.

□

**Theorem 4.5.** *Let $K_{p,q} \oplus S$ be connected partial complement of complete bipartite graph of order $n = p+q$ such that $p = a+b, q = c+d$ and $k = a+c$ with $b,d \neq 0$. Then characteristic polynomial of $D_p(K_{p,q} \oplus S)$ is $\lambda^{n-4}\{\lambda^4 + (k-2n+4)\lambda^3 + (ad - 2ab - 8ac - 4n + bc + 3bd - 2cd + 4)\lambda^2 + (2ad - 4ab - 32ac - 4k + 2bc - 4cd + 9abc + 3abd + 9acd + 3bcd)\lambda - 32ac + 18abc + 18acd - 9abcd\}$.*

*Proof.* Distance partial complement matrix of $K_{p,q} \oplus S$ is

$$D_p(K_{p,q} \oplus S) = \begin{bmatrix} J_{a\times a} & 2J_{a\times b} & 3J_{a\times c} & J_{a\times d} \\ 2J_{b\times a} & 2B_{b\times b} & J_{b\times c} & J_{b\times d} \\ 3J_{c\times a} & J_{c\times b} & J_{c\times c} & 2J_{c\times d} \\ J_{d\times a} & J_{d\times b} & 2J_{d\times c} & 2B_{d\times d} \end{bmatrix}_{n\times n}$$

, where $B$ is the matrix of complete subgraph.

Characteristic polynomial of $K_{p,q} \oplus S$ is

$$|\lambda I - D_p(K_{p,q} \oplus S)| = \begin{vmatrix} \lambda I - J & -2J & -3J & -J \\ -2J & \lambda I - 2B & -J & -J \\ -3J & -J & \lambda I - J & -2J \\ -J & -J & -2J & \lambda I - 2B \end{vmatrix}_{n\times n}. \tag{11}$$

On applying row operation $R_i = R_i - R_{i+1}, i = 1,2,\ldots,a-1,a+1,\ldots,b-1,b+1,\ldots,c-1,$ $c+1\ldots,d-1$ and column operations $C_i = C_i + C_{i+1},\ldots,C_a, C_j = C_j + C_{j+1},\ldots,C_b, C_l = C_l +$ $C_{l+1},\ldots,C_c,$
$C_m = C_m + C_{m+1},\ldots,C_d$, where $i = 1,2,\ldots,a-1, j = a+1,\ldots,b-1, l = b+1,\ldots,c-1,$ $m = c+1,\ldots,d-1$ in equation $(11)$ and further simplifying the determinant, it reduces to order 4.

i.e., $$|\lambda I - D_p(K_{p,q} \oplus S)| = \lambda^{k-2}\lambda^{n-k-2} \begin{vmatrix} \lambda - a & -2b & -3c & -d \\ -2a & \lambda - 2b + 2 & -c & -d \\ -3a & -b & \lambda - c & -2d \\ -a & -b & -2c & \lambda - 2d + 2 \end{vmatrix}_{4\times 4}.$$

The result is obtained by further expansion of determinant. □

## 5. Conclusion

The distance partial complement energy depends on the chosen induced set of graph such that resultant partial complement is connected. It is quite interesting to find the distance energy of connected partial complement graph for various induced subgraph in a graph. We found few bounds for distance energy of connected partial complements of graph and derived the spectrum for class of graphs.

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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