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FIXED POINT THEOREMS FOR MULTIPLICATIVE CONTRACTION MAPPINGS ON MULTIPLICATIVE METRIC SPACE

MONIKA*, NAWNEET HOODA
Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonepat 131039, Haryana, India

Abstract: The purpose of this paper is to prove some fixed point theorems in multiplicative metric space using Kannan contraction mapping and other contraction mappings.

Keywords: multiplicative metric space; fixed points; subsequently convergent.

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1. INTRODUCTION

In 2008, Bashirov et.al [3] introduced a new kind of space, called multiplicative metric space. In this space the usual triangular inequality was replaced by a multiplicative triangular inequality as follows.

Definition 1.1.[3] Let X be a non empty set. A mapping $d: X \times X \rightarrow R_+$ is said to be multiplicative metric on X if it satisfies the following conditions –

(1) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(3) $d(x, y) \leq d(x, z).d(z, y)$ for all $x, y, z \in X$ (multiplicative triangle Inequality)

Then the pair (X, d) is called multiplicative metric space.

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*Corresponding author
E-mail address: mverma192@gmail.com
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In 1968, Kannan [7] established fixed point theorem for mapping satisfying

$d(Sx, Sy) \leq \lambda\, [d(x, Sx) + d(y, Sy)]$ for all $x, y \in X$ where $\lambda \in [0, \frac{1}{2})$

After that, in 2008, Azam and Arshad [2] extended the Kannan's theorem for generalised metric spaces introduced by Branciari in 2000 [4].

In 2009, S. Moradi [8] extended Kannan's theorem [7] and then extended the theorem due to Azam and Arshad [2] on complete metric spaces and on generalised metric spaces depended on another function.

In this paper we prove 'Extended Kannan theorem' [7] and some other theorems in multiplicative metric space.

2. PRELIMINARIES

Example 2.1.[9] Let R^n_+ be the collection of all n-tuples of positive real numbers.

Let d: $R^n_+ \times R^n_+ \to R$ be defined as

$$d(x, y) = \left(\left|\frac{x_1}{y_1}\right| . \left|\frac{x_2}{y_2}\right| , \ldots \ldots \ldots \left|\frac{x_n}{y_n}\right| \right)$$

where $x = (x_1, x_2, \ldots \ldots, x_n)$ and $y = (y_1, y_2, \ldots \ldots . y_n) \in R^n_+$ and $|\,.\,| : R_+ \to R_+$ is defined by

$|a| = \begin{cases} a & if\ a \geq 1 \\ \frac{1}{a} & if\ a < 1 \end{cases}$, here all the conditions of multiplicative metric are satisfied. Therefore

(R^n_+, d) is a multiplicative metric space.

One can refer to [5] and [9] for detailed multiplicative metric topology.

Definition 2.2. ([9]) A sequence $\{x_n\}$ in multiplicative metric space (X, d) is said to be multiplicatively convergent to $x \in X$ if and only if $d(x_n, x) \to 1$ as $n \to \infty$.

Definition 2.3. ([9]) Let (X, d) be a multiplicative metric space. Then a sequence $\{x_n\}$ in (X, d) is called multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1$ as $n, m \to \infty$.

Definition 2.4. ([9]) Let (X, d) be a multiplicative metric space then it is said to be complete if every multiplicative Cauchy sequence is multiplicatively convergent.

Theorem 2.5.([7]) Let (X, d)be a complete metric space and T: $X \to X$ be a Kannan contraction mapping that is $d(Tx, Ty) \leq k\, [d(x, Tx) + d(y, Ty)$ for all $x, y \in X$ where $k \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Theorem 2.6. Let (X,d) be a complete metric space and T: $X \to X$ be a chatterjea-contraction mapping that is $d(Tx,Ty) \leq k\,[d(x,Ty) + d(y,Tx)]$ for all $x,y \in X$ where $k \in [0,\frac{1}{2})$. Then T has a unique fixed point.

Theorem 2.7. Let (X,d) be a multiplicative metric space. A self mapping f is said to be multiplicative Kannan contraction if

$$d(fx,fy) \leq (d(fx,x).\,d(fy,y))^{\lambda} \text{ for all } x,y \in X \text{ where } \lambda \in [0,\tfrac{1}{2}).$$

Theorem2.8. Let (X,d) be a multiplicative metric space. A self mapping f is said to be multiplicative chatterjea contraction if

$$d(fx,fy) \leq (d(fx,y).\,d(fy,x))^{\lambda} \text{ for all } x,y \in X \text{ where } \lambda \in [0,\tfrac{1}{2}).$$

After that S.Moradi proved ‘Extended Kannan’s Theorem’ on complete metric space as follows:

Theorem 2.9. (Extended Kannan’s theorem) Let (X,d) be a complete metric space and T, S : $X \to X$ be a mapping such that T is continuous one to one and subsequentially convergent.

If $\lambda \in [0,\frac{1}{2})$ and

$$d(TSx,TSy) \leq \lambda\,[d(Tx,TSx) + d(Ty,TSy)], \text{ for all } x,y \in X,$$

then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in$ X, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

3. MAIN RESULTS

In this section we prove extended Kannan theorem in multiplicative metric space as follows –

Theorem 3.1. Let (X,d) be a multiplicative metric space and T, S : $X \to X$ be a mapping such that T is continuous, one to one and subsequentially convergent. If $\lambda \in [0,\frac{1}{2})$ and $d(TSx,TSy) \leq [d(Tx,TSx).\,d(Ty,TSy)]^{\lambda}$ for all $x,y \in X$ then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Proof. Let x_0 be any arbitrary point in X. We define $\{x_n\}$ such that $x_{n+1} = Sx_n$

Now we have,

$d(Tx_n,Tx_{n+1}) = d(TSx_{n-1},TSx_n)$

$$\leq [d(Tx_{n-1}, TSx_{n-1}).\, d(Tx_n, TSx_n)]^{\lambda}$$

$$\leq d^{\lambda}\,(Tx_{n-1}, TSx_{n-1}).\, d^{\lambda}\,(Tx_n, TSx_n)$$

$$= d^{\lambda}\,(Tx_{n-1}, Tx_n).\, d^{\lambda}\,(Tx_n, Tx_{n+1})$$

implies

$$d^{1-\lambda}\,(Tx_n, Tx_{n+1}) \leq d^{\lambda}\,(Tx_{n-1}, Tx_n)$$

$$d(Tx_n, Tx_{n+1}) \leq (d(Tx_{n-1}, Tx_n))^{\frac{\lambda}{1-\lambda}}$$

by same argument-

$$d(Tx_n, Tx_{n+1}) \leq (d(Tx_0, Tx_1))^{\left(\frac{\lambda}{1-\lambda}\right)^n} \tag{3.1}$$

Now by (3.1), for every m, n $\in$ N such that m > n , we have

$$d(Tx_m, Tx_n) = d(Tx_m, Tx_{m-1})\, d(Tx_{m-1}, Tx_{m-2}) \ldots d(Tx_{n+1}, Tx_n)$$

$$\leq (d(Tx_0, Tx_1))^{\left(\frac{\lambda}{1-\lambda}\right)^{m-1}}.(d(Tx_0, Tx_1))^{\left(\frac{\lambda}{1-\lambda}\right)^{m-2}} \ldots (d(Tx_0, Tx_1))^{\left(\frac{\lambda}{1-\lambda}\right)^{n}}$$

$$\leq (d(Tx_0, Tx_1))^{\left(\frac{\lambda}{1-\lambda}\right)^{m-1}+\left(\frac{\lambda}{1-\lambda}\right)^{m-2}+\cdots+\left(\frac{\lambda}{1-\lambda}\right)^{n}}.$$

$$\leq (d(Tx_0, Tx_1))^{\left(\frac{\lambda}{1-\lambda}\right)^{n}+\left(\frac{\lambda}{1-\lambda}\right)^{n+1}+\cdots}$$

$$= (d(Tx_0, Tx_1))^{\left(\frac{\lambda}{1-\lambda}\right)^{n}\left(\frac{1-\lambda}{1-2\lambda}\right)} \tag{3.2}$$

Letting m, n $\to \infty$ in (3.2), we have $d(Tx_m, Tx_n) \to 1$.

So $\{Tx_n\}$ is a Cauchy sequence and since X is a complete multiplicative metric space , there exist $v \in X$ such that

$$\lim_{n\to\infty} Tx_n = v \tag{3.3}$$

Since T is subsequentially convergent so $\{x_n\}$ has a convergent subsequence.

So there exist $u \in X$ and $\{x_{n(k)}\}^{\infty}_{k=1}$ such that $\lim_{k\to\infty} x_{n(k)} = u$.

using continuity of T and $\lim_{k\to\infty} Tx_{n(k)} = Tu$

by (3.3) we conclude that $Tu = v$

so, $d(Tu, TSu) \leq d\big(TSu, Tx_{n(k)}\big).d\big(Tx_{n(k)}, Tx_{n(k)+1}\big).\, d\big(Tx_{n(k)+1}, Tu\big).$

$$\leq d\big(TSu, TSx_{n(k)-1}\big).d\big(Tx_{n(k)}, Tx_{n(k)+1}\big).\, d\big(Tx_{n(k)+1}, Tu\big).$$

$\leq \{d(Tu,TSu).d(Tx_{n(k)-1},TSx_{n(k)-1})\}^{\lambda}.(\mathrm{d}(Tx_1,Tx_0))^{\left(\frac{\lambda}{1-\lambda}\right)^{n(k)}}.d(Tx_{n(k)+1},Tu)$.

Implies, $((d(Tu,TSu))^{1-\lambda} \leq \{d(Tx_{n(k)-1},Tx_{n(k)})\}^{\lambda}.(\mathrm{d}(Tx_1,Tx_0))^{\left(\frac{\lambda}{1-\lambda}\right)^{n(k)}}.d(Tx_{n(k)+1},Tu)$.

$d(Tu,TSu) \leq \{d(Tx_{n(k)-1},Tx_{n(k)})\}^{\left(\frac{\lambda}{1-\lambda}\right)}.(\mathrm{d}(Tx_1,Tx_0))^{\left(\frac{\lambda}{1-\lambda}\right)^{n(k)}.\left(\frac{1}{1-\lambda}\right)}.d(Tx_{n(k)+1},Tu)^{\left(\frac{1}{1-\lambda}\right)}$

Letting $k \to \infty$, $d(Tu,TSu) \to 1$, since T is one to one, we get $Su = u$.

So, S has a fixed point. Since (3.1) holds and T is one to one, S has a unique fixed point.

Now if T is sequentially convergent, then by replacing n with $\{n(k)\}$, we conclude that

$\lim_{n\to\infty} x_n = u$ and this shows that $\{x_n\}$ converges to the fixed point of S.

Remark 3.2. By taking $Tx = x$ in theorem (3.1), we can conclude the Kannan's theorem for multiplicative space.

Theorem 3.3. Let (X,d) be a multiplicative metric space and T, S : $X \to X$ be a mapping such that T is continuous, one to one and subsequentially convergent. If $\lambda \in [0,\frac{1}{2})$ and $d(TSx,TSy) \leq [d(Tx,TSy).d(Ty,TSx)]^{\lambda}$ for all $x,y \in X$ then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Theorem 3.4. Let (X,d) be a complete multiplicative metric space and T,S : $X \to X$ be a mapping such that T is continuous one to one and subsequentially convergent and

$d(TSx,TSy) \leq d(Tx,TSx)^p.d(Ty,TSy)^q.d(Tx,Ty)^r$,

where p, q, $r \in [0,\frac{1}{2})$ with $p + q + r < 1$.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X such that $x_n = Sx_{n-1} = S^n x_0$

Now,

$\mathrm{d}(Tx_{n+1},Tx_n) = \mathrm{d}(TSx_n,TSx_{n-1})$

$\leq \mathrm{d}(Tx_n,TSx_n)^p.\ \mathrm{d}(Tx_{n-1},TSx_{n-1})^q.\ \mathrm{d}(Tx_n,Tx_{n-1})^r$

$\mathrm{d}(Tx_n,Tx_{n+1})^{1-p} \leq \mathrm{d}(Tx_{n-1},Tx_n)^{q+r}$

$\mathrm{d}(Tx_n,Tx_{n+1}) \leq \mathrm{d}(Tx_{n-1},Tx_n)^{\frac{q+r}{1-p}}$, where $\lambda = \frac{q+r}{1-p} \in [(0,1)]$

Thus $\mathrm{d}(Tx_n,Tx_{n+1}) \leq \mathrm{d}(x_n,x_{n-1})^{\lambda} \leq \ldots \leq \mathrm{d}(Tx_1,Tx_0)^{\lambda^n}$

Now for every m, n $\in N$ such that m $>$ n,

we have,

$$
\begin{aligned}
d(Tx_m, Tx_n) &= d(Tx_m, Tx_{m-1}).d(Tx_{m-1}, Tx_{m-2})\ldots d(Tx_{n+1}, Tx_n) \\
&\le d(Tx_1, Tx_0)^{\lambda^{m-1}}.\ d(Tx_1, Tx_0)^{\lambda^{m-2}}\ldots\ d(Tx_1, Tx_0)^{\lambda^{n}} \\
&\le (d(Tx_0, Tx_1))^{(\lambda)^{m-1}+(\lambda)^{m-2}+\cdots+(\lambda)^{n}} \\
&\le (d(Tx_0, Tx_1))^{(\lambda)^{n}+(\lambda)^{n+1}+\cdots} \\
&= (d(Tx_0, Tx_1))^{\lambda^{n}[1+\lambda+\lambda^2+\cdots]} \\
&= (d(Tx_0, Tx_1))^{\frac{\lambda^n}{1-\lambda}}
\end{aligned}
$$

Taking m, n $\to \infty$, we have $d(Tx_m, Tx_n) \to 1$. So $\{Tx_n\}$ is a Cauchy sequence and since X is a complete metric space, there exists $v \in X$ such that

$$\lim_{n\to\infty} Tx_n = v$$

Since T is subsequentially convergent, so $\{x_n\}$ has a convergent subsequence. So there exist , $u \in X$ and $\{x_{n(k)}\}$ such that $\lim_{k\to\infty} Tx_{n(k)} = Tu$

By (3.3) we conclude that $Tu = v$

So,

$$
\begin{aligned}
d(Tu, TSu) &\le d(TSu, Tx_{n(k)}).d(Tx_{n(k)}, Tx_{n(k)+1}).\,d(Tx_{n(k)+1}, Tu). \\
&\le d(TSu, TSx_{n(k)-1}).d(Tx_{n(k)}, Tx_{n(k)+1}).\,d(Tx_{n(k)+1}, Tu). \\
&\le d(Tu, TSu)^p.d(Tx_{n(k)-1}, TSx_{n(k)-1})^q d(Tu, TSx_{n(k)-1})^r.\,(d(Tx_0, Tx_1))^{\lambda^{n(k)}}
\end{aligned}
$$

So,

$$d(Tu, TSu)^{1-p} \le d(Tx_{n(k)-1}, Tx_{n(k)})^q.\,d(Tu, Tx_{n(k)})^r.\,(d(Tx_0, Tx_1))^{\lambda^{n(k)}}.\,d(Tx_{n(k)+1}, Tu).$$

Letting k $\to \infty$, $d(Tu, TSu) \to 1$, since T is one to one we get Su = u.

So, S has a fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

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