Available online at http://scik.org

J. Math. Comput. Sci. 10 (2020), No. 5, 1360-1374

https://doi.org/10.28919/jmcs/4622

ISSN: 1927-5307

$L(1,1,1)$-LABELING OF PATH, BOUQUET OF CYCLES AND SUN GRAPH

NASREEN KHAN*

Department of Mathematics, Santal Bidroha Sardha Satabarshiki Mahavidyalaya, Goaltore, Paschim

Medinipur-721128, West Bengal, India

Abstract: For a given graph $G(V,E)$, $L(1,1,1)$-labeling problem is an assignment from vertex set V to the set of non negative integers. If Z^+ be the non negative integers then $L(1,1,1)$-labeling is a function $f: V \rightarrow Z^+$ such that for any two vertices x and y, $|f(x) - f(y)| \geq 1$, when $d(x,y) = 1$; $|f(x) - f(y)| \geq 1$, when $d(x,y) = 2$; and $|f(x)f(y)| \geq 1$, when $d(x,y) = 3$. The $L(1,1,1)$-chromatic number $\lambda_{1,1,1}$ is the smallest positive integer such that G has an $L(1,1,1)$-labeling with $\lambda_{1,1,1}$ as the maximum label. In this paper we determine the $L(1,1,1)$-chromatic number for a path, a cycle, bouquet of cycles joining at a vertex (all are of finite lengths) and sun graph. We also present a lower and upper bounds for $\lambda_{1,1,1}$ in terms of the maximum degree of G.

Keywords: distance labeling; radio labeling; graph colouring; λ-labeling; $L(h,k)$-labeling; $L(d,1,1)$-labeling; $L(d,2,1)$-labeling.

2010 AMS Subject Classification: 05C12.

1. INTRODUCTION

In 1980 Hale introduced channel assignment problem, which is nothing but an assignment to

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*Corresponding author

E-mail address: nasreen.khan10@gmail.com

Received April 14, 2020

assign a channel (non-negative integer) to each radio transmitter (radio, T.V, cell phone, etc.) so that interfering transmitters are assigned channels whose separations is not in a set of disallowed separations. A variation of the channel assignment problem in which “close” transmitters must receive different channels and “very close” transmitters must receive channels that are at least two channels apart. This problem can be modelled as a graph labelling/colouring problem where the vertices represent the transmitters, two vertices are “very close” if they are adjacent and “close” if they are of distance two in the graph. This type of channel assignment is known in the literature as an $L(2,1)$-labeling. Frequency assignment problem has been widely studied in the past [6, 7, 9, 12, 19, 21, 20].

$L(3,2,1)$-labeling naturally extends from $L(2,1)$-labeling by taking into consideration vertices which are within a distance of three apart, but it remains less difficult than radio-labeling. An $L(3,2,1)$-labelling of a graph G is a function f from its vertex set V to the set of non-negative integers such that $|f(x) - f(y)| \geq 3$ if $d(x,y) = 1$, $|f(x) - f(y)| \geq 2$ if $d(x,y) = 2$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 3$. The $L(3,2,1)$-labelling number, $k_3(G)$, of G is the smallest natural number k_3 such that G has an $L(3,2,1)$-labelling with k_3 as the maximum label. An $L(3,2,1)$-labeling of a graph G is called a minimal $L(3,2,1)$-labeling of G if, under the labeling, the highest label of any vertex is $k_3(G)$.

In 2007, Bertossi et al. have studied approximate $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring of trees and interval graphs [2] and Bodlaender et al. have studied about approximations for λ-colorings of graphs [3]. Also in [16], Khan et al. have studied $L(2,1)$-total labeling of cactus graphs and in [17] they studied $L(0,1)$-labeling of cactus graphs. Later Calamoneri [4] studied $L(\delta_1, \delta_2, 1)$-labeling of eight grids, Amanathulla and Pal studied $L(3,2,1)$-labeling and $L(4,3,2,1)$-labeling of circular-arc graphs [1]. The upper bound of $\lambda_{p,1}(G)$ of any graph G is $\Delta^2 - (p-1)\Delta - 2$ [5], where Δ is the degree of the graph. In [10], Clipperton et al. showed that $\lambda_{3,2,1}(G) \leq \Delta^3 + \Delta^2 + 3\Delta$ for any graph. Later Chai et al. [8] improved this upper bound and showed that $\lambda_{3,2,1}(G) \leq \Delta^3 + 2\Delta$ for any graph. In [18], Liu and Shao studied the $L(3,2,1)$-labeling of planer graph and showed that $\lambda_{3,2,1}(G) \leq 15(\Delta^2 - \Delta + 1)$. In [8], Chia et al. also showed that $\lambda_{3,2,1}(G) = 2n + 5$

if T is a complete n-ary tree of height $h \geq 3$ and for any tree $2\Delta + 1 \leq \lambda_{3,2,1}(G) \leq 2\Delta + 3$. In [11], Jean studied about $L(d, 2,1)$-labeling of simple graph and showed that $\lambda_{d,2,1}(K_n) = d(n-1) + 1$, where K_n is complete graph with n vertices and also shown that $\lambda_{d,2,1}(K_{m,n}) = d + 2(m + n) - 3$. Kim et al. [15] show that $\lambda_{3,2,1}(K_3 \square C_n) = 15$ when $n \geq 28$ and $n \equiv 0 \pmod 5$, where $K_3 \square C_n$ is the Cartesian product of complete graphs K_3 and the cycle C_n.

An $L(1,1,1)$-labeling is a simplified model for the channel assignment problem. $L(1,1,1)$-labeling is a function $f: V \to Z^+$ such that for any two vertices x and y, $|f(x) - f(y)| \geq 1$, when $d(x, y) = 1$; $|f(x) - f(y)| \geq 1$, when $d(x, y) = 2$; and $|f(x)f(y)| \geq 1$, when $d(x, y) = 3$. The $L(1,1,1)$-chromatic number $\lambda_{1,1,1}$ is the smallest positive integer such that G has an $L(1,1,1)$-labeling with $\lambda_{1,1,1}$ as the maximum label.

In this paper we only focus on $L(1,1,1)$-labeling of paths, a cycle, bouquet of cycles (joining at a common cut vertex) and sun graph. And also we find out the lower and upper bounds of $L(1,1,1)$-labeling number $\lambda_{1,1,1}(G)$ such that G has an $L(1,1,1)$-labeling with λ as the maximum label.

2. L(1,1,1)-LABELING OF PATH

Lemma 1 Let P_n be a path of length $n \geq 2$. Then $\lambda_{1,1,1}$ lies between $\Delta + 1$ and $\Delta + 2$, where Δ is the degree of path.

Proof. Let us consider v_i; $i = 0,1,2, \ldots, n-1$ be the vertices of P_n. Here we label the vertices of P_2 as $f(v_0) = 1, f(v_1) = 2$; P_3 as $f(v_0) = 1, f(v_1) = 2, f(v_2) = 3$ respectively. For $n \geq 4$ the labeling procedure of the vertices of P_n is as follows

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 4; \\ 2, & \text{if } i \equiv 1 \pmod 4; \\ 3, & \text{if } i \equiv 2 \pmod 4; \\ 4, & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

Here $\lambda_{1,1,1}(P_2) = 2 = \Delta + 1$, $\lambda_{1,1,1}(P_3) = 3 = \Delta + 1$ and $\lambda_{1,1,1}(P_n) = 4 = \Delta + 2$ (for $n \geq 4$).

We have proved that for P_n, $\Delta + 1 \leq \lambda_{1,1,1}(P_n) \leq \Delta + 2$. ∎

3. L(1,1,1)-LABELING OF A CYCLE

Lemma 2 If C_n be a cycle of finite length n, then $\lambda_{1,1,1}$ lies between $\Delta + 1$ and $\Delta + 5$, where $\Delta = 2$.

Proof. Let us consider C_n be a cycle of finite length n and $v_0, v_1, \dots, v_{n-1}$ be the vertices. Here degree of the cycle is 2, that is $\Delta = 2$. Now we classify C_n into five groups, viz., C_3, C_{4k}, C_{4k+1}, C_{4k+2} and C_{4k+3} respectively. Then the labeling procedures of $L(1,1,1)$ are as follows.

Case 1. For n=3.

We label the vertices of C_3 as $\{1,2,3\}$. Here $\lambda_{1,1,1}(C_3) = 3 = \Delta + 1$.

Case 2. For $n = 4k \equiv 0 \pmod 4$.

The labeling technique for the case is

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod 4; \\ 2, & \text{if } i \equiv 1 \pmod 4; \\ 3, & \text{if } i \equiv 2 \pmod 4; \\ 4, & \text{if } i \equiv 3 \pmod 4). \end{cases}$$

Here we have found $\lambda_{1,1,1}(C_{4k}) = 4 = \Delta + 2$.

Case 3. For $n = 4k + 1 \equiv 1 \pmod 4$.

First we label the vertices $v_0, v_1, v_2, \dots, v_{4k-1}$ using the procedure as developed in case 2. Then we label the remaining vertex v_{4k} as

$$f(v_{S4k}) = 5.$$

Here we can see that the maximum non negative integer is 5, so, $\lambda_{1,1,1}(C_{4k+1}) = 5 = \Delta + 3$.

Case 4. For $n = 4k + 2 \equiv 2 \pmod 4$.

The labeling technique of first $n - 2 = 4k$ vertices is same as given in case 2 of this lemma. For remaining two vertices V_{4k} and V_{4k+1}, we label them as

$$f(v_{4k}) = 5 \text{ and } f(v_{4k+1}) = 6.$$

Here $\lambda_{1,1,1}(C_{4k+2}) = 6 = \Delta + 4$.

Case 5. For $n = 4k + 3 \equiv 3 \pmod 4$.

Using the same procedure as developed in case 2, we can label first $n - 3 = 4k$ vertices. And we label the last three vertices as

$$f(v_{4k}) = 5, \quad f(v_{4k+1}) = 6 \text{ and } f(v_{4k+2}) = 7.$$

Here, $\lambda_{1,1,1}(C_{4k+3}) = 7 = \Delta + 5$.

Thus, from all above cases we see that $L(1,1,1)$-chromatic number $\lambda_{1,1,1}$ lies between $\Delta + 1$ and $\Delta + 5$, that is, $\Delta + 1 \leq \lambda_{1,1,1}(C_n) \leq \Delta + 5$, where $\Delta = 2$.

Hence the proof. ■

4. L(1,1,1)-LABELING OF TWO CYCLES (JOINED AT A COMMON CUTVERTEX)

Lemma 3. Let G be a graph consists of two cycles C_n and C_m of finite lengths n and m respectively, joined at a common cutvertex. Then $\lambda_{1,1,1}$ lies between $\Delta + 1$ and $\Delta + 3$, where $\Delta = 4$ is the degree of the common cutvertex.

Proof. Let us consider $G = C_n \cup_{v_0} C_m$, and v_i; $i = 0,1, \dots, n-1$ and v_0, v'_j; $j = 1, \dots, m-1$ be the vertices of C_n and C_m respectively. Here v_0 is the common cutvertex. We will prove this lemma by using six cases. Now, we discuss all the cases as follows.

Case 1. **For $n = 3$ and $m = 3$.**

By using the technique as given in case 1 of lemma 2, we label the vertices of first C_3. After that we label the remaining vertices of the second cycles as

$f(v'_1) = 4$ and $f(v'_2) = 5$.

Here $\lambda_{1,1,1}(G) = 5 = \Delta + 1$.

Case 2. **For $n = 4k + i$ and $m = 3$, for $i = 0, 1, 2, 3$.**

Case 2.1. For $n = 4k \equiv 0 \pmod 4$ and $m = 3$.

Here the labelling procedure of C_n is same as developed in case 2 of lemma 2. Now we label the remaining vertices of C_3 as

$f(v'_1) = 5$ and $f(v'_2) = 6 = \Delta + 2$.

Case 2.2. For $n = 4k + 1 \equiv 1 \pmod 4$ and $m = 3$.

Using the labeling technique given in case 3 of lemma 2, we first label the vertices of C_{4k+1}. After that we label the second vertices as

$f(v'_1) = 6$ and $f(v'_2) = 7 = \Delta + 3$.

Case 2.3. For $n = 4k + 2 \equiv 2 \pmod 4$ and $m = 3$.

The labeling procedure of C_{4k+2} is same as given in case 4 of lemma 2. Now for the vertices v'_1, v'_2, we label them by

$f(v'_1) = 4$ and $f(v'_2) = 7$ respectively.

Here $\lambda_{1,1,1}(G) = 7 = \Delta + 3$.

Case 2.4. For $n = 4k + 3 \equiv 3$ (mod 4) and $m = 3$.

First we label the vertices of C_{4k+3} by the rule, developed in case 5 of lemma 2. And for the vertices v'_1, v'_2, we label them by

$f(v'_1) = 4$ and $f(v'_2) = 5$ respectively.

We get, $\lambda_{1,1,1}(G) = 7 = \Delta + 3$.

Case 3. For $n = 4k \equiv 0$ (mod 4) and $m = 4k + i$, for $i = 0, 1, 2, 3$.

First we label the vertices of C_n by the rule as given in case 2 of lemma 2. After that the procedure to label the remaining vertices of C_m are given in the following subcases.

Case 3.1. For $m = 4k \equiv 0$ (mod 4).

$f(v'_1) = 5$, $f(v'_2) = 3$, $f(v'_3) = 4$ and for the vertices v'_j; $j = 4,5,6, \dots, m - 2 = 4k - 2$, the labeling procedure is as follows

$$f(v'_j) = \begin{cases} 1, & \text{if } j \equiv 0 \pmod 4; \\ 2, & \text{if } j \equiv 1 \pmod 4; \\ 3, & \text{if } j \equiv 2 \pmod 4; \\ 4, & \text{if } j \equiv 3 \pmod 4; \end{cases}$$

and $f(v'_{4k-1}) = 6 = \Delta + 2$.

Case 3.2. For $m = 4k + 1 \equiv 1$ (mod 4).

Using the procedure as developed in subcase 3.1, we label the first $4k - 1$ vertices of C_{4k+1}. Then we label last two vertices as

$f(v'_{4k-1}) = 6$ and $f(v'_{m-1}) = f(v'_{4k}) = 7$ respectively.

So, $\lambda_{1,1,1}(G) = 7 = \Delta + 3$.

Case 3.3. For $m = 4k + 2 \equiv 2$ (mod 4).

For the vertices v'_j; $j = 1,2, \dots, 4k - 1$, the labeling procedure is same same as developed in subcase 3.1. Now we label last two vertices as

$f(v'_{4k}) = 6$ and $f(v'_{m-1}) = f(v'_{4k+1}) = 7 = \Delta + 3$.

Case 3.4. $m = 4k + 3 \equiv 3$ (mod 4).

The labeling rule of first $4k$ vertices of C_{4k+3} is same as given in subcase 3.1. Then we label last three vertices as

$f(v'_{4k}) = 5$, $f(v'_{4k+1}) = 6$ and $f(v'_{4k+2}) = 7 = \Delta + 3$.

Case 4. For $n = 4k + 1 \equiv 1$ (mod 4) and $m = 4k + i$, for $i = 1, 2, 3$.

Here also we label the first cycle C_n by using case 3 in lemma 2. After that we label the vertices (except v_0) of C_m and the rule are discuss in the following subcases.

Case 4.1. For $m = 4k + 1 \equiv 1$ (mod 4).

$f(v'_1) = 6,\ f(v'_2) = 3,\ f(v'_3) = 4,$ for the vertices $v'_j;\ j = 4,5,6,\dots, m-2 = 4k-1$, the labelling procedure is as follows

$$f(v'_j) = \begin{cases} 1, & if\ \ j \equiv 0\ \ (mod\ \ 4); \\ 2, & if\ \ j \equiv 1\ \ (mod\ \ 4); \\ 3, & if\ \ j \equiv 2\ \ (mod\ \ 4); \\ 4, & if\ \ j \equiv 3\ \ (mod\ \ 4); \end{cases}$$

and $f(v'_{4k}) = 7 = \Delta + 3$.

Case 4.2. For $m = 4k + 2 \equiv 2$ (mod 4).

The labeling technique of the vertices $v'_j;\ j = 1,2,\dots,4k-2$ is same as given in subcase 4.1. And for the remaining vertices, we label them as

$f(v'_{4k-1}) = 5,\ f(v'_{4k}) = 4$ and $f(v'_{4k+1}) = 7 = \Delta + 3$.

Case 4.3. For $m = 4k + 3 \equiv 3$ (mod 4).

Using the procedure given in subcase 4.1, we label the vertices $v'_j;\ j = 1,2,\dots,4k-2$. For the remaining vertices, we label them as

$f(v'_{4k-1}) = 5,\ f(v'_{4k}) = 6,\ f(v'_{4k+1}) = 4$ and $f(v'_{4k+2}) = 7$ respectively.

Thus, $\lambda_{1,1,1}(C_{4k+1} \cup_{v_0} C_{4k+3}) = 7 = \Delta + 3$.

Case 5. For $\boldsymbol{n = 4k + 2 \equiv 2}$ (mod 4) and $\boldsymbol{m = 4k + i}$, for $\boldsymbol{i = 2, 3}$.

Using case 4 of lemma 2, we label the vertices of C_n. And the labeling of second cycle are discussed in the following subcases.

Case 5.1. For $m = 4k + 2 \equiv 2$ (mod 4).

$f(v'_1) = 4,\ f(v'_2) = 3,\ f(v'_3) = 5,$ for the vertices $v'_j;\ j = 4,5,6,\dots, m-3 = 4k-1$, the labeling procedure is as follows

$$f(v'_j) = \begin{cases} 1, & if\ \ \mathrm{j} \equiv 0\ \ (mod\ \ 4); \\ 2, & if\ \ j \equiv 1\ \ (mod\ \ 4); \\ 3, & if\ \ j \equiv 2\ \ (mod\ \ 4); \\ 4, & if\ \ j \equiv 3\ \ (mod\ \ 4); \end{cases}$$

and $f(v'_{4k}) = 5,\ f(v'_{4k+1}) = 7 = \Delta + 3$.

Case 5.2. For $m = 4k + 3 \equiv 3$ (mod 4).

The labeling rule of the vertices $v'_j;\ j = 1,2,\dots,4k-1$ is same as developed in subcase 5.1. For the remaining vertices the procedure is as

$f(v'_{4k}) = 6,\ f(v'_{4k+1}) = 5$ and $f(v'_{4k+2}) = 7$ respectively.

Thus, $\lambda_{1,1,1}(G) = 7 = \Delta + 3$.

Case 6. For $n = 4k + 3 \equiv 3 \pmod 4$ and $m = 4k + 3 \equiv 3 \pmod 4$.

Using case 5 of lemma 2, we label the vertices of C_n. For the vertices $v'_j;\ j = 1,2,\dots,4k-2$, the labeling procedure is same as given in subcase 3.1. And we label last three vertices as

$f(v'_{4k-1}) = 5,\ f(v'_{4k}) = 7,\ f(v'_{4k+1}) = 4$ and $f(v'_{4k+2}) = 6$.

Here, $\lambda_{1,1,1}(G) = 7$.

So, from all above cases we have found that $\Delta + 1 \le \lambda_{1,1,1}(G) \le \Delta + 3$.

Hence the proof. ■

5. L(1,1,1)-LABELING OF THREE CYCLES (JOINED AT A COMMON CUTVERTEX)

Lemma 4. Let G be a graph consists of three cycles C_n, C_m and C_l of finite lengths n, m and l, joined at a common cutvertex . Then $\Delta + 1 \le \lambda_{1,1,1}(G) \le \Delta + 3$, where $\Delta = 6$ is the degree of the common cutvertex.

Proof. Let us consider $v_i;\ i = 0,1,2,\dots,n-1;\ v_0,\ v'_j;\ j = 1,2,\dots,m-1;\ v_0,\ v'_p;\ p = 1,2,\dots,l-1$ be the vertices of C_n, C_m and C_l respectively. Here v_0 is the common cutvertex.

So, $G = C_n \cup_{v_0} C_m \cup_{v_0} C_l$.

Case 1. For $n = 3,\ m = 3$ and $l = 3$.

Using the procedure as given in case 1 of lemma 3, we label first two cycles. Next we label remaining vertices of the third cycle as

$f(v''_1) = 6$ and $f(v''_2) = 7$.

Here $f(v''_2) = 7 = \Delta + 1$.

Case 2. For $n = 4k + i;\ i = 0,1,2,3,\ m = 3$ and $l = 3$.

The labeling technique for first two cycles has been developed in case 2 of lemma 3. Now labeling procedure of the remaining vertices of third cycle for different values of i are given in the following subcases.

Case 2.1. For $n = 4k \equiv 0 \pmod 4$, $m = 3$ and $l = 3$.

$f(v''_1) = 7$ and $f(v''_2) = 8 = \Delta + 2$.

Case 2.2. For $n = 4k + i;\ i = 1,2,3,\ m = 3$ and $l = 3$.

$f(v''_1) = 8$ and $f(v''_2) = 9 = \Delta + 3$.

Case 3. **For $n = 4k \equiv 0 \pmod 4$, $m = 4k + j$; $j = 0, 1, 2, 3$ and $l = 3$.**

Here first we label first two cycles using the procedure given in case 3 of lemma 3. Next we label the remaining vertices of third cycle for different values of j as given in the following subcases.

Case 3.1. For $n = 4k \equiv 0 \pmod 4$, $m = 4k \equiv 0 \pmod 4$ and $l = 3$.

The labeling procedure of v''_1 and v''_2 is same as developed in subcase 2.1.

Case 3.2. For $n = 4k \equiv 0 \pmod 4$, $m = 4k + j$; $j = 1,2,3$ and $l = 3$.

We can label the vertices v''_1 and v''_2 by the technique given in subcase 2.2.

Case 4. For $n = 4k + i$; $i = 1,2,3$ $m = 4k + j$; $j = 1,2,3$ and $l = 3$.

Here also we label the vertices v''_1 and v''_2, using the technique given in subcase 2.2.

Case 5. **For $n = 4k \equiv 0 \pmod 4$, $m = 4k \equiv 0 \pmod 4$ and $l = 4k + p$; $p = 0, 1, 2, 3$.**

Using the procedure of labeling as developed in case 3.1 of lemma 3, we label C_n and C_m first. After that we label C_l and the procedure is discussed in the following subcases.

Case 5.1. For $l = 4k \equiv 0 \pmod 4$.

$f(v''_1) = 7$; $f(v''_2) = 3$; $f(v''_3) = 4$;

for the vertices $p = 4,5, \ldots, l-2 = 4k-2$,

$$f(v''_p) = \begin{cases} 1, & if\ p \equiv 0 \ (mod\ 4); \\ 2, & if\ p \equiv 1 \ (mod\ 4); \\ 3, & if\ p \equiv 2 \ (mod\ 4); \\ 4, & if\ p \equiv 3 \ (mod\ 4); \end{cases}$$

and $f(v''_{l-1}) = f(v''_{4k-1}) = 8 = \Delta + 2$.

Case 5.2. For $l = 4k + 1 \equiv 1 \pmod 4$.

The labeling procedure of v''_p; $p = 1,2, \ldots, l-4 = 4k-3$ is same as given in the above subcase except last three vertices. We label them as

$f(v''_{l-3}) = f(v''_{4k-2}) = 5$,

$f(v''_{l-2}) = f(v''_{4k-1}) = 3$,

$$f(v''_{l-1}) = f(v''_{4k}) = 8 = \Delta + 2$$

But when $l = 5$, then

$f(v''_1) = 7$, $f(v''_2) = 3$, $f(v''_3) = 8$ and $f(v''_4) = 9 = \Delta + 3$.

Case 5.3. For $l = 4k + 2 \equiv 2 \pmod 4$.

For the vertices v''_p; $p = 1,2, \ldots, l-3 = 4k-1$, we label them by using the same technique as given in case 5.1. And for the remaining vertices, we label them as

$f(v''_{l-2}) = f(v''_{4k}) = 8,$
$f(v''_{l-1}) = f(v''_{4k+1}) = 9 = \Delta + 3.$

Case 5.4. For $l = 4k + 3 \equiv 3 \pmod 4$.

The labeling rule for first $l - 3 = 4k + 3 - 3 = 4k$ vertices is same as given in subcase 5.1. For the last three vertices the rule is as follows

$f(v''_{4k}) = 5,\ f(v''_{4k+1}) = 8$ and $f(v''_{4k+2}) = 9 = \Delta + 3.$

Case 6. For $\boldsymbol{n = 4k \equiv 0}$ (mod 4), $\boldsymbol{m = 4k + 1 \equiv 1}$ (mod 4) and $\boldsymbol{l = 4k + p;\ p = 1, 2, 3}$.

The labeling rule for the vertices C_n and C_m is same as given in subcase 3.2 of lemma 3. After that we label C_l and the procedure is given in the following subcases.

Case 6.1. For $l = 4k + 1 \equiv 1 \pmod 4$.

$f(v''_1) = 8;\ f(v''_3) = 3;\ f(v''_3) = 4;$

for the vertices $p = 4,5, \dots, l - 3 = 4k - 2,$

$$f(v''_p) = \begin{cases} 1, & if\ \ p \equiv 0\ \ (mod\ \ 4); \\ 2, & if\ \ p \equiv 1\ \ (mod\ \ 4); \\ 3, & if\ \ p \equiv 2\ \ (mod\ \ 4); \\ 4, & if\ \ p \equiv 3\ \ (mod\ \ 4); \end{cases}$$

and $f(v''_{l-1}) = 6\ \ f(v''_{4k-1}) = 9 = \Delta + 3.$

Case 6.2. For $l = 4k + 2 \equiv 2 \pmod 4$.

The labeling of the vertices $v''_p;\ p = 1,2, \dots, l - 3 = 4k - 1$ is same as given in subcase 6.1. And for the remaining vertices, we label the vertices as follows

$f(v''_{4k}) = 6,\ f(v''_{4k+1}) = 9 = \Delta + 3.$

Case 6.3. For $l = 4k + 3 \equiv 3 \pmod 4$.

We label the vertices $v''_p;\ p = 1,2, \dots, l - 4 = 4k - 1$ as the same procedure as given in subcase 6.1. Then we label last three vertices as

$f(v''_{l-3}) = f(v''_{4k}) = 5,$
$f(v''_{l-2}) = f(v''_{4k+1}) = 6,$
$f(v''_{l-1}) = f(v''_{4k+2}) = 9 = \Delta + 3.$

Case 7. For $\boldsymbol{n = 4k \equiv 0}$ (mod 4), $\boldsymbol{m = 4k + 2 \equiv 2}$ (mod 4) and $\boldsymbol{l = 4k + p;\ p = 2, 3}$.

The rule of vertex labeling of C_n and C_m is same as mentioned in subcase 3.3 of lemma 3. Now the labeling technique for labeling the vertices of C_l of different cases are discussed in the following subcases.

Case 7.1. For $l = 4k + 2 \equiv 2 \pmod 4$.

The labeling procedure of $C_l = C_{4k+2}$ is same as given in subcase 6.2 of this lemma.

Case 7.2. For $l = 4k + 3 \equiv 3 \pmod 4$.

Here also the procedure of vertex labeling of C_l is same as given subcase 6.3.

Case 8. For $n = 4k \equiv 0 \pmod 4$, $m = 4k + 3 \equiv 3 \pmod 4$ and $l = 4k + 3 \equiv 3 \pmod 4$.

We label the vertices of C_l using the rule as developed in subcase 6.3 of this lemma.

Case 9. For $n = 4k + 1 \equiv 1 \pmod 4$, $m = 4k + 1 \equiv 1 \pmod 4$ and $l = 4k + p$; $p = 1, 2, 3$.

We label the vertices of first two cycles using the technique as given in subcase 4.1 of lemma 3. For the third cycle C_l, the rules are discussed below.

Case 9.1. For $l = 4k + 1 \equiv 1 \pmod 4$.

Using the technique of vertex labeling as given in subcase 6.1 of this lemma, we label first $4k$ vertices. After that we label last vertex as

$$f(v''_{4k}) = 9 = \Delta + 3.$$

Case 9.2. For $l = 4k + 2 \equiv 2 \pmod 4$.

For the vertices v''_p; $p = 1, 2, \dots, l - 4 = 4k - 2$, we label them by using the rule as given in subcase 6.1. And for the remaining vertices, we label them as

$$f(v''_{4k-1}) = 5,\ f(v''_{4k}) = 4 \text{ and } f(v''_{4k+1}) = 9 = \Delta + 3.$$

Case 9.3. For $l = 4k + 3 \equiv 3 \pmod 4$.

For the vertices v''_p; $p = 1, 2, \dots, l - 4 = 4k - 1$, we label them by using the rule as given in subcase 6.1. And for the remaining vertices, we label them as

$$f(v''_{4k}) = 5,\ f(v''_{4k+1}) = 3 \text{ and } f(v''_{4k+2}) = 9 = \Delta + 3.$$

Case 10. For $n = 4k + 1 \equiv 1 \pmod 4$, $m = 4k + 2 \equiv 2 \pmod 4$ and $l = 4k + p$; $p = 2, 3$.

We label the vertices of C_n and C_m using the process as in subcase 4.2 of lemma 3. For the third cycle C_l, the labeling procedures are discussed below.

Case 10.1. For $l = 4k + 2 \equiv 2 \pmod 4$.

The labeling procedure of C_l is same as mentioned in subcase 9.2.

Case 10.2. For $l = 4k + 3 \equiv 3 \pmod 4$.

The labeling procedure of C_l is same as in subcase 9.3.

Case 11. For $n = 4k + 1 \equiv 1 \pmod 4$, $m = 4k + 3 \equiv 3 \pmod 4$ and $l = 4k + 3 \equiv 3 \pmod 4$.

Here we first label the vertices of C_n and C_m using the process as given in subcase 4.3

of lemma 3. Then using the rule as given in subcase 9.3, we label the vertices of C_l.

Case 12. **For $n = 4k + 2 \equiv 1 \pmod 4$, $m = 4k + 2 \equiv 2 \pmod 4$ and $l = 4k + p$; $p = 2, 3$.**

The labeling procedure of C_n and C_n is same as given in subcase 5.1 of lemma 3.

Case 12.1. For $l = 4k + 2 \equiv 2 \pmod 4$.

$f(v''_1) = 8$; $f(v''_3) = 3$; $f(v''_3) = 4$;

for the vertices $p = 4,5, \dots, l - 2 = 4k$,

$$f(v''_p) = \begin{cases} 1, & if\ p \equiv 0\ (mod\ 4); \\ 2, & if\ p \equiv 1\ (mod\ 4); \\ 3, & if\ p \equiv 2\ (mod\ 4); \\ 4, & if\ p \equiv 3\ (mod\ 4); \end{cases}$$

and $f(v''_{l-1}) = 6$ $f(v''_{4k+1}) = 9 = \Delta + 3$.

Case 12.2. For $l = 4k + 3 \equiv 3 \pmod 4$.

The procedure given in subcase 6.3, we label the vertices of C_l.

Case 13. **For $n = 4k + 2 \equiv 2 \pmod 4$, $m = 4k + 3 \equiv 3 \pmod 4$ and $l = 4k + 3 \equiv 3 \pmod 4$.**

The labeling procedure of C_n and C_n is same as given in subcase 5.2 of lemma 3. Now we label C_l by using the technique as given in subcase 6.3.

Case 14. **For $n = 4k + 3 \equiv 3 \pmod 4$, $m = 4k + 3 \equiv 3 \pmod 4$ and $l = 4k + 3 \equiv 3 \pmod 4$.**

We label the vertices of C_n and C_m using the rule as given in subcase 5.3. The we label C_l using the rule as given in subcase 6.3.

So, from all the cases we see that the values of $\lambda_{1,1,1}$ lies between $\Delta + 1$ and $\Delta + 3$.

That is $\Delta + 1 \leq \lambda_{1,1,1}(G) \leq \Delta + 3$, $\Delta = 6$ is the degree of G.

Hence the proof. ■

6. L(1,1,1)-LABELING OF BOUQUET OF CYCLES (JOINED AT A COMMON CUTVERTEX)

Definition 1. If $G = C_n \cup_{v_0} C_m \cup_{v_0} C_l \cup_{v_0} \dots$, where $C_n, C_m, \dots$, are cycles of finite lengths $n, m, \dots$ respectively and they joined at a common cutvertex v_0, then the graph G is said to be bouquet of cycles.

By using the results of lemma 2, lemma 3 and lemma 4 we can conclude the result, that is, the range of chromatic number for graph which consists of a bouquet of finite length cycles (the cycles are joined at a common cutvertex). And the result is given below.

Lemma 5. If a graph G contains finite number of cycles of finite lengths (joined at a common cutvertex), then the range of $L(1,1,1)$-chromatic number $\lambda_{1,1,1}(G)$ is $\Delta+1$ and $\Delta+3$.

7. L(1,1,1)-LABELING OF SUN GRAPH

Definition 2. Let v_i; $i=0,1,\dots,n-1$ be the vertices of cycle C_n. Then sun graph S_{2n} is a graph which is obtained by adding an edge (v_i,v'_i); $i=0,1,\dots,n-1$ to every vertex of C_n.

Lemma 6. If S_{2n} be a sun graph, then the value of $L(1,1,1)$-chromatic number $\lambda_{1,1,1}$ lies between $\Delta+3$ and $\Delta+4$, where $\Delta=3$ is the degree of the sun graph.

Proof. Let $v_0,v_1,\dots,v_{n-1}$ be the vertices C_n. And the edges of S_{2n} are (v_i,v'_i) for $i=0,1,\dots,n-1$. We can label the vertices of C_n by using the technique as given in different cases of lemma 2. Now we label the pendent vertices of the sun graph and that are discussed in the following subcases.

Case 1. **For n=3**.

$f(v'_0)=4$, $f(v'_1)=5$ and $f(v'_2)=6$.

Here $\lambda_{1,1,1}=6=\Delta+3$.

Case 2. **For $n=4k\equiv 0 \pmod 4$**.

$$f(v'_i)=\begin{cases}5, & \text{if } i\equiv 0 \pmod 2;\\ 6, & \text{if } i\equiv 1 \pmod 2).\end{cases}$$

Here $\lambda_{1,1,1}=6=\Delta+3$.

Case 3. **For $n=4k+1\equiv 1 \pmod 4$**.

$f(v'_0)=4$, $f(v'_1)=6$;

for $j=2,3,\dots,n-4=4k-3$

$$f(v'_i)=\begin{cases}5, & \text{if } i\equiv 0 \pmod 2;\\ 6, & \text{if } i\equiv 1 \pmod 2;\end{cases}$$

and $f(v'_{4k-2})=7$, $f(v'_{4k-1})=6$, $f(v'_{4k})=7$.

So, $\lambda_{1,1,1}=7=\Delta+4$.

Case 4. **For $n=4k+2\equiv 2 \pmod 4$**.

$f(v'_0)=4$, $f(v'_1)=7$;

for $j=2,3,\dots,n-5=4k-3$

$$f(v'_i) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 2; \\ 6, & \text{if } i \equiv 1 \pmod 2; \end{cases}$$

and $f(v'_{4k-2}) = 7,\ f(v'_{4k-1}) = 1,\ f(v'_{4k}) = 2,\ f(v'_{4k+1}) = 3.$
So, $\lambda_{1,1,1} = 7 = \Delta + 4.$

Case 5. For $\mathbf{n = 4k + 3 \equiv 3 \pmod 4}$.
For $j = 0,1,\ldots,n - 6 = 4k - 3$

$$f(v'_i) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 2; \\ 6, & \text{if } i \equiv 1 \pmod 2; \end{cases}$$

and $f(v'_{4k-2}) = 7,\ f(v'_{4k-1}) = 1,\ f(v'_{4k}) = 2,\ f(v'_{4k+1}) = 3,\ f(v'_{4k+2}) = 4.$

Thus, from all above cases we see that $L(1,1,1)$-chromatic number $\lambda_{1,1,1}$ lies between $\Delta + 3$ and $\Delta + 4$, that is, $\Delta + 3 \le \lambda_{1,1,1}(S_{2n}) \le \Delta + 4$, where $\Delta = 3$.

Hence the proof. ■

8. CONCLUSION

In this paper we provide very close lower and upper bounds of $L(1,1,1)$-labeling number (chromatic number) for path of finite length, a cycle of finite length, bouquet of cycles of finite lengths and sun graph. In near future, we shall develop an algorithm to label the vertices by using $L(1,1,1)$-labeling technique. Also we shall find the range of $L(1,1,1)$-chromatics number for a cactus graph.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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