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GENERATION OF PROTO-FUZZY CONCEPTS FROM FIXPOINTS OF FUZZY CLOSURE OPERATORS

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Abstract. It is well known in Formal Concept Analysis that computation of all formal concepts from data table with graded attributes can be reduced to the problem of computing fixpoints of two fuzzy closure operators, $\uparrow\downarrow$ and $\downarrow\uparrow$. It is also true that as the size of datasets grows, the fuzzy concepts generated from fuzzy context become larger in number. Therefore for large and complex datasets, it is very hard to deal with such a large number of fuzzy concepts. To handle large and complex datasets, several alternative approaches were proposed to the fuzzy concept lattice theory by researchers. The fuzzy concepts introduced by Kridlo et al (2008) are known as proto-fuzzy concepts. In point of view of applications in different domain, significance of proto-fuzzy concepts is very much effective. But as far as our knowledge is concerned, there is no general method to generate proto-fuzzy concepts. In this paper, we present an algorithm for finding proto-fuzzy concepts directly from the input data. The algorithm we present generates proto-fuzzy concepts form the fixpoints of the fuzzy closure operators, $\uparrow\downarrow$ and $\downarrow\uparrow$.

Keywords: fuzzy sets; *t*-cut of fuzzy sets; fixpoints; formal concept analysis; fuzzy concepts; proto-fuzzy concepts.

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1. INTRODUCTION

Formal Concept Analysis (FCA) is a data analysis technique based on lattice theory. Formal Concept Analysis (FCA) was proposed by Wille in 1982 [31]. Over last three decades the theoretical development of FCA [9, 10, 22, 23, 24, 25] has established the core theory to a stable state. The Wille's theory in fuzzy setting was first introduced by Burusco and Fuentes-Ganzáles [7]. Later a generalization of this theory from the point of view of fuzzy logic has been presented in [1, 2]. Generating fuzzy concepts from a given data with fuzzy attributes is one of the fundamental problems in the theory of fuzzy concept lattice. In [3, 5], the authors presented two algorithm for generating fuzzy concepts along with their concept hierarchy. In [6], authors showed that the problem of generating fuzzy concepts is a problem of computing all fixpoints of a fuzzy closure operator.

The theory of concept lattices is nowadays an efficient mathematical method for acquiring rules and expressing knowledge, and has been applied successively in many fields such as decision systems, information retrieval, data mining, knowledge discovery, software and so on [21, 26, 28, 29, 14]. Nevertheless, the downside of FCA has been the existence of a large number of concepts [2] combined with the fact that many of the existing approaches require computation of a whole fuzzy concept lattice which, often, is too large. Handling such large amount of clusters become an difficult task and usually impossible. To cope with this situation, different techniques has been proposed [4, 18]. However, as the size of datasets grows, the fuzzy concept lattices proposed in [4, 18] continues to grow inexorably in size. Recently, many researches [30, 31, 11] focus on reduction of formal concepts and concept lattice in formal concept analysis with fuzzy setting.

This paper presents an algorithm for finding proto-fuzzy concepts directly from the input data. In point of view of applications in different domain, significance of proto-fuzzy concepts is very high. The proto-fuzzy concept is introduced by Kridlo and Krajci [19, 20]. Recently, in [12] a fuzzy graph based technique for computing proto-fuzzy concepts has been proposed. In [12], representing the fuzzy context by a fuzzy graph, the authors showed one-to-one correspondence between proto-fuzzy concepts and maximal cliques of level graphs of the defined fuzzy graph.

In this paper, the proposed algorithm generates all proto- fuzzy concepts form the fixpoints of the fuzzy closure operators, $\uparrow\downarrow$ and $\downarrow\uparrow$. At the beginning of this paper, a theorems is introduced for computing fixpoints of fuzzy closure operators $\uparrow\downarrow$ and $\downarrow\uparrow$ directly from **L**-context. Then we present another theorem to show how all of these fixpoints could be used to generate all the proto-fuzzy concepts of different degrees $t \in L$. Finally, we proposed an algorithm for computing proto-fuzzy concepts with an illustration.

The paper is organized as follows. In Section 2, some preliminary notion on fuzzy logic, fuzzy sets and Fuzzy concept lattice are recalled. In Section 3, the theorems for computing fixpoints of the fuzzy closure operators, $\uparrow\downarrow$ and $\downarrow\uparrow$ and proto-fuzzy concepts are introduced and established. Finally in Section 4, the algorithm for computing proto-fuzzy concepts is presented and the proposed algorithm is discussed with an illustration.

2. Mathematical Background – Explanations on the Fundamentals Applied

2.1. Basics of fuzzy logic, fuzzy sets. In order to better interpret our approach, we briefly recall some basic terminologies of fuzzy sets and fuzzy logic (for more extensive overviews see the references [33, 8, 13, 15, 16, 17]) as far as they are needed for this paper.

Since fuzzy logic are developed using general structure of truth degree. We use a complete residuated lattice as a basic structure of truth degree. A complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \bigotimes, \rightarrow, 0, 1\rangle$ such that (1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; (2) $\langle L, \otimes, 1\rangle$ is a monoid; (3) $\otimes$ and $\rightarrow$ satisfy so called adjointness property, i.e., $a \otimes b \leq c$ if and only if $a \leq b \rightarrow c$, for each $a, b, c \in L$. Operations $\otimes$ and $\rightarrow$ are known as "fuzzy conjunction" and "fuzzy implication". All elements a of L are called truth degrees. Usually, the common choice of **L** is a structure with $L = [0,1]$, with $\vee$ and $\wedge$ being maximum and minimum, respectively, $\otimes$ being a left-continuous $t-$norm with the corresponding $\rightarrow$. One of the most important pairs of adjoint operation on $[0,1]$ is by Gödel: $a \otimes b = min(a,b)$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = b$ else. One may consider a finite set $\{a_0 = 0, a_1, ..., a_n = 1\}(a_0 < a_1 < ..... < a_n)$ as the set of truth values with $\otimes$ given by $a \otimes b = a_{min(k,l)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_n$ for $a_k \leq a_l$ and $a_k \rightarrow a_l = a_l$ otherwise. Such an **L** is called a finite Gödel chain.

Now based on the structure of complete residuated Lattice **L**, we present the basic notions of

L-set and fuzzy relation. An **L**-set [13] A in a universe set X is a mapping $A: X \to L$. $A(x)$ is called the truth value (or membership value) of x in A which maps X to the membership space L. Similarly, an **L**-relation I is a mapping $I: X \times Y \to L$ assigning to any $x \in X$ and $y \in Y$ a truth value $I(x,y)$ to which x and y is related under I. The collection of all **L**-sets in X is denoted by the set L^X. For every $t \in L$, $A^t = \{x \in X \mid A(x) \geq t\}$ are called level sets or t-cut of A. We let $supp(A) = \{x \in X \mid A(x) > 0\}$. We call $supp(A)$ the *support* of A. An **L**-set A is nontrivial if $supp(A) \neq \phi$. We use the notation $\vee$ for supremum and $\wedge$ for infimum. Let A_1 and A_2 be any two **L**-sets of X. Then $A_1 \subseteq A_2$ if $A_1(x) \leq A_2(x)$ for all $x \in X$. The union $A_1 \cup A_2$ of $A_1, A_2 \in L^X$ is a subset of X defined by $(A_1 \cup A_2)(x) = A_1(x) \vee A_2(x)$ for all $x \in X$ and intersection $A_1 \cap A_2$ of A_1, A_2 is also a subset of X defined by $(A_1 \cap A_2)(x) = A_1(x) \wedge A_2(x)$ for all $x \in X$.

2.2. Basics of formal concept analysis. Formal Concept Analysis is a mathematical tool for analysis of data based on lattice theory. Formal Concept Analysis aims to formulate the philosophical understanding of a concept as a unit of two parts: its extent (the set of the objects which fall under this concept) and its intent (the set of attributes covered by this concept). In addition, certain objects have certain attributes; in other words, objects are related to attributes. The sets of objects and attributes together with their relation to each other form a "formal context". Ganter-Wille's approach was based on bivalent logic, in which objects (attributes) either crisply belong or not to the extent (intent) of the concept. But many of the information people facing are usually fuzzy and imprecise, so can not be described by a concept in the formal setting, e.g., if we consider the concept "POLITICAL LEADER" then the attributes (say, contributions in the society, contributions in the economy, power of leadership etc.) of "POLITICAL LEADER" can not be delineated. Therefore, it would not be the proper way to analyze the intent by bivalent logic. By introducing Fuzzy sets into formal context, one can express the fuzzy characteristic between the objects and attributes. In order to better interpret our approach, the following is a brief presentation of the FCA framework.

2.3. Formal context and formal Concept. Definition A formal context is a triplet $\langle X, Y, I \rangle$, where X and Y are sets and $I \subseteq X \times Y$ is a binary relation. The elements of X are called objects and the elements of Y are called attributes. $I \subseteq X \times Y$ is a binary relation between objects and attributes, i.e., the inclusion $(x,y) \in I$ means that object x has attribute y.

For $A \subseteq X$ and $B \subseteq Y$, if we define

$$A^{\uparrow} = \{y|\text{ for all } x \in A : (x,y) \in I\}$$
$$B^{\downarrow} = \{x|\text{ for all } y \in B : (x,y) \in I\}$$

With the above notation we define the concept.

Definition A formal concept in $\langle X,Y,I\rangle$ is a pair $\langle A,B\rangle$ of a set $A \subseteq X$ of objects and a set $B \subseteq Y$ of attributes such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$. A is called extent and B is called intent of the concept $\langle A,B\rangle$.

If $B\langle X,Y,I\rangle$ denotes the set of all concepts, i.e., $B\langle X,Y,I\rangle = \{\langle A,B\rangle\, |A^{\uparrow} = B, B^{\downarrow} = A\}$ and $\leq$ is a partial order relation on $B\langle X,Y,I\rangle$ defined by $\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle$ iff $A_1 \subseteq A_2$(or, equivalently$B_1 \supseteq B_2$), then the $(B\langle X,Y,I\rangle\,,\leq)$ is an ordered set. It has some important properties:

$(B\langle X,Y,I\rangle\,,\leq)$ is a complete lattice, the concept lattice of $\langle X,Y,I\rangle$ [32].

2.4. Fuzzy contexts and fuzzy concepts. We start with a set X of objects, a set Y of attributes, a complete residuated lattice $\mathbf{L}$ and a fuzzy relation I between X and Y. The key idea of a fuzzy context ($\mathbf{L}$-context) is as follows: it is a triplet $\langle X,Y,I\rangle$, where $I(x,y) \in L$ (the set of truth values of complete residuated lattice L) is interpreted as the truth value of the fact, "the object $x \in X$ has the attribute $y \in Y$ ". For fuzzy sets $A \in L^X$ and $B \in L^Y$, Belohlavek [1] and, independently, Pollandt [27] defined the fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ according to the formulae

$$A^{\uparrow}(y) = \Lambda_{x\in X}\{A(x) \to I(x,y)\}$$
$$B^{\downarrow}(x) = \Lambda_{y\in Y}\{B(y) \to I(x,y)\}$$

One can easily interpret the element $A^{\uparrow}(y) \in A^{\uparrow}$ as the truth degree of "y is shared by all objects from A"and $B^{\downarrow}(x) \in B^{\downarrow}$ as the truth degree of "x has all attributes from B".

A fuzzy concept $\langle A,B\rangle$ consists of a fuzzy set A of objects (the extent of the concept) and a fuzzy set B of attributes (the intent of the concept) such that $A^{\uparrow}=B$ and $B^{\downarrow}=A$. If $B\langle X,Y,I\rangle = \{\langle A,B\rangle \,|A^{\uparrow}=B, B^{\downarrow}=A\}$ denotes the set of all fuzzy concepts of the fuzzy context $\langle X,Y,I\rangle$, then the set $B\langle X,Y,I\rangle$ with the order relation:

$\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle$ if and only if $A_1 \subseteq A_2$ (or, equivalently $B_1 \supseteq B_2$) is a complete lattice. The lattice $(B\langle X,Y,I\rangle\,,\leq)$ is called a fuzzy concept lattice.

2.5. Proto-fuzzy concepts. Let $\langle X,Y,I\rangle$ be an **L**-context, where X and Y are set of objects (X) and set of properties (Y), respectively and I is a fuzzy relation between X and Y. Since the value $I(x,y)$ express the degree to which the object x carries the attribute y. If we set a threshold value $t \in L$ to eliminate the lower degree membership value from fuzzy relation then the resulting relation is called t-cut of **L**-context which is basically a binary relation between X and Y and is denoted by I_t. For every confidence threshold $t \in L$, consider two sets: $A' = \{y \in Y \mid \forall x \in A : I(x,y) \geq t\}$ for $A \subseteq X$, i.e., the set of all attributes from Y shared by all objects of A at least with the degree t and $B' = \{x \in X \mid \forall y \in B : I(x,y) \geq t\}$ for $B \subseteq Y$, i.e., the set of all objects from X sharing attributes from B at least in the degree t. The pair $\langle A,B\rangle \in 2^X \times 2^Y$ is called t-concept iff $A' = B$, $B' = A$. The set of all t-concept in the t-cut is denoted by C_t.
The Triples $\langle A,B,t\rangle \in 2^X \times 2^Y \times L$ such that$\langle A,B\rangle \in \bigcup_{k\in L} C_k$ and $l = sup\{k \in L : \langle A,B\rangle \in C_k\}$ are called proto-fuzzy concepts. i.e., the proto-fuzzy concept is triple of a subset of objects, a subsets of attributes and a value as a best common degree of membership of all pairs of objects and attributes from the above-mentioned sets to the $L-$context. The set of proto-fuzzy concepts denoted by C^P.

3. Generation of Fixpoints of the Fuzzy Closure Operators $\uparrow\downarrow$ and $\downarrow\uparrow$ Directly from L-Context

In this section we present two theorems for computing fixpoints of the fuzzy closure operators $\uparrow\downarrow$ corresponding to each property $y \in Y$ as well as fixpoints of $\downarrow\uparrow$ corresponding to each object $x \in X$. In order to compute all proto-fuzzy concept, it is necessary to prove the following theorems.

Theorem 1. Let $\langle X,Y,I\rangle$ be an **L**-context. Then for any $y \in Y$, A_y is a fixpoint of $\uparrow\downarrow$, where $A_y(x_i) = I(x_i,y)$ for each $x_i \in X$.
Analogously, let $\langle X,Y,I\rangle$ be an **L**-context. Then for any $x \in X$, B_x is a fixpoint of $\downarrow\uparrow$, where $B_x(y) = I(x,y_j)$ for each $y_j \in Y$.

Proof. Let $A_y \in L^X$ be a fuzzy set. To show that $A_y \in L^X$ is a fixpoint of $\uparrow\downarrow$, we have to prove $A_y^{\uparrow\downarrow} = A_y$.

For any $y' \in Y$, let we consider the set $A_{y'} \in L^X$, where $A_{y'}(x_i) = I(x_i,y')$ for each $x_i \in X$. Then, $A_{y'}^{\uparrow}(y') = \bigwedge_{x_i\in X}\{A_{y'}(x_i) \to I(x_i,y')\} = \bigwedge_{x_i\in X}\{A_{y'}(x_i) \to I(x_i,y')\} = 1$, and correspondingly $A_{y'}^{\uparrow}(y') \to I(x_i,y') = I(x_i,y')$ for $x_i \in X$. Also for any value of $A_{y'}^{\uparrow}(y)$, $y \in Y - \{y'\}$, $A_{y'}^{\uparrow}(y) \to I(x_i,y) = 1$, or $I(x_i,y)$ for $x_i \in X$. Now, if $A_{y'}^{\uparrow}(y) \to I(x_i,y) = 1$ for all $y \in Y - \{y'\}$, then $A_{y'}^{\uparrow\downarrow}(x_i) = \bigwedge_{y\in Y}\{A_{y'}^{\uparrow}(y) \to I(x_i,y)\} = I(x_i,y') = A_{y'}(x_i)$ for each $x_i \in X$. Hence $A_{y'}$ is a fixpoint of $\uparrow\downarrow$ for any $y' \in Y$. For $x_i \in X$, if we let $A_{y'}^{\uparrow}(y) \to I(x_i,y) = I(x_i,y)$ for some or all $y \in Y - \{y'\}$, then Gödel fuzzy logical connectives gives $A_{y'}^{\uparrow}(y) > I(x_i,y)$ for each $y \in Y - \{y'\}$. Again, $A_{y'}^{\uparrow}(y) > I(x_i,y)$ gives $I(x_i,y') \not> I(x_i,y)$ for $x_i \in X$. Because $A_{y'}^{\uparrow}(y) > I(x_i,y)$ implies that $A_{y'}^{\uparrow}(y) = 1$, or $I(x',y)(> I(x_i,y)$ for $x' \in \{X - x_i\}$. Now, if $I(x_i,y') > I(x_i,y)$, then $I(x_i,y') \to I(x_i,y) = I(x_i,y)$ and thus $A_{y'}^{\uparrow}(y) = I(x_i,y)$. This contradicts that $A_{y'}^{\uparrow}(y) > I(x_i,y)$. So, if $A_{y'}^{\uparrow}(y) \to I(x_i,y) = I(x_i,y)$ for some or all $y \in Y - \{y'\}$, then $I(x_i,y) \geq I(x_i,y')$. Therefore, $A_{y'}^{\uparrow\downarrow}(x_i) = \bigwedge_{y\in Y}\{A_{y'}^{\uparrow}(y) \to I(x_i,y)\} = I(x_i,y') = A_y$ for any $y' \in Y$. Hence A is a fixpoint of $\uparrow\downarrow$.

Proof is similar for analogous part.

Using the above theorem, from **L**-context we can directly determine the fixpoints of $\uparrow\downarrow$ corresponding to each $y \in Y$ as well as fixpoints of $\downarrow\uparrow$ corresponding to each $x \in X$. Now, The following theorem shows how all of these fixpoints could be used to generate all the proto-fuzzy concepts of different degrees $t \in L$.

Theorem 2. Let $\langle X,Y,I\rangle$ be an **L**-context and X^* be the collection of all fixpoints, $A_y \in L^X$, corresponding to each properties $y \in Y$. Then for $t \in L$ and for each $A_y \in X^{**} \subseteq X^*$, $\langle A_y^t, B^t\rangle$ is a proto fuzzy concept of degree t, where $X^{**} = \{A_y \in X^* : A_y(x) = I(x,y) \text{ for each } x \in X \text{ and } \exists\, x \in X \text{ such that } A_y(x) = t\}$ and $B(y) = min_{x\in A_y^t}\{I(x,y)\}$ for each $y \in Y$.
Analogously, let $\langle X,Y,I\rangle$ be an **L**-context and and Y^* be the collection of all fixpoints, $D_x \in L^Y$, corresponding to each objects $x \in X$. Then for $t \in L$ and for each $D_x \in Y^{**} \subseteq Y^*$, $\langle C_x^t, D_x^t\rangle$ is a proto fuzzy concept of degree t, where $Y^{**} = \{D_x \in Y^* : D_x(y) = I(x,y) \text{ for each } y \in Y \text{ and } \exists\, y \in Y \text{ such that } D_x(y) = t\}$ and $C(x) = min_{y\in D_x^t}\{I(x,y)\}$.

proof. For $t \in L$, let $A_{y'} \in X^{**} \subseteq X^*$ be any fixpoint corresponding to $y' \in Y$. Since $X^{**} = \{A_y \in X^* : A_y(x) = I(x,y) \text{ for each } x \in X \text{ and } \exists\, x' \in X \text{ such that } A_{y'}(x') = t\}$. Therefore $x' \in A_{y'}^t \neq \emptyset$, and

$$X \supseteq A_{y'}^t = \{x \in X : A_{y'}(x) = I(x,y') \geq t\}.$$

Now, we consider the set $B \in L^Y$, where $B(y) = min_{x\in A_{y'}^t}\{I(x,y)\}$ for each $y \in Y$. For $t \in L$, it is obvious that $y' \in B^t$. Therefore, $B^t \neq \emptyset$, and

$$B^t = \{y \in Y : \quad B(y) = I(x,y) \geq t \text{ for all } x \in A_{y'}^t\} \qquad (1)$$

Again, there does not exists any $y'' \in Y - \{B^t\}$ so that $I(x,y'') \geq t$ for all $x \in A_{y'}^t$. If any such y'' exists, then $B(y'') = min_{x\in A_{y'}^t}\{I(x,y'')\} \geq t$, i.e., $y'' \in B^t$. This contradicts the fact that

$y'' \in Y - \{B^t\}$. So for any $x \in A^t_{y'}$, $I(x,y) \geq t$ for all $y \in B^t$. Now for the pair $\left\langle A^t_{y'}, B^t \right\rangle$ it is obvious that

$$(A^t_{y'})^{\uparrow} = B^t \text{ and } (B^t)^{\downarrow} = A^t_{y'}$$

Therefore, the pair $\left\langle A^t_{y'}, B^t \right\rangle$ is a formal concept in the t-cut of $\langle X, Y, I \rangle$. It is also to be noted that $A_{y'} \in X^{**} = \{A_y \in X^* : A_y(x) = I(x,y)$ for each $x \in X$ and $\exists\, x' \in X$ such that $A_{y'}(x') = t\}$. Thus $\left\langle A^t_{y'}, B^t \right\rangle$ can not be a formal concept in any t'-cut, where $t' > t$. Hence $\left\langle A^t_{y'}, B^t \right\rangle$ is a proto-fuzzy concept of degree t and theorem is proved.

Proof is similar for analogous part.

4. Generation of Proto-Fuzzy Concepts Directly from Input Data

Using the theorems established in above section, we now develop an algorithm to compute proto-fuzzy concepts by an illustration.

Example: Consider the following **L**-context, where X contain five objects $\{o_1, o_2, o_3, o_4, o_5\}$ and Y contain five properties $\{p_1, p_2, p_3, p_4, p_5\}$. Consider $L = \{0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1\}$ and Gödel fuzzy logical connectives. Also let **L** be the residuated lattice with the Gödel operations over $L = \{0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1\}$. The **L**-context and proto-fuzzy concepts are given in Table 1 and Table 2, respectively. In Table 2, we denote the proto-fuzzy concepts of degree t as $\left\langle \mathscr{C}^{\mathscr{P}}_{p_j}, t \right\rangle$ for each properties $p_j \in Y$ and $\left\langle \mathscr{C}^{\mathscr{P}}_{o_i}, t \right\rangle$ for each objects $o_i \in X$.

TABLE 1. Fuzzy context of the given example

	p_1	p_2	p_3	p_4	p_5
o_1	0.9	0.7	0.2	0.4	1
o_2	0.8	1	0.3	0.7	0.9
o_3	0.2	0.2	0.2	0.1	0.3
o_4	0.3	0.6	0.3	0.2	0.2
o_5	0.5	0.8	0.4	0.3	0.4

Let $\langle X,Y,I\rangle$ be an **L**-context. To compute all proto-fuzzy concepts, directly from input data, we proceed as follows:

Step 1. For all $y \in Y$, compute the fixpoints, $A_y \in L^X$, where $A_y(x_i) = I(x_i,y)$ for each $x_i \in X$ and $y \in Y$. Similarly, For all $x \in X$, compute the fixpoints, $D_x \in L^Y$, where $D_x(y_j) = I(x,y_j)$ for each $y_j \in Y$ and $x \in X$.

Step 2. For $t \in L$, find all those fixpoints, $A_{y'} \in L^X$, from the fixpoints obtained in step 1, in which there exists at least one $x \in X$ such that $A_{y'}(x) = t$. Similarly, For $t \in L$, find all those fixpoints, $D_{x'} \in L^Y$ in which there exists at least one $y \in Y$ such that $D_{x'}(y) = t$.

Step 3. For each $A_{y'} \in L^X$ obtained in Step 2, compute t-cut, $A^t_{y'} = \{x \in X : A(x) \geq t\}$. Similarly, for each $D_{x'} \in L^Y$, compute t-cut, $D^t_{x'} = \{y \in Y : D(y) \geq t\}$.

Step 4. For each t-cut, $A^t_{y'}$, compute $B \in L^Y$ where $B(y) = min_{x\in A^t_{p'}}\{I(x,y)\}$ for each $y \in Y$ and obtain the proto-fuzzy concepts $\left\langle A^t_{y'}, B^t, t\right\rangle$ of degree t. Similarly, for each t-cut, $D^t_{x'}$, compute $C \in L^X$ where $C(x) = min_{y\in D^t_{x'}}\{I(x,y)\}$ for each $x \in X$ and obtain the proto-fuzzy concepts $\left\langle C^t, D^t_{x'}, t\right\rangle$ of degree t. Omit the repeated proto-fuzzy concepts.

Step 5. For each truth value $t \in L$, repeat the step 2 to step 4.

TABLE 2. Proto-fuzzy concepts of the fuzzy context given in Table 1

Truth value t	proto-fuzzy concepts of degree t
0.1	$\langle \mathscr{C}^{\mathscr{P}}_{p_4},0.1\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_3},0.1\rangle = \langle\{o_1,o_2,o_3,o_4,o_5\},\{p_1,p_2,p_3,p_4,p_5\},0.1\rangle$
0.2	$\langle \mathscr{C}^{\mathscr{P}}_{p_1},0.2\rangle = \langle \mathscr{C}^{\mathscr{P}}_{p_2},0.2\rangle = \langle \mathscr{C}^{\mathscr{P}}_{p_3},0.2\rangle = \langle \mathscr{C}^{\mathscr{P}}_{p_5},0.2\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_3},0.2\rangle$
	$= \langle\{o_1,o_2,o_3,o_4,o_5\},\{p_1,p_2,p_3,p_5\},0.2\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{p_4},0.2\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_1},0.2\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_4},0.2\rangle = \langle\{o_1,o_2,o_4,o_5\},\{p_1,p_2,p_3,p_4,p_5\},0.2\rangle$
0.3	$\langle \mathscr{C}^{\mathscr{P}}_{p_1},0.3\rangle = \langle\{o_1,o_2,o_4,o_5\},\{\{p_1,p_2\},0.3\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{p_3},0.3\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_4},0.3\rangle = \langle\{o_2,o_4,o_5\},\{\{p_1,p_2,p_3\},0.3\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{p_4},0.3\rangle = \langle\{o_1,o_2,o_5\},\{\{p_1,p_2,p_4,p_5\},0.3\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{p_5},0.3\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_3},0.3\rangle = \langle\{o_1,o_2,o_3,o_5\},\{\{p_5\},0.3\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{o_2},0.3\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_5},0.3\rangle = \langle\{o_2,o_5\},\{\{p_1,p_2,p_3,p_4,p_5\},0.3\rangle$
0.4	$\langle \mathscr{C}^{\mathscr{P}}_{p_3},0.4\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_5},0.4\rangle = \langle\{o_5\},\{\{p_1,p_2,p_3,p_5\},0.4\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{p_4},0.4\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_1},0.4\rangle = \langle\{o_1,o_2\},\{\{p_1,p_2,p_4,p_5\},0.4\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{p_5},0.4\rangle = \langle\{o_1,o_2,o_5\},\{\{p_1,p_2,p_5\},0.4\rangle$
0.5	$\langle \mathscr{C}^{\mathscr{P}}_{p_1},0.5\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_5},0.5\rangle = \langle\{o_1,o_2,o_5\},\{p_1,p_2\},0.5\rangle$
0.6	$\langle \mathscr{C}^{\mathscr{P}}_{p_2},0.6\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_4},0.6\rangle = \langle\{o_1,o_2,o_4,o_5\},\{p_2\},0.6\rangle$
0.7	$\langle \mathscr{C}^{\mathscr{P}}_{p_2},0.7\rangle = \langle\{o_1,o_2,o_5\},\{p_2\},0.7\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{p_4},0.7\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_2},0.7\rangle = \langle\{o_2\},\{p_1,p_2,p_4,p_5\},0.7\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{o_1},0.7\rangle = \langle\{o_1,o_2\},\{p_1,p_2,p_5\},0.7\rangle$
0.8	$\langle \mathscr{C}^{\mathscr{P}}_{p_1},0.8\rangle = \langle\{o_1,o_2\},\{p_1,p_5\},0.8\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{p_2},0.8\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_5},0.8\rangle = \langle\{o_5\},\{p_2\},0.8\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{o_2},0.8\rangle = \langle\{o_2\},\{p_1,p_2,p_5\},0.8\rangle$
0.9	$\langle \mathscr{C}^{\mathscr{P}}_{p_1},0.9\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_1},0.9\rangle = \langle\{o_1\},\{p_1,p_5\},0.9\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{p_5},0.9\rangle = \langle\{o_1,o_2\},\{p_5\},0.9\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{o_2},0.9\rangle = \langle\{o_2\},\{p_2,p_5\},0.9\rangle$
1	$\langle \mathscr{C}^{\mathscr{P}}_{p_2},1\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_2},1\rangle = \langle\{o_2\},\{p_2\},1\rangle$
	$\langle \mathscr{C}^{\mathscr{P}}_{p_5},1\rangle = \langle \mathscr{C}^{\mathscr{P}}_{o_1},1\rangle = \langle\{o_1\},\{p_5\},1\rangle$

5. Conclusion

This paper provides a simplest new technique of generating all proto-fuzzy concepts. It shows the use of fixpoints of the fuzzy closure operators, $\uparrow\downarrow$ and $\downarrow\uparrow$, is a good tool of generating all proto-fuzzy concepts since all fixpoints which are used to generate proto-fuzzy concepts can be obtained directly from L-context.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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