A NEW ONE PARAMETER RAYLEIGH MAXWELL DISTRIBUTION

SAURAV SARMA*, KISHORE K. DAS

Department of Statistics, Gauhati University, Guwahati, Assam, India

Abstract. In this article, a new one-parameter lifetime distribution has been suggested. The distribution is a two-component finite mixture of Rayleigh and Maxwell distribution. The various statistical properties of the distribution such as moments, skewness, kurtosis, moment generating function, and characteristics function have been discussed. Survival function and hazard function are also studied. The maximum likelihood estimate for the unknown parameter under the proposed model is derived. Finally, the model is fitted to a real-life failure data, and outcomes were compared with some standard statistical probability distributions.

Keywords: finite mixture; Rayleigh distribution; Maxwell-Boltzman distribution; life time distribution; failure data.


1. INTRODUCTION

The temperature of a physical system is determined by the velocity of the particles (atoms or molecules) contributing to the system. These particles have different velocities and they also constantly changes due to collisions among themselves. Maxwell (1860) [13] showed that the energy of such particles exhibits a certain probability pattern. Boltzmann (1872) [4] later simplified the probability distribution and investigated its physical origin.

The Rayleigh distribution was introduced by Lord Rayleigh (1880) [16] to study a problem in the field of acoustics. The distribution is related to several well-known distributions such as Chi-Square, Exponential, Gamma, and Weibull. The distribution has a wide range of application and hence extensive work has been done in various fields of science and technology (Johnson et. al. 1994) [11].

A physical system, for example, industrial equipment or vehicles are comprised of many individual and vital parts. All of these parts of a system may exhibit a completely different failure pattern. Considering a single probability distribution to explore the survival function of such a system may not always give us the desired outcome. A finite mixture of some known
and suitable probability distributions can help in understanding the sub-populations of a system with different properties. Finite mixtures are found to be useful in various fields of physics, chemistry, biology, and social sciences.

In this study, a finite mixture of Rayleigh and Maxwell (RMM) distribution is proposed. The various statistical properties of the mixture are discussed. The parameter of the proposed mixture is estimated under the maximum likelihood method. Finally, the mixture is fitted to a real-life data set.

2. **One Parameter Rayleigh-Maxwell Distribution**

Let us consider a two-component mixture of Rayleigh distribution with parameter \( a \) and Maxwell-Boltzmann distribution with parameter \( a \) with their mixing components \( \frac{1}{1+a} \) and \( \frac{a}{1+a} \) respectively. The probability distribution of the new distribution can be written as

\[
f(x; a) = \frac{(1 + \epsilon x)x}{(1 + a)a} e^{-\frac{x^2}{2a^2}}
\]

The corresponding cdf is given by

\[
F(x) = \frac{1 - e^{-z}}{1 + a} + \frac{\beta a}{(1 + a)} \gamma\left(\frac{3}{2}, z\right)
\]

where, \( \epsilon = \sqrt{\frac{2}{\pi}} \approx 0.8 \), \( \beta = \frac{2}{\sqrt{\pi}} \approx 1.13 \), \( z = \frac{x^2}{2a^2} \) and \( \gamma(a, b) \) is the lower incomplete gamma integral.

2.1. **Moments of RMM Distribution.** The \( r^{th} \) raw moment is given by

\[
\mu'_r = \frac{2^r \epsilon^r}{1 + a} \left[ \Gamma\left(\frac{r + 2}{2}\right) + \beta a \Gamma\left(\frac{r + 3}{2}\right) \right]
\]

\( r = 1, 2, 3, ... \)

Replacing particular values of \( r \) (\( r = 1, 2, 3, 4 \)) in (3) we get the first four raw moments as

\[
\mu'_1 = \frac{a}{1+a} (1.2533 + 1.5958a) = \text{Mean}
\]

\[
\mu'_2 = \frac{a^2}{1+a} (2 + 3a)
\]
Figure 1. Probability density function of the RMM distribution for different values of the parameter.

\[ \mu_3' = \frac{a^3}{1 + a} \left(3.7599 + 6.3831a\right) \]  

\[ \mu_4' = \frac{a^4}{1 + a} \left(8 + 15a\right) \]

The corresponding central moments are

\[ \mu_2 = \frac{a^2}{(1 + a)^2} \left(0.43a^2 + a + 0.4375\right) = \text{Variance} \]

\[ \mu_3 = \frac{a^3}{(1 + a)^3} \left[4.93a^3 + 12.20a^2 + 10.02a + 2.6840\right] \]

\[ \mu_4 = \frac{a^4}{(1 + a)^4} \left[0.63a^4 + 2.8a^3 + 4.24a^2 + 2.73a + 0.6\right] \]
2.1.1. Skewness and Kurtosis. The skewness and kurtosis of RMM distribution were found out to be

\[
\text{Skewness} = \frac{\mu_3}{\mu_2^3} = \frac{[4.93a^3 + 12.20a^2 + 10.02a + 2.6840]^2}{[0.43a^2 + a + 0.4375]^3}
\]

(11)

\[
\text{Kurtosis} = \frac{\mu_4}{\mu_2^2} = \frac{[0.63a^4 + 2.8a^3 + 4.24a^2 + 2.73a + 0.6]}{[0.43a^2 + a + 0.4375]^2}
\]

(12)

2.1.2. Harmonic Mean. The harmonic mean of RMM distribution is

\[
\text{Harmonic Mean}\left(\frac{1}{H}\right) = E\left[\frac{1}{X}\right] = \int_0^\infty \frac{1}{x} f(x) dx
\]

(13)

\[
= \frac{1.25 + 0.8a}{a(1 + a)}
\]

2.1.3. Mode of RMM distribution. The mode of a distribution is obtained by solving the equation

\[
\frac{\partial}{\partial x} \log[f(x)] = 0
\]

(14)
On simplification, the equation to obtain mode of RMM distribution reduces to

(15) \[0.8x^3 + x^2 - 1.6a^2x - a^2 = 0\]

2.2. Generating Functions.

2.2.1. Moment Generating Function. The moment generating function (mgf) of RMM distribution is

\[M_X(t) = E[e^{tX}]\]

\[= \int_0^{\infty} e^{tx}f(x)dx\]

(16)

\[= \int_0^{\infty} \sum_{r=0}^{\infty} \frac{t^rx^r}{r!}f(x)dx\]

\[= \sum_{r=0}^{\infty} \frac{t^r 2^{r/2} a^r}{r!} \left[ \Gamma\left(\frac{r+2}{2}\right) + \frac{2a}{\sqrt{\pi}} \Gamma\left(\frac{r+3}{2}\right) \right]\]

2.2.2. Characteristic Function. The characteristic function (cf) of RMM distribution is

\[\Phi_X(t) = E[e^{itX}]\]

\[= \int_0^{\infty} e^{itx}f(x)dx\]

(17)

\[= \int_0^{\infty} \sum_{r=0}^{\infty} \frac{(it)^rx^r}{r!}f(x)dx\]

\[= \sum_{r=0}^{\infty} \left( \frac{(it)^r}{r!} \right) \frac{2^{r/2} a^r}{1+a} \left[ \Gamma\left(\frac{r+2}{2}\right) + \frac{2a}{\sqrt{\pi}} \Gamma\left(\frac{r+3}{2}\right) \right]\]

2.3. Survival Function. The survival function of our proposed RMM model is given by

(18) \[S(x) = 1 - F(x)\]

\[= a + e^{-z} - ac \gamma(3/2, z)\]

\[= a + e^{-z} - ac \gamma(3/2, z)\]

Where,
\[c = \sqrt{\frac{2}{\pi}}, z = \frac{x^2}{2a^2}\] and \[\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt\], lower incomplete gamma integral.
2.4. Hazard Function. The hazard function of RMM distribution is obtained as

\[
h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}
\]

\[= \frac{x(1 + cx)e^{-z}}{a^2[a + e^{-z} - ac\gamma(3/2, z)]}; \quad x > 0\]

The reverse hazard rate function of RMM distribution is

\[
\Phi(x) = \frac{f(x)}{F(x)}
\]

\[= \frac{x(1 + cx)e^{-z}}{a^2[1 - e^{-z} + ac\gamma(3/2, z)]}; \quad x > 0\]
Figure 4. Hazard function of the RMM distribution.

Table 1. Descriptive statistics for different values of the parameter

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>Variance</th>
<th>Harmonic Mean</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.6837</td>
<td>0.1161</td>
<td>2.2000</td>
<td>113.0907</td>
<td>3.1266</td>
</tr>
<tr>
<td>0.6</td>
<td>0.8290</td>
<td>0.1677</td>
<td>1.8021</td>
<td>118.1770</td>
<td>3.1309</td>
</tr>
<tr>
<td>0.7</td>
<td>0.9760</td>
<td>0.2286</td>
<td>1.5210</td>
<td>123.0796</td>
<td>3.1361</td>
</tr>
<tr>
<td>0.8</td>
<td>1.1244</td>
<td>0.2988</td>
<td>1.3125</td>
<td>127.7937</td>
<td>3.1418</td>
</tr>
<tr>
<td>0.9</td>
<td>1.2740</td>
<td>0.3783</td>
<td>1.1520</td>
<td>132.3193</td>
<td>3.1478</td>
</tr>
<tr>
<td>1.0</td>
<td>1.4246</td>
<td>0.4669</td>
<td>1.0250</td>
<td>136.6599</td>
<td>3.1541</td>
</tr>
<tr>
<td>1.5</td>
<td>2.1882</td>
<td>1.0458</td>
<td>0.6533</td>
<td>155.8035</td>
<td>3.1845</td>
</tr>
<tr>
<td>2.0</td>
<td>2.9633</td>
<td>1.8478</td>
<td>0.4750</td>
<td>171.3434</td>
<td>3.2109</td>
</tr>
<tr>
<td>2.5</td>
<td>3.7449</td>
<td>2.8699</td>
<td>0.3714</td>
<td>184.1038</td>
<td>3.2327</td>
</tr>
<tr>
<td>3.0</td>
<td>4.5305</td>
<td>4.1105</td>
<td>0.3042</td>
<td>194.7252</td>
<td>3.2506</td>
</tr>
<tr>
<td>3.5</td>
<td>5.3189</td>
<td>5.5685</td>
<td>0.2571</td>
<td>203.6830</td>
<td>3.2654</td>
</tr>
</tbody>
</table>
3. **Inferential Procedures**

The likelihood equation corresponding to (1) is

\[
L(x;a) = \left\{ \frac{1}{(1+a)a^2} \right\}^n \prod_{i=1}^{n} x_i (1 + cx_i) e^{-\sum_{i=1}^{n} \frac{x_i^2}{2a^2}}
\]  

(21)

Where, \( c = \sqrt{\frac{2}{\pi}} \approx 0.8 \)

The log likelihood function is

\[
\log L = -2n \log (a) - n \log (1 + a) + \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log (1 + cx_i) - \frac{1}{2a^2} \sum_{i=1}^{n} x_i^2
\]  

(22)

Differentiating (22) w.r.t. \( a \) and equating to zero we get

\[
3a^3 + 2a^2 - (1 + a)T_x = 0
\]  

(23)

Where, \( T_x = \frac{1}{n} \sum_{i=1}^{n} x_i^2 \)

Solving (23) for \( a \) numerically, we can get the maximum likelihood estimate of the parameter.

4. **Application**

The proposed model is fitted to a data set related to the number of miles to the first major motor failure of 191 buses operated by a large city bus company (Davis [6]). The outcomes are compared with Rayleigh, Maxwell-Boltzman, Gamma, Chi-square, and Exponential distributions. To compare the performance of our proposed model with other distributions, different discrimination criteria such as AIC, BIC, AICC, HQIC, and CAIC are constructed under the log-likelihood function. Table (2) presents the data related to the motor failure of 191 buses. Table (3) presents the discriminating criteria under different distributions.
TABLE 2. Data related to motor failure of 191 city buses

<table>
<thead>
<tr>
<th>Distance Interval (in Thousands of Miles)</th>
<th>Number of Failures</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 20</td>
<td>6</td>
</tr>
<tr>
<td>20 - 40</td>
<td>11</td>
</tr>
<tr>
<td>40 - 60</td>
<td>16</td>
</tr>
<tr>
<td>60 - 80</td>
<td>25</td>
</tr>
<tr>
<td>80 - 100</td>
<td>34</td>
</tr>
<tr>
<td>100 - 120</td>
<td>46</td>
</tr>
<tr>
<td>120 - 140</td>
<td>33</td>
</tr>
<tr>
<td>140 - 160</td>
<td>16</td>
</tr>
<tr>
<td>160 - 180</td>
<td>2</td>
</tr>
<tr>
<td>180+</td>
<td>2</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>191</strong></td>
</tr>
</tbody>
</table>

TABLE 3. Results of AIC, AICC, HQIC and CAIC for different probability distribution considering the data related to motor failure of city buses

<table>
<thead>
<tr>
<th>Test</th>
<th>RMM</th>
<th>Rayleigh</th>
<th>Maxwell</th>
<th>Gamma</th>
<th>Chi-Square</th>
<th>Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIC</td>
<td>1945.70</td>
<td>1963.46</td>
<td>1947.30</td>
<td>1977.73</td>
<td>3377.02</td>
<td>2130.36</td>
</tr>
<tr>
<td>BIC</td>
<td>1948.95</td>
<td>1966.72</td>
<td>1950.56</td>
<td>1984.23</td>
<td>3380.27</td>
<td>2133.61</td>
</tr>
<tr>
<td>AICC</td>
<td>1945.72</td>
<td>1963.49</td>
<td>1949.32</td>
<td>1977.79</td>
<td>3377.04</td>
<td>2130.38</td>
</tr>
<tr>
<td>HQIC</td>
<td>1947.01</td>
<td>1964.78</td>
<td>1950.62</td>
<td>1980.36</td>
<td>3378.33</td>
<td>2131.68</td>
</tr>
<tr>
<td>CAIC</td>
<td>1949.95</td>
<td>1967.72</td>
<td>1953.56</td>
<td>1986.23</td>
<td>3381.27</td>
<td>2134.61</td>
</tr>
</tbody>
</table>

Our proposed model is performing better in explaining the data set than the remaining distributions since the values of AIC, AICC, HQIC, and CAIC are less compare to Rayleigh, Maxwell - Boltzman, Gamma, Chi-square and Exponential distribution.

5. CONCLUSION

The superior performance of our proposed model can be confirmed from the different discrimination criteria since the best model is the one that gives the minimum values of those criteria. The distribution can be used in cases where we observed a high rate of failure as we
move towards the mode of the data and then failure rate decreases drastically. The various statistical properties of the proposed model were also discussed. Further extension of the proposed model was also possible and will be studied in future work.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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