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SOLUTIONS OF CLASSES OF DIFFERENTIAL EQUATIONS USING MODIFIED DIFFERENTIAL TRANSFORM METHOD

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Abstract. This paper proposed a numerical approximation scheme for solving singular two-point boundary value problems. This approach is based on a truncated series obtained by differential transform (DT) method, the Laplace transformation (LT) approach and process of Padé approximants. The modified differential transform method (MDTM) aims to overcome the difficulty of solving these types of problems and gives a good approximation for the true solution in a large region. Some illustrative examples are presented to demonstrate the effectiveness and applicability of the new method and the obtained results are compared with the exact solution.

Keywords: Emden-Fowler type of equations; singular two-point boundary value problems; Lane Emden equation of index *m*; Van Der Pol equation; modified differential transform method.

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1. Introduction

The study of intrinsic properties and systematic behavior, and the mathematical modelling of problems related to certain phenomenon occurring on numerous fields pertaining to astronomy,

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physiology, electro hydrodynamics, radiography, and other multidisciplinary sciences have been acknowledged by various researchers [1]. These problems are modelled mathematically to formulate a class of second order singular differential equations of the form:

$$\frac{1}{p(t)}y^{''}(t)+\frac{1}{q(t)}y^{'}(t)+\frac{1}{r(t)}y(t)=g(t),\ 0<t\leq 1, \tag{1}$$

satisfying the boundary conditions

$$y(0)=\alpha_1,\ y(1)=\beta_1, \tag{2}$$

or the initial conditions

$$y(0)=\alpha_2,\ y^{'}(0)=\beta_2, \tag{3}$$

where $p,\ q,\ r$, and g are real valued functions that are continuous on $(0,\ 1]$ and the parameters $\alpha_1,\ \alpha_2,\ \beta_1$ and β_2 are real constants. These types of problems are traced to the fields of , fluid mechanics, medicine, optimal control, chemical kinetics, and other areas of applied mathematics [2, 3].

However, the universe is also occupied by interesting phenomenon such as kinetics of combustion along with a concentration of a reactant, the situation celestial scientific observation where the distance of center of cloud is said to be proportional to gravitational potential, self-gravitating star temperature variation, thermal behavior of a spherical cloud of gas, and many more. These types of collective processes induce equation (1) to the Lane-Emden and Emden-Fowler type of special and utmost equations of the form:

$$y^{''}(t)+\frac{k}{t}y^{'}+f(t,y)=g(t),\ 0<t\leq 1,\ k\geq 0, \tag{4}$$

with boundary and initial conditions (2) or (3).

Suppose $f(t,y)$ in (4), annihilates to be a pure function of real variable 'y' say $h(y)$ in a defined and appropriate domain and give rise to a special class of differential equations known as Lane Emden equation type of problem.

The Emden-Fowler class of equations are singular initial value problems use in modelling phenomena related to astrophysics and mathematical physics [4, 5, 6]. This type of equations related to the second-order ordinary differential equations (ODE's). Numerous problems in mathematical physics can be characteristically expressed as Emden– Fowler type equations:

$$y'' + \frac{2}{t}y' + af(t)g(y) = 0,\ y(0) = y_0,\ y'(0) = 0. \tag{5}$$

where $f(t)$ and $g(y)$ are functions of t and y. Due to the significance and applicability of this equations, various researchers have tried to investigate different forms of the function $g(y)$.
On the other hand, the Lane- Emden equation of index m is important in the theory of stellar structure. This equation is used to describe the variation of temperature in a spherical gas cloud that is under the mutual attraction of its molecules. However, this is subject to the laws of thermodynamics [7]. The Lane- Emden equation is defined as:

$$y''(t) + \frac{2}{t}y'(t) + y^m(t) = 0,\ y(0) = 1,\ y'(0) = 0. \tag{6}$$

And the general oscillator differential equation:

$$y'' + f(t,y,y') = 0, y'(0) = a,\ y(0) = b, \tag{7}$$

where y is the displacement and t is time.
Using vacuum tubes, Van Der Pol [8] studied the oscillations of electrical circuits. The results of the experiments show that the oscillations are in a stable state. The stable oscillations are currently referred to the limit cycles which are found in problems of nonlinear dynamics. His derived equation for oscillator is among the systems with nonlinear damping forces and defines as:

$$\frac{d^2y}{dt^2} - k(1-y^2)\frac{dy}{dt} + y = 0. \tag{8}$$

where k controls the process of voltage flows. This nonlinear damping force is capable of increasing the amplitude for small velocities but to decease in cases of large velocities.
Based on the concept of Taylor series [9, 10, 11], Zhou [12] proposed the differential transform method (DTM) for solutions differential equations. The method is also used to obtain the solution of initial value problems, difference equations, and boundary value problems. The analytical solution obtained by the DTM which are in the form of polynomial and sufficiently

differentiable are used as an approximation to exact solutions. However, the truncated series solution obtained by the DTM is a good approximation in a very small region only [13].
Thus, we proposed the MDTM as an alternative scheme with good approximation to problems in a large region. This method modifies the series solution for singular two-point initial value problem by first applying the Laplace transformation to the DTM truncated, then, employing Padé approximants to convert the transformed series into a meromorphic function and finally obtaining the analytic solution using inverse Laplace transform (ILT). This approach can be a periodic or better approximation solution to the truncated series solution obtained by DTM.

In this paper: Section 2 discusses a brief overview and some fundamental results of DTM, Padé approximants and Laplace transform. We apply the new scheme to various examples to illustrate the simplicity and efficiency of the proposed MDTM in section 3. Lastly, section 4 presents the conclusion and discussion for further reference.

2. PRELIMINARIES

This section presents some definitions of DTM and Padé approximants.

2.1. Differential Transform Method.

Definition 2.1. [14]
Consider a function $f(x)$ which is analytical at x_0, then

$$F(k) = \frac{f^{(k)}(x_0)}{k!}. \tag{9}$$

In the domain of interest and it inverse differential transform is given as

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k. \tag{10}$$

To know more about the main operations of the DTM see [15].

Theorem 2.1. [16] *If $f(y) = y^m$, then*

$$F(k) = \begin{cases} (Y(0))^m, & k = 0; \\ \frac{1}{Y(0)} \sum_{r=1}^{k} \left(\frac{(m+1)r-k}{k}\right) Y(r)F(k-r), & k \geq 1. \end{cases}$$

Proof. From the definition (9)

$$F(0) = [y^m(x)]_{x=0} = y^m(0) = (Y(0)^m \tag{11}$$

Differentiating $f(y) = y^m$ with respect to x, we have

$$\frac{df(y)}{dx} = my^{m-1}(x)\frac{dy(x)}{dx}$$

or,

$$y(x)\frac{df(y(x))}{dx} = mf(y(x))\frac{dy(x)}{dx} \tag{12}$$

Applying differential transform to (12) gives

$$\sum_{r=0}^{k} Y(r)(k-r+1)F(k-r+1) = m\sum_{r=0}^{k}(r+1)Y(r+1)F(k-r) \tag{13}$$

From Eq.(13),

$$(k+1)Y(0)F(k+1) = m\sum_{r=0}^{k}(r+1)Y(r+1)F(k-r) - \sum_{r=1}^{k} Y(r)(k-r+1)F(k-r+1).$$

or,

$$(k+1)Y(0)F(k+1) = m\sum_{r=1}^{k+1} rY(r)F(k-r+1) - \sum_{r=1}^{k} Y(r)(k-r+1)F(k-r+1).$$

. Thus,

$$(k+1)Y(0)F(k+1) = \sum_{r=1}^{k+1}\{(m+1)r-k-1\}Y(r)F(k-r+1).$$

Replacing $k+1$ by k yields

$$kY(0)F(k) = \sum_{r=1}^{k}\{(m+1)r-k\}Y(r)F(k-r) \tag{14}$$

From Eq. (14), we get

$$F(k) = \frac{1}{Y(0)}\sum_{r=1}^{k}\left(\frac{(m+1)r-k}{k}\right)Y(r)F(k-r) \tag{15}$$

Combining Eqs.(11) and (15), we obtain the transformed function of $f(y) = y^m$ as

$$F(k) = \begin{cases} (Y(0))^m, & k = 0; \\ \frac{1}{Y(0)}\sum_{r=1}^{k}\left(\frac{(m+1)r-k}{k}\right)Y(r)F(k-r), & k \geq 1. \end{cases}$$

□

By this proof, it is obvious that the DT of various complex function can be obtained in a similar manner. Some nonlinear functions and their transforms are given in the following Theorems.

Theorem 2.2. [16] *If $f(y) = e^{ay}$, then*

$$F(k) = \begin{cases} e^{aY(0)}, & k = 0; \\ a\sum_{r=0}^{k-1}(\frac{r+1}{k})Y(r+1)F(k-1-r), & k \geq 1. \end{cases}$$

Theorem 2.3. [16] *If $f(y) = sin(\alpha y)$ and $g(y) = cos(\alpha y)$, then*

$$F(k) = \begin{cases} sin(\alpha Y(0)), & k = 0; \\ \alpha\sum_{r=0}^{k-1}(\frac{k-r}{k})G(r)Y(k-r), & k \geq 1. \end{cases}$$

and

$$G(k) = \begin{cases} cos(\alpha Y(0)), & k = 0; \\ -\alpha\sum_{r=0}^{k-1}(\frac{k-r}{k})F(r)Y(k-r), & k \geq 1. \end{cases}$$

The differential equation in the domain of interest can be transformed into an equation of algebraic nature in the K-domain using the differential transform, and the finite-term Taylor series expansion is used to obtain $f(x)$ with a remainder, as

$$f(x) = \sum_{i=0}^{N} F(i)\frac{(t-t_0)^i}{i!} + R_{N+1}(t) \tag{16}$$

The series solution (16) converges swiftly in a small region, however, the convergence rate may be very slow in the wide region and thus, produce an inaccurate truncation result. For further reference on DTM see [14, 16, 17, 18, 19, 20].

2.2. Padé approximation.

Given the Taylor series expansion of a function $y(x)$, the ratio of two polynomials derived from the coefficients of this function is known as Padé approximant. The $[L/M]$ Padé approximants to $y(x)$ is defined as:

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)}$$

where $P_L(x)$ and $Q_M(x)$ are polynomials of degree at most M and L respectively whose formal power series is:

$$y(x) = \sum_{m=1}^{\infty} a_m x^m,$$

and the coefficients of $P_L(x)$ and $Q_M(x)$ are determined using

$$y(x) - \frac{P_L(x)}{Q_M(x)} = \mathrm{O}\left(x^{L+M+1}\right) \tag{17}$$

Since multiplying both the denominator and numerator by a constant leave $[L/M]$ unchanged, it implies we impose the condition of normalization

$$Q_M(0) = 1. \tag{18}$$

To this end, it is required that there are no common factors between $P_L(x)$ and $Q_M(x)$. However, writing the coefficient of $P_L(x)$ and $Q_M(x)$ as

$$\begin{cases} P_L(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L \\ Q_M(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_M x^M \end{cases} \tag{19}$$

and may be multiplying (17) by $Q_M(x)$ linearizes the coefficient equations. The detailed expression of Eq.(17) is as follows

$$\begin{cases} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0 \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0 \\ . \\ . \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0 \end{cases} \tag{20}$$

$$\begin{cases} a_0 = p_0 \\ a_0 + a_0 q_1 = p_1 \\ a_2 + a_1 q_1 + a_0 q_2 = p_2 \\ . \\ . \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L \end{cases} \tag{21}$$

To obtain the solutions of these equations, we compute for all the unknowns $q's$ in the set of linear equations (20). The values of $q's$ would give the explicit formula for the unknown $p's$ in (21) and thus completes the solution.

If equation (20) and equation (21) are non-singular, then, they can be solved directly to obtain

$$
\left[\frac{L}{M}\right] = \frac{det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M}x^j & \sum_{j=M-1}^{L} a_{j-M+1}x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}} \tag{22}
$$

If (22) holds and the lower index on the sum exceeds that of the upper, then, we can replace the sum by zero [21]. Using the MATLAB symbolic software, we can obtain the diagonal elements of Padé approximants of orders including $[2\ /2]$, $[4/4]$ or $[6/6]$. Note that typically, the Padé approximant, obtained from a partial Taylor sum, is more accurate than the latter. However; the Padé, being a rational expression, has poles, which are not present in the original function. It is a simple algebraic task to expand the form of an $[N,\ M]$ Padé in a Taylor series and compute the Padé coefficients by matching with the above [22, 23, 24, 25].

3. Main Results

Problem 3.1. *Consider the Isothermal Gas Spheres Equation whose model is used for viewing the isothermal gas sphere with a constant temperature* [7]. *For*

$$a = 1,\ f(t) = 1 \text{ and } g(y) = e^y,$$

equation (5) *is the isothermal gas spheres equation.*

$$y'' + \frac{2}{t}y' + e^{y} = 0,\ (t > 0). \tag{23}$$

Satisfying the initial conditions

$$y(0) = 0,\ y'(0) = 0. \tag{24}$$

A series solution of Eq. (23) *is obtained by* [7, 26, 27].

$$y(t) = -\frac{1}{6}t^{2} + \frac{1}{120}t^{4} - \frac{1}{1890}t^{6} + \frac{61}{1632960}t^{8} - \dots \tag{25}$$

Now, taking the DT for both sides of equation (23) *gives*

$$(k+1)(k+2)Y(k+2) + 2(k+2)Y(k+2) + G(k) = 0, \tag{26}$$

then

$$Y(k+2) = -\frac{1}{(k+2)(k+3)}G(k). \tag{27}$$

where $G(k)$ *is the DT of* $g(y) = e^{y}$.

Transforming Eq.(24) *gives*

$$Y(0) = Y(1) = 0.$$

By Theorem (2.2)*, the differential transform* $G(k)$ *in equation* (27) *is*

$$G(0) = e^{Y(0)} = 1, \tag{28}$$

$$G(k) = \sum_{r=0}^{k-1} \frac{r+1}{k} Y(r+1)G(k-1-r),\ k \geq 1. \tag{29}$$

Therefore,

$G(0) = 1,$ *then* $Y(2) = -\frac{1}{6}G(0) = -\frac{1}{6}$. *Consequently,*

$$Y(3) = 0,\ y(4) = \frac{1}{120},\ y(5) = 0,\ y(6) = -\frac{1}{1890},\ Y(7) = 0,\ Y(8) = \frac{61}{1632960}.$$

Using (10)*, we obtain an approximate solution of* (23) *as follows*

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^{k} = -\frac{1}{6}t^{2} + \frac{1}{120}t^{4} - \frac{1}{1890}t^{6} + \frac{61}{1632960}t^{8} - \dots. \tag{30}$$

It is interestingly the obtained series is the same as obtained by other methods. However, to improve the accuracy of (30)*, we employ the modified DTM as follows*

Applying the Laplace transform to (30) [32]*, we obtain*

$$\mathscr{L}(y(t)) = -\frac{1}{3}\frac{1}{s^3} + \frac{1}{5}\frac{1}{s^5} - \frac{8}{21}\frac{1}{s^7} + \frac{122}{81}\frac{1}{s^9} - \ldots.$$

Let $s = \frac{1}{z}$ *for simplicity, then, we have*

$$\mathscr{L}(y(t)) = -\frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{8}{21}z^7 + \frac{122}{81}z^9 - \ldots. \tag{31}$$

The Padé approximants $\left[\frac{5}{5}\right]$ *gives*

$$\left[\frac{5}{5}\right] = \frac{-556171z^5 - 129465z^3}{697050\, z^4 + 1901550z^2 + 388395}$$

Recalling $z = \frac{1}{s}$*, we obtain* $\left[\frac{5}{5}\right]$ *in terms of s*

$$\left[\frac{5}{5}\right] = \frac{-129465s^2 - 556171}{388395s^5 + 1901550s^3 + 697050}$$

To obtain the modified solution, we apply the inverse Laplace transform to the $\left[\frac{5}{5}\right]$ *Padé approximant.*

Therefore, we have

$$y(t) = \frac{7154043465016223}{9007199254740992} cos\left(\frac{711283029649915}{1125899906842624}t\right)$$
$$+\frac{2094956233865021}{576460752303423488} cos\left(\frac{2387548870573799}{1125899906842624}t\right) - \frac{1796694289042591}{2251799813685248}.$$

Table (1) *shows comparison between values of* $y(t)$ *for problem* (3.1) *using MDTM wth DTM and ADM, and since the exact solution for problem* (3.1) *is unknown, figure* (1) *shows comparison for the residual error.*

The following Algorithm explains the MDTM procedures for solving classes of second order differential equations in case of the exact solution be a known.

TABLE 1. Comparison of the solution using MDTM with DTM and ADM for Problem (3.1)

t_i	*MDTM*	*DTM* [28]	*ADM* [7]
0.0	*0*	*0*	*0*
0.2	*-0.0066533671*	*-0.0066533671*	*-0.0066533671*
0.4	*-0.0264554762*	*-0.0264554760*	*-0.0264554760*
0.6	*-0.0589440692*	*-0.0589440582*	*-0.0589440582*
0.8	*-0.1033859603*	*-0.10338576665*	*-0.10338576665*
1.0	*-0.1588268608*	*-0.1588250783*	*-0.1588250783*

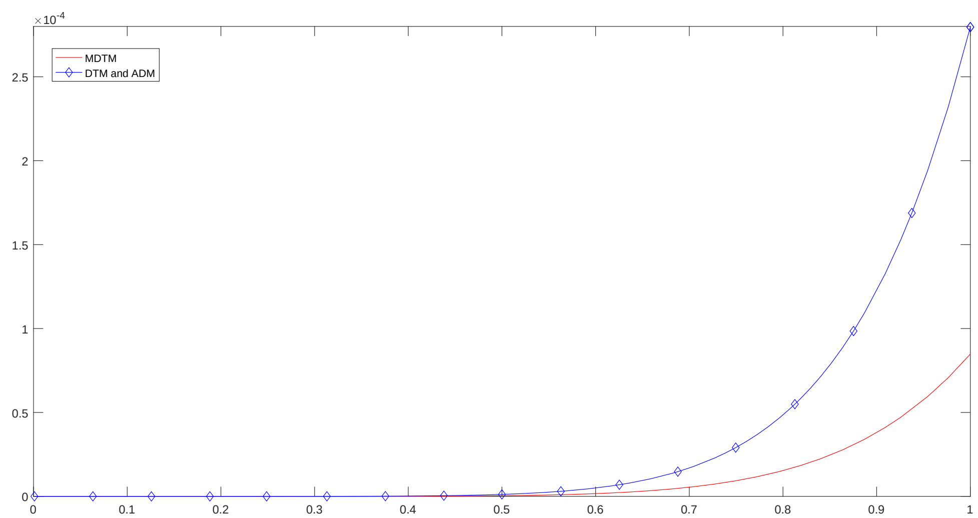

FIGURE 1. Comparison between DTM, ADM and MDTM Residual errors for Problem (3.1)

ALGORITHM 1

Step 1: Given $y^{''} = f(t, y, y^{'}), 0 \leq t \leq 1$, the exact solution $y(t)$

Step 2: Set $M = 2$,

Step 3: Compute $Y(k) = \frac{1}{k!}[\frac{d^k}{dx^k}y(t)]_{t=0}, k = 0, 1, 2, \ldots, N$

Step 4: Expand $y_1(t) = \sum_{k=0}^{N} Y(k)t^k$.

Step 5: Obtain $\mathscr{L}(y_1(t)) = \mathscr{L}(\sum_{k=0}^{N} Y(k)t^k)$.

Step 6: Substitute $1/s = z$ in Step 5

Step 7: Compute $[M/M]\mathscr{L}(y_1(t))$

Step 8: Substitute $z = 1/s$ in Step 7

Step 9: Compute $y_2(t) = \mathscr{L}^{-1}([M/M]\mathscr{L}(y_1(t)))$

Step 10: If $|y(t) - y_2(t)| \leq 10^{-2}$ Then Stop. Else $M = M+1$ and go to step 7.

Problem 3.2. *Given the singular two-point initial value problem* [29]

$$u^{''}(t) + \frac{2}{t}u^{'}(t) = 2(2t^2+3)u(t), \quad 0 \leq t \leq 1, \tag{32}$$

with initial conditions

$$u(0) = 1, u^{'}(0) = 0, \tag{33}$$

and exact solution $u(t) = e^{t^2}$.

Transforming Eq. (32)*, we obtain*

$$(k+1)(k+2)U(k+2) + 2(k+2)U(k+2) = 6U(k) + 4\sum_{i=0}^{k}\delta(i-2)Y(k-i).$$

Thus,

$$U(k+2) = \frac{1}{(k+2)(k+3)}\Big[6U(k) + 4\sum_{i=0}^{k}\delta(i-2)Y(k-i)\Big]. \tag{34}$$

$$U(0) = 1, U(1) = 0. \tag{35}$$

Substituting (35) *in* (34)*, yields*

$$U(2) = 1,\ U(3) = 0,\ U(4) = \frac{1}{2},\ U(5) = 0,\ U(6) = \frac{1}{6},\ U(7) = 0,\ U(8) = \frac{1}{24}.$$

Using (10)*, an approximate solution of* (3.2) *is obtained as follows*

$$u_1(t) = \sum_{k=0}^{\infty} U(k)t^k = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{1}{24}t^8 + \ldots. \tag{36}$$

which yields the exact solution of (3.2) *in the limit of infinitely many terms.*

Applying the MDTM, we can improve the accuracy of (36) *as follows. Taking just the first five terms from* (36) *and applying the Laplace transform for both sides, we obtain*

$$\mathscr{L}(u_1(t)) = \frac{1}{s} + 2\frac{1}{s^3} + 12\frac{1}{s^5} + 120\frac{1}{s^7} + 1680\frac{1}{s^9}. \tag{37}$$

TABLE 2. Absolute error for Problem 3.2

t_i	*Exact Solution*	*MDTM Solution*	*Absolute Error MDTM*	*Absolute Error DTM*	*Absolute Error* [29]
0.0	*1.00000*	*1.00000*	*0*	*0*	*0*
0.2	*1.04081*	*1.04081*	*1.10E-10*	*8.59E-10*	*0*
0.4	*1.17351*	*1.17351*	*1.20E-7*	*8.977E-7*	*7.26E-7*
0.6	*1.43333*	*1.43332*	*7.58E-6*	*5.358E-5*	*4.15E-5*
0.8	*1.89648*	*1.89633*	*1.54E-4*	*9.997E-4*	*7.10E-4*
1.0	*2.71828*	*2.71659*	*1.70E-3*	*9.949E-3*	*5.95E-3*

Let's denote $s = \frac{1}{z}$*; then*

$$\mathscr{L}(u_1(t)) = z + 2z^3 + 12z^5 + 120z^7 + 1680z^9. \tag{38}$$

The Padé approximants $\left[\frac{5}{5}\right]\mathscr{L}(u_1(t))$ *gives*

$$\left[\frac{5}{5}\right]\mathscr{L}(u_1(t)) = \frac{32t^5 - 18t^3 + t}{60t^4 - 20t^2 + 1}.$$

Recalling $z = \frac{1}{s}$*, we obtain* $\left[\frac{5}{5}\right]\mathscr{L}(u_1(t))$ *in terms of s*

$$\left[\frac{5}{5}\right]\mathscr{L}(u_1(t)) = \frac{s^4 - 18s^2 + 32}{s^5 - 20s^3 + 60s}.$$

To obtain the modified solution, we apply the ILT to the $\left[\frac{5}{5}\right]\mathscr{L}(u_1(t))$*.*

Therefore,

$$u_2(t) = \frac{2000282079184873}{4503599627370496}\, cosh\Big(\frac{4317026594442411}{2251799813685248}\, t\Big) + \frac{3244727901483475}{144115188075855872}\, cosh\Big(\frac{4549047411303143}{1125899906842624}\, t\Big) + \frac{8}{15}$$

Table (2) *shows comparison between absolute error for problem* (3.2) *using MDTM, DTM and Padé Approximation* [29]*, and figure* (2) *shows the graphs of approximated solution and exact solution* $u(t)$*.*

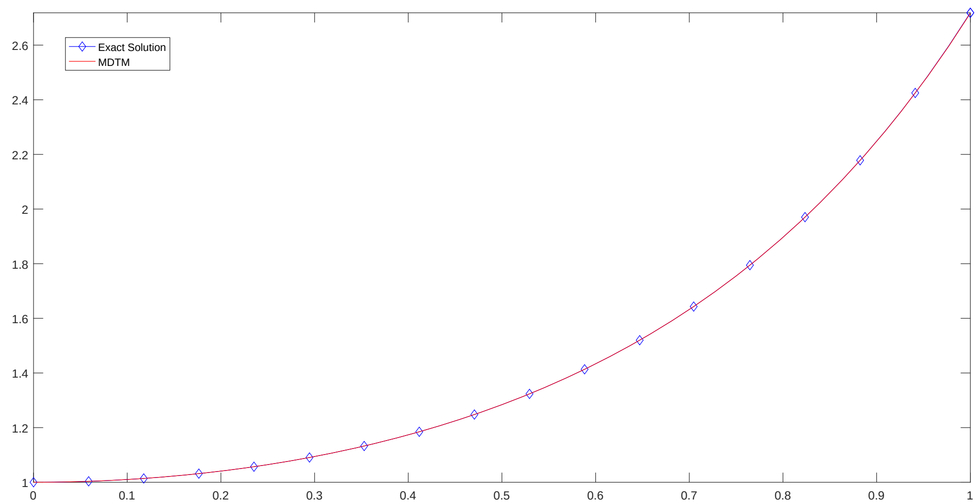

FIGURE 2. the graphs of approximated solution and exact solution $u(t)$ for Problem 3.2

Problem 3.3. *Consider the Lane-Emden equation of index m is of the form* [7]

$$u^{''}(t)+\frac{2}{t}u^{'}(t)+u^{m}(t)=0,\ \ u(0)=1,\ u^{'}(0)=0 \tag{39}$$

Substituting $m=0;\ 1$ *and* 5 *into* (39) *and solving the resulting differential equations leads to the exact solution* $u(t)=1-\frac{t^2}{6}, u(t)=\frac{sin(t)}{t}$ *and* $u(t)=\sqrt{\frac{3}{3+t^2}}$ *respectively. Transforming Eq.* (39) *with the conditions, we obtain*

$$(k+1)(k+2)U(k+2)+2(k+2)U(k+2)+G(k)=0.$$

Thus,

$$U(k+2)=-\frac{G(k)}{(k+2)(k+3)},\ U(0)=1,U(1)=0 \tag{40}$$

where $G(k)$ *is the DT of* $g(u)=u^{m}(t)$. *By Theorem* (2.1)*, the differential transform* $G(k)$ *in equation* (40) *is*

$$G(0)=(U(0))^2=1, \tag{41}$$

$$G(k)=\sum_{r=1}^{k}\Big(\frac{(m+1)r-k}{k}\Big)U(r)G(k-r),\ \ k\geq 1. \tag{42}$$

Substituting (41) *in* (40) *gives* $U(2) = -\frac{1}{6}$. *Consequently,*

$$U(3) = 0,\ U(4) = \frac{m}{120},\ U(5) = 0,\ U(6) = \frac{5m-8m^2}{15120},\ U(7) = 0.$$

Using the (10), *we obtain an approximate solution of* (39) *as follows:*

$$u_1(t) = \sum_{k=0}^{\infty} U(k)t^k = 1 - \frac{t^2}{6} + \frac{m}{120}t^4 + \frac{5m-8m^2}{15120}t^6 + \dots. \tag{43}$$

Applying MDTM, we obtain an improve solution of differential transform series (43) *as follows Substitute* $m = 1$ *in* (43), *and applying the Laplace transform to the subsequent series, we get*

$$\mathscr{L}(u_1(t)) = \frac{1}{s} - \frac{1}{3}\frac{1}{s^3} + \frac{1}{5}\frac{1}{s^5} - \frac{1}{7}\frac{1}{s^7} + \dots.$$

Let $s = \frac{1}{z}$*; then*

$$\mathscr{L}(u_1(t)) = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 + \dots. \tag{44}$$

The Padé approximants $\left[\frac{4}{4}\right]\mathscr{L}(u_1(t))$ *gives*

$$\left[\frac{4}{4}\right]\mathscr{L}(u_1(t)) = \frac{55z^3+105z}{9z^4+90z^2+105}.$$

Since $z = \frac{1}{s}$, *we can get* $\left[\frac{4}{4}\right]\mathscr{L}(u_1(t))$ *in terms of s*

$$\left[\frac{4}{4}\right]\mathscr{L}(u_1(t)) = \frac{105s^3+55s}{105s^4+90s^2+9}.$$

To obtain the modified solution, we apply the ILT to the $\left[\frac{4}{4}\right]\mathscr{L}(u_1(t))$.

Therefore,

$$u_2(t) = \frac{2937000676430437}{4503599627370496}cos\Big(\frac{1531138501201791}{4503599627370496}t\Big) + \frac{6266395803760235}{18014398509481984}cos\Big(\frac{7756426344020357}{9007199254740992}t\Big)$$

Now, substitute $m = 5$ *in* (43), *and applying the LT to the resulting series, yields*

$$\mathscr{L}(u_1(t)) = \frac{1}{s} - \frac{1}{3}\frac{1}{s^3} + \frac{1}{s^5} - \frac{25}{3}\frac{1}{s^7} + \dots.$$

TABLE 3. Absolute error for Problem (3.3) for $m = 1$

t_i	*Exact Solution*	*MDTM Solution*	*Absolute Error MDTM*	*Absolute Error DTM* [30]	*Absolute Error* [7]
0.0	*1.0000000000000*	*1.0000000000000*	*0*	*0*	*0*
0.2	*0.9933466539753*	*0.9933466539749*	*3.6826E-13*	*7.0521E-12*	*7.0521E-12*
0.4	*0.9735458557716*	*0.9735458556776*	*9.3982E-11*	*1.8034E-09*	*1.8034E-09*
0.6	*0.9410707889917*	*0.9410707865949*	*2.3968E-09*	*4.6135E-08*	*4.6135E-08*
0.8	*0.8966951136244*	*0.8966950898485*	*2.3776E-08*	*4.5966E-07*	*4.5966E-07*
1.0	*0.8414709848079*	*0.8414708443480*	*1.4046E-07*	*2.7308E-06*	*2.7308E-06*

Since $s = \frac{1}{z}$; *then*

$$\mathscr{L}(u_1(t)) = z - \frac{1}{3}z^3 + z^5 - \frac{25}{3}z^7 + \dots. \tag{45}$$

The Padé approximants $\left[\frac{4}{4}\right]\mathscr{L}(u_1(t))$ *gives*

$$\left[\frac{4}{4}\right]\mathscr{L}(u_1(t)) = \frac{26z^3 + 3z}{6z^4 + 27z^2 + 3}.$$

However, $z = \frac{1}{s}$, *thus, we obtain* $\left[\frac{4}{4}\right]\mathscr{L}(u_1(t))$ *in terms of* s

$$\left[\frac{4}{4}\right]\mathscr{L}(u_1(t)) = \frac{3s^3 + 26s}{3s^4 + 27s^2 + 6}.$$

The modified approximate solution is obtained in a similar pattern.

$$u_2(t) = \frac{4448076759099869}{4503599627370496} cos\left(\frac{4300861033607317}{9007199254740992}t\right) + \frac{222091473082509}{18014398509481984} cos\left(\frac{6669282851725225}{2251799813685248}t\right)$$

Table (3) *and Table* (4) *show absolute error for problem* (3.3) *for* $m = 1$ *and* $m = 5$,*respectively. While figure* (3), *and figure* (4) *show the graphs of approximated solutions and exact solutions* $u(t)$ *for* $m = 1$ *and* $m = 5$.

TABLE 4. Absolute error for Problem (3.3) for $m = 5$

t_i	*Exact Solution*	*MDTM Solution*	*Absolute Error MDTM*	*Absolute Error DTM* [30]	*Absolute Error* [7]
0.0	*1.0000000000000*	*1.0000000000000*	*0*	*0*	*0*
0.2	*0.9933992677988*	*0.9933992638762*	*3.9226E-09*	*8.5395E-09*	*8.5395E-09*
0.4	*0.9743547036924*	*0.9743537608245*	*9.4287E-07*	*2.1111E-06*	*2.1111E-06*
0.6	*0.9449111825231*	*0.9448893678248*	*2.1815E-05*	*5.1183E-05*	*5.1183E-05*
0.8	*0.9078412990032*	*0.9076515145267*	*1.8978E-04*	*4.7537E-04*	*4.7537E-04*
1.0	*0.8660254037844*	*0.8650711644611*	*9.5424E-04*	*2.5995E-03*	*2.5995E-03*

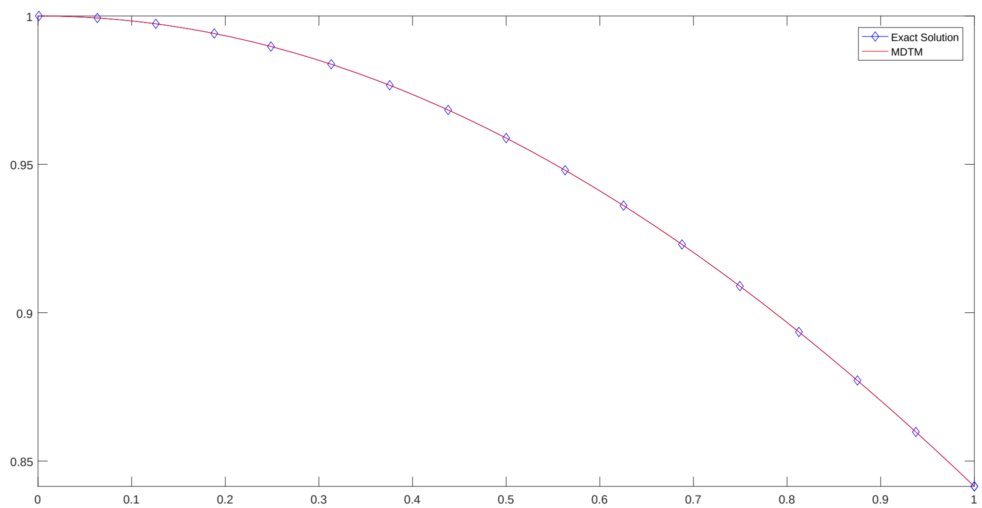

FIGURE 3. the graphs of approximated solutions and exact solutions $u(t)$ at $m = 1$ for Problem (3.3)

Problem 3.4. *Given the nonlinear Van Der Pol oscillator differential equation* [31]

$$\frac{d^2u}{dt^2} - u + u^2 + \left(\frac{du}{dt}\right)^2 - 1 = 0 \tag{46}$$

with initial conditions:

$$u(0) = 2,\ u^{'}(0) = 0. \tag{47}$$

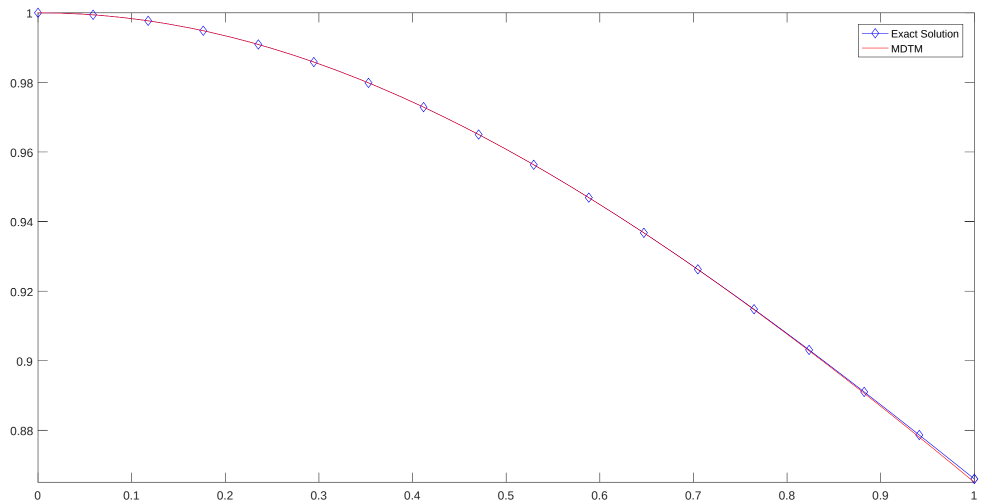

FIGURE 4. the graphs of approximated solutions and exact solutions $u(t)$ at $m=5$ for Problem (3.3)

and exact solution $u(t) = 1 + cos(t)$.

To solve the differential equation (47). *Taking differential transform for both sides, to get*

$$(k+1)(k+2)U(k+2) - U(k) + \sum_{i=0}^{k} U(i)U(k-i)$$

$$+ \sum_{m=0}^{k} (m+1)U(m+1)(k-m+1)U(k-m+1) - \delta(k) = 0.$$

Therefore,

$$U(k+2) = \frac{1}{(k+1)(k+2)}$$

$$\times \Big[U(k) - \sum_{i=0}^{k} U(i)U(k-i) - \sum_{m=0}^{k} (m+1)U(m+1)(k-m+1)U(k-m+1) + \delta(k) \Big].$$

From initial conditions (47) *and definition* (9), *we get*

$$U(0) = 2,\ U(1) = 0.$$

Then recurrence relation gives

$$U(2) = \frac{1}{2}[U(0) - (U(0))^2 - (U(1))^2 + \delta(0)] = -\frac{1}{2}.$$

$$U(3) = \frac{1}{6}[U(1) - 2U(0)U(1) - 4U(1)U(2)] = 0.$$

Consequently,

$$U(4) = \frac{1}{4!}\ ,\ U(5) = 0 \text{ and } U(6) = -\frac{1}{6!},\ U(7) = 0 \text{ and } U(8) = \frac{1}{8!}.$$

Using (10)*, the approximate solution of* (46) *is obtained as follows.*

$$u(t) = \sum_{k=0}^{\infty} U(k)t^k$$

Thus,

$$u_1(t) = 2 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \dots \tag{48}$$

But in order to improve the accuracy of the differential transform solution (48)*, and to prove the power and efficiency for the MDTM, and by taking just the first three terms from* (48)*, we implement the MDTM as follows:*

Applying the LT to the first three terms from (48)*, we obtain*

$$\mathscr{L}(u(t)) = \frac{2}{s} - \frac{1}{s^3} + \frac{1}{s^5}.$$

Let $s = \frac{1}{z}$*; then*

$$\mathscr{L}(u_1(t)) = 2z - z^3 + z^5. \tag{49}$$

The Padé approximants $\left[\frac{3}{3}\right]\mathscr{L}(u_1(t))$ *gives*

$$\left[\frac{3}{3}\right]\mathscr{L}(u_1(t)) = \frac{z^3+2z}{z^2+1}$$

But $z = \frac{1}{s}$*, thus we have* $\left[\frac{3}{3}\right]\mathscr{L}(u_1(t))$ *in terms of s*

$$\left[\frac{3}{3}\right]\mathscr{L}(u_1(t)) = \frac{2s^2+1}{s^3+s}$$

To obtain the modified solution, we apply the ILT to the $\left[\frac{3}{3}\right]\mathscr{L}(u_1(t))$*.*

Therefore, we have

$$u_2(t) = 1 + cos(t).$$

Problem 3.5. *Consider the Van Der Pol nonlinear oscillator equation* [31]

$$\frac{d^2u}{dt^2} + \frac{du}{dt} + u + u^2\frac{du}{dt} = 2cos(t) - cos^3(t) \tag{50}$$

with the initial conditions:

$$u(0)=0,\ u^{'}(0)=1 \tag{51}$$

the exact solution is $u(t)=sin(t)$. *To solve the differential equation* (50).

Taking differential transform for both sides, to get

$$(k+1)(k+2)U(k+2)+(k+1)U(k+1)+U(k)+\sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}(k_1+1)U(k_1+1)U(k_1-k_2)U(k-k_1)$$

$$=\frac{2}{k!}cos\Big(\frac{\pi k}{2}\Big)-\sum_{k_3=0}^{k}\sum_{k_4=0}^{k_3}\frac{1}{k_3!}cos\Big(\frac{\pi k_3}{2}\Big)\frac{1}{(k_3-k_4)!}cos\Big(\frac{\pi(k_3-k_4)}{2}\Big)\frac{1}{(k-k_3)!}cos\Big(\frac{\pi(k-k_3)}{2}\Big).$$

From initial conditions (51) *and definition* (9)*, we get*

$$U(0)=0,\ U(1)=1.$$

Then recurrence relation gives

$$U(2)=0\ ,\ U(3)=-\frac{1}{3!}\text{ and }U(4)=0,\ U(5)=\frac{1}{5!},\ U(6)=0\text{ and }U(7)=-\frac{1}{7!}.$$

Using (10)*, the approximate solution of equation* (50) *is obtained in the form*

$$u_1(t)=\sum_{k=0}^{\infty}U(k)t^k$$

Thus,

$$u_1(t)=t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots \tag{52}$$

which gives the exact solution of (50) *in limit of infinitely many terms.*

To prove the power and efficiency for the MDTM, and by taking just the first two terms from (52)*, we implement the MDTM as follows*

Applying the LT to the first two terms from (52)*, we obtain*

$$\mathscr{L}(u(t))=\frac{1}{s^2}-\frac{1}{s^4}.$$

Let $s=\frac{1}{z}$*; then*

$$\mathscr{L}(u_1(t))=z^2-z^4. \tag{53}$$

The Padé approximants $\left[\frac{2}{2}\right]\mathscr{L}(u_1(t))$ *gives*

$$\left[\frac{2}{2}\right]\mathscr{L}(u_1(t)) = \frac{z^2}{z^2+1}$$

Since $z = \frac{1}{s}$*, then we obtain* $\left[\frac{2}{2}\right]\mathscr{L}(u_1(t))$ *in terms of s*

$$\left[\frac{2}{2}\right]\mathscr{L}(u_1(t)) = \frac{1}{s^2+1}$$

To obtain the modified solution, we apply the inverse Laplace transform to the $\left[\frac{2}{2}\right]\mathscr{L}(u_1(t))$. *Therefore, we have*

$$u_2(t) = sin(t).$$

4. Conclusions and Discussions

The MDTM presents an efficient scheme for computing the approximate solutions of boundary value problem. Especially, for cases of multipoint boundary value problems. All the solutions obtained has shown that the computed results of the proposed method are in perfect agreement with results obtained by the exact solution. These examples illustrate that the MDTM has improved the DTM's truncated series solution in the convergence rate with true analytic solution. The obtained results also show that the proposed MDTM is very promising and capable of finding wider applications.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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