ON GRAPH CLIQUISH FUNCTIONS

PIYALI MALLICK

Department of Mathematics, Government General Degree College, Kharagpur -II, West Bengal, India

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Abstract. In the present paper we introduce a new notion of graph cliquish functions from a topological space to a metric space and study its relation with other types of generalized continuity. We also give a characterization of that new notion of generalized continuity on a dense set of points.

Keywords: graph continuity; graph quasi-continuity; quasi-continuity; cliquish functions; graph cliquish functions.

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1. INTRODUCTION AND BASIC NOTATIONS


In what follows $X$ is a topological space and $Y$ is a metric space with metric $d$. For a subset $A \subseteq X$, $f|_A$ denotes the restriction of a function $f: X \rightarrow Y$ on $A$. If $G(f)$ denotes the graph of $f: X \rightarrow Y$ then the symbol $cl(G(f))$ denotes the closure of $G(f)$ in the product topology of $X \times Y$. By $C(f)$
we denote the set of all points at which \( f: X \to Y \) is continuous. The letters \( \mathbb{R}, \mathbb{Q}, \mathbb{Z} \) stand for the set of all reals, rationals and integers respectively and \( S(x,r) \) denotes the open sphere with centre \( x \) and radius \( r \).

A function \( f: X \to Y \) is said to be

- graph continuous if there exists a continuous function \( g: X \to Y \) such that \( G(g) \subseteq cl(G(f)) \) [7].
- graph quasi-continuous if there exists a quasi-continuous function \( g: X \to Y \) such that \( G(g) \subseteq cl(G(f)) \) [3].
- quasi-continuous at a point \( x_0 \in X \) if for each \( \varepsilon > 0 \) and each open neighbourhood \( U \) of \( x_0 \), there exists a non-empty open set \( G \subseteq U \) such that \( d(f(x), f(x_0)) < \varepsilon \) for each \( x \in G \) [4].
- cliquish at a point \( x_0 \in X \) if for each \( \varepsilon > 0 \) and each open neighbourhood \( U \) of \( x_0 \), there exists a non-empty open set \( G \subseteq U \) such that \( d(f(x), f(y)) < \varepsilon \) whenever \( x, y \in G \) [10].

\( f \) is called quasi-continuous (cliquish) if it has this property at each point.

**Definition 1.1:** A function \( f: X \to Y \) is said to be graph cliquish if there exists a cliquish function \( g: X \to Y \) such that \( G(g) \subseteq cl(G(f)) \).

Evidently every cliquish function is graph cliquish. Also, it follows that

**Remark 1.1:** If a function \( f: X \to Y \) is graph cliquish with closed graph then \( f \) is cliquish.

### 2. The Graph Cliquish and Other Continuity Types

The following implications follow from the above definitions:

Continuity \( \Rightarrow \) quasi-continuity \( \Rightarrow \) cliquish

\[ \downarrow \quad \downarrow \quad \downarrow \]

Graph continuity \( \Rightarrow \) graph quasi-continuity \( \Rightarrow \) graph cliquish

And all of these are not invertible.

**Example 2.1:** Consider the real line \( \mathbb{R} \). Let \( f: \mathbb{R} \to \mathbb{R} \) be defined by

\[
f(x) = \begin{cases} 
1, & \text{if } x \in \mathbb{Q} \\
0, & \text{otherwise}
\end{cases}
\]

Here \( f \) is not cliquish but graph continuous.

**Example 2.2:** Consider the real line \( \mathbb{R} \). Let \( f: \mathbb{R} \to \mathbb{R} \) be defined by
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\[
f(x) = \begin{cases} 
1, & x \in \mathbb{Z} \\
0, & x \in \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Z}) \\
2, & \text{otherwise} 
\end{cases}
\]

Here \( f \) is graph cliquish but not cliquish. Also \( f \) is not graph quasi-continuous.

3. RESULTS

The following results are known:

Result 3.1: If \( f: X \to Y \) is cliquish then \( X \setminus C(f) \) is of first category [5]. Also, we know that

Result 3.2: In a Baire space the complement of every set of first category is dense [6].

Using these two results it easily follows that

Result 3.3: If \( X \) is a Baire space and if \( f: X \to Y \) is cliquish then \( C(f) \) is dense in \( X \).

Now we can formulate the following properties of a graph cliquish function.

Theorem 3.1: Let \( f: X \to Y \) be graph cliquish. Then for any \( \varepsilon > 0 \) the set \( A(f, g, \varepsilon) = \{ x \in X : d(f(x), g(x)) < \varepsilon \} \) is dense in \( X \), for any cliquish function \( g: X \to Y \) with \( G(g) \subseteq cl(G(f)) \).

Proof: Let \( \varepsilon > 0 \) and \( U \) be a non-empty open set in \( X \). Let \( x_0 \in U \). Since \( g \) is cliquish at \( x_0 \), there exists a non-empty open set \( U_1 \subseteq U \) such that \( d(g(x), g(y)) < \frac{\varepsilon}{2} \) whenever \( x, y \in U_1 \).

Let \( x_1 \in U_1 \). Then \( (x_1, g(x_1)) \in cl(G(f)) \). So, \( \left[U_1 \times S(g(x_1), \frac{\varepsilon}{2})\right] \cap G(f) \neq \emptyset \).

Choose \( x_2 \in U_1 \) such that \( d(f(x_2), g(x_1)) < \frac{\varepsilon}{2} \).

Now, \( d(f(x_2), g(x_2)) \leq d(f(x_2), g(x_1)) + d(g(x_1), g(x_2)) < \varepsilon \)

So, \( x_2 \in A(f, g, \varepsilon) \).

Hence \( A(f, g, \varepsilon) \) is dense in \( X \).

Remark 3.1: Let \( f: X \to Y \) be given and \( g: X \to Y \) be a cliquish function such that for any \( \varepsilon > 0 \), the set \( A(f, g, \varepsilon) \) is dense in \( X \). Then it is not necessarily true that \( G(g) \subseteq cl(G(f)) \).

Example 3.1: Consider \( \mathbb{R} \) with the topology \( \tau = \{A \subseteq \mathbb{R}: 0 \in A\} \cup \{\emptyset\} \) and \( \mathbb{R} \) with the usual metric \( d \).
The functions $f, g: (\mathbb{R}, \tau) \to (\mathbb{R}, d)$ are defined as

$$f(x) = 0; \forall x \in \mathbb{R} \text{ and } g(x) = \begin{cases} 0, & x = 0 \\ 1, & \text{otherwise} \end{cases}$$

$g$ is cliquish. Now, $A(f, g, \varepsilon) = \begin{cases} \{0\}, & 0 < \varepsilon \leq 1 \\ \mathbb{R}, & \varepsilon > 1 \end{cases}$

$A(f, g, \varepsilon)$ is dense in $(\mathbb{R}, \tau)$ for any $\varepsilon > 0$. But, $G(g) \not\subseteq cl(G(f))$.

**Remark 3.2:** In example 3.1, $C(g) = \{0\}$ and $G(g|_{C(g)}) \subseteq cl(G(f|_{C(g)})$.

**Result 3.4:** Let $A(\subseteq X)$ be dense in $X$. If $f: X \to Y$ is cliquish then $f|_A$ is also cliquish.

**Proof:** Let $x_0 \in A$, $U$ be an open neighbourhood of $x_0$ in $A$ and $\varepsilon > 0$.

Now, $U = A \cap U_1$, $U_1$ is open in $X$.

Since $f$ is cliquish at $x_0$, $\exists$ a non-empty open set $G(\subseteq U_1)$ in $X$ such that $d(f(x), f(y)) < \varepsilon$ whenever $x, y \in G$.

Since $A$ is dense in $X$, $A \cap G \neq \varnothing$. Also $A \cap G$ is open in $A$ and $d((f|_A)(x), (f|_A)(y)) < \varepsilon$ whenever $x, y \in A \cap G$.

So, $f|_A$ is cliquish.

Now we can formulate the following characterization of graph cliquish function on a dense set.

**Theorem 3.2:** Let $X$ be a Baire space and $f: X \to Y$ be given. For a cliquish function $g: X \to Y$ the following conditions are equivalent:

a) $G(g|_{C(g)}) \subseteq cl(G(f|_{C(g)})$

b) For any $\varepsilon > 0$, $A(f|_{C(g)}, g|_{C(g)}, \varepsilon)$ is dense in $X$.

**Proof:**

a) $\Rightarrow$ b):

It follows from the Result 3.4 and Theorem 3.1.

b) $\Rightarrow$ a):

Let $x_0 \in C(g)$, $U$ be an open neighbourhood of $x_0$ and $\varepsilon > 0$. It is sufficient to show that $[U \times S(g(x_0), \varepsilon)] \cap G(f|_{C(g)}) \neq \varnothing$. 
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Since \( g \) is continuous at \( x_0 \), there exists an open neighbourhood \( U_1 \) of \( x_0 \) such that \( U_1 \subseteq U \) and \( g(U_1) \subseteq S(g(x_0), \frac{\varepsilon}{2}) \).

Now \( A \left( f|_{C(g)}, g|_{C(g)}, \frac{\varepsilon}{2} \right) = \{ x \in C(g): d(f(x), g(x)) < \frac{\varepsilon}{2} \} \) is dense in \( X \).

So, \( U_1 \cap A \left( f|_{C(g)}, g|_{C(g)}, \frac{\varepsilon}{2} \right) \neq \emptyset \).

Choose \( x_1 \in U_1 \cap C(g) \) such that \( d(f(x_1), g(x_1)) < \frac{\varepsilon}{2} \).

Now, \( d(f(x_1), g(x_0)) \leq d(f(x_1), g(x_1)) + d(g(x_1), g(x_0)) < \varepsilon \).

So, \((x_1, f(x_1)) \in [U \times S(g(x_0), \varepsilon)] \cap G(f|_{C(g)})\).

**Theorem 3.3:** Let \( f:X \to Y \) be cliquish. Then for any \( \varepsilon > 0 \) the set \( B(f, g, \varepsilon) = \{ x \in X: d(f(x), g(x)) \geq \varepsilon \} \) is nowhere dense in \( X \) for any cliquish function \( g:X \to Y \) with \( G(g) \subseteq cl(G(f)) \).

**Proof:** Let \( \varepsilon > 0 \) and \( U \) be a non-empty open set in \( X \).

Let \( x_0 \in U \). Since \( g \) is cliquish at \( x_0 \), \( \exists \) a non-empty open set \( U_1 \subseteq U \) such that \( d(g(x), g(y)) < \frac{\varepsilon}{3} \) whenever \( x, y \in U_1 \).

Let \( x_1 \in U_1 \). Since \( f \) is cliquish at \( x_1 \), \( \exists \) a non-empty open set \( U_2 \subseteq U_1 \) such that \( d(f(x), f(y)) < \frac{\varepsilon}{3} \) whenever \( x, y \in U_2 \).

By Theorem 3.1, \( U_2 \cap A(f, g, \frac{\varepsilon}{3}) \neq \emptyset \).

Choose \( x_2 \in U_2 \) such that \( d(f(x_2), g(x_2)) < \frac{\varepsilon}{3} \).

Let \( x_3 \in U_2 \).

Then, \( d(f(x_3), g(x_3)) \leq d(f(x_3), f(x_2)) + d(f(x_2), g(x_2)) + d(g(x_2), g(x_3)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \).

So, \( x_3 \in X \setminus B(f, g, \varepsilon) \).

Hence, \( U_2 \cap B(f, g, \varepsilon) = \emptyset \).

Thus, \( B(f, g, \varepsilon) \) is nowhere dense in \( X \).
Corollary 3.1: If \( f: X \to Y, g: X \to Y \) are cliquish functions such that \( G(g) \subseteq cl(G(f)) \) then \( A(f, g, \varepsilon) \) is semi-open for any \( \varepsilon > 0 \).

It follows from the result that the complement of a no-where dense set is semi-open [1].

Theorem 3.4:
Let \( f: X \to Y \) be such that the set \( B(f, g, \varepsilon) \) is nowhere dense for any \( \varepsilon > 0 \) and for any cliquish function \( g: X \to Y \). Then \( f \) is cliquish on \( X \).

Proof: Let \( x_0 \in X, U \) be an open neighbourhood of \( x_0 \) and \( \varepsilon > 0 \).

Since \( g: X \to Y \) is cliquish at \( x_0 \), there exists a non-empty open set \( U_1 \subseteq U \) such that \( d(g(x), g(y)) < \frac{\varepsilon}{3} \) for \( x, y \in U_1 \).

As, \( B(f, g, \frac{\varepsilon}{3}) \) is nowhere dense, we can find a non-empty open set \( U_2 \subseteq U_1 \) such that
\[ U_2 \cap B(f, g, \frac{\varepsilon}{3}) = \emptyset \]

Then \( d(f(x), g(x)) < \frac{\varepsilon}{3} \) for \( x \in U_2 \).

Let \( x_1, x_2 \in U_2 \).

Then \( d(f(x_1), f(x_2)) \leq d(f(x_1), g(x_1)) + d(g(x_1), g(x_2)) + d(g(x_2), f(x_2)) < \varepsilon \).

Then \( f \) is cliquish.

Theorem 3.5: Let \( f: X \to Y \) and \( g: X \to Y \) be two cliquish functions such that \( G(g) \subseteq cl(G(f)) \).
Then the set \( \{ x \in X: f(x) \neq g(x) \} \) is of first category.

Proof:

Now, \( \{ x \in X: f(x) \neq g(x) \} = \bigcup_{n=1}^{\infty} B(f, g, \frac{1}{n}) \). The sets \( B(f, g, \frac{1}{n}) \) is nowhere dense by Theorem 3.3 and so the proof is completed.

Corollary 3.2: Let \( X \) be a Baire space. If \( f: X \to Y \) and \( g: X \to Y \) are cliquish functions such that \( G(g) \subseteq cl(G(f)) \) then the set \( \{ x \in X: f(x) = g(x) \} \) is dense in \( X \).

Now, \( W = \{ x \in X: f(x) = g(x) \} = X \setminus \{ x \in X: f(x) \neq g(x) \} \) is residual. Since \( X \) is a Baire space, \( W \) is dense in \( X \).
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CONFLICT OF INTERESTS
The author declares that there is no conflict of interests.

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