COMPOSITION OPERATORS IN SOME FUNCTION SPACES OF HYPERBOLIC TYPE

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Abstract. In this paper, we introduce natural metrics in the weighted hyperbolic \((\alpha, \omega)\)-Bloch class and \(F^*(p, q, s; \omega)\) classes. These classes are shown to be complete metric spaces with respect to the corresponding metrics. Moreover, Lipschitz continuous and boundedness of the composition operator \(C_{\phi}\) acting from the hyperbolic \((\alpha, \omega)\)-Bloch class to the classes \(F^*(p, q, s; \omega)\) are characterized by conditions depending on an analytic self-map \(\phi: D \to D\).

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1. Introduction

Let \(D := \{z \in \mathbb{C} : |z| < 1\}\) be the open unit disc of the complex plane \(\mathbb{C}\), \(\partial D\) it’s boundary. Let \(\mathcal{H}(D)\) denote the space of all analytic functions in \(D\) and let \(B(D)\) be the subset of \(\mathcal{H}(D)\) consisting of those \(f \in \mathcal{H}(D)\) for which \(|f(z)| < 1\) for all \(z \in D\). Also, \(dA(z)\) be the normalized area measure on \(D\) so that \(A(D) \equiv 1\).

Let the Green’s function of \(D\) be defined as \(g(z, a) = \log \frac{1}{|\varphi_a'(z)|^\alpha}\), where \(\varphi_a(z) = \frac{a-z}{1-\overline{a}z}\), for \(z, a \in D\) is the Möbius transformation related to the point \(a \in D\).
If \((X, d)\) is a metric space, we denote the open and closed balls with center \(x\) and radius \(r > 0\) by \(B(x, r) := \{y \in X : d(y, x) < r\}\) and \(\overline{B}(x, r) := \{y \in X : d(x, y) \leq r\}\), respectively.

Hyperbolic function classes are usually defined by using either the hyperbolic derivative 
\[ f^*(z) = \frac{|f'(z)|}{1 - |f(z)|^2} \]
of \(f \in B(\mathbb{D})\), or the hyperbolic distance 
\[ \rho(f(z), 0) := \frac{1}{2} \log\left(\frac{1 + |f(z)|}{1 - |f(z)|}\right) \]
between \(f(z)\) and zero.

A function \(f \in B(\mathbb{D})\) is said to belong to the hyperbolic \(\alpha\)-Bloch class \(B^*_\alpha\) if 
\[ \|f\|_{B^*_\alpha} = \sup_{z \in \mathbb{D}} f^*(z)(1 - |z|^2)^\alpha < \infty. \]
The little hyperbolic Bloch-type class \(B^*_{\alpha, 0}\) consists of all \(f \in B^*_\alpha\) such that 
\[ \lim_{|z| \to 1} f^*(z)(1 - |z|^2)^\alpha = 0. \]
The usual \(\alpha\)-Bloch spaces \(B_\alpha\) and \(B_{\alpha, 0}\) are defined as the sets of those \(f \in \mathcal{H}(\mathbb{D})\) for which 
\[ \|f\|_{B_\alpha} = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2)^\alpha < \infty, \]
and 
\[ \lim_{|z| \to 1} |f'(z)|(1 - |z|^2)^\alpha = 0, \]
respectively.

It is obvious that \(B^*_{\alpha, 0}\) is not a linear space since the sum of two functions in \(B(\mathbb{D})\) does not necessarily belong to \(B(\mathbb{D})\). From [5, 16, 17], we have the following:

For a given reasonable function \(\omega : (0, 1] \to (0, \infty)\) and for \(0 < \alpha < \infty\). An analytic function \(f\) on \(\mathbb{D}\) is said to belong to the \(\alpha\)-weighted Bloch space \(B^\omega_\alpha\) (see [16, 17]) if 
\[ \|f\|_{B^\omega_\alpha} = \sup_{z \in \mathbb{D}} \frac{(1 - |z|)^\alpha}{\omega(1 - |z|)} |f'(z)| < \infty. \]
Also, for a given reasonable function \(\omega : (0, 1] \to (0, \infty)\) and for \(0 < \alpha < \infty\). An analytic function \(f\) on \(\mathbb{D}\) is said to belong to the little weighted Bloch space \(B^{\omega, 0}_{\alpha, 0}\) if 
\[ \|f\|_{B^{\omega, 0}_{\alpha, 0}} = \lim_{|z| \to 1} \frac{(1 - |z|)^\alpha}{\omega(1 - |z|)} |f'(z)| = 0. \]
Throughout this paper and for some techniques we consider the case of \(\omega \neq 0\).

Now, we give the following definitions.
Definition 1.1. For a given reasonable function $\omega : (0, 1] \to (0, \infty)$ and for $0 < \alpha < \infty$. A function $f \in B(\mathbb{D})$ is said to belong to the $(\alpha, \omega)$-weighted hyperbolic Bloch space $B^*_{\omega,\alpha}$ if
\[
\|f\|_{B^*_\omega,\alpha} = \sup_{z \in \mathbb{D}} \frac{(1 - |z|)^\alpha}{\omega(1 - |z|)} (f^*(z)) < \infty.
\]
Also, for a given reasonable function $\omega : (0, 1] \to (0, \infty)$ and for $0 < \alpha < \infty$. A function $f \in B(\mathbb{D})$ is said to belong to the little weighted hyperbolic Bloch space $B^*_{\omega,\alpha,0}$ if
\[
\|f\|_{B^*_\omega,\alpha} = \lim_{|z| \to 1} \frac{(1 - |z|)^\alpha}{\omega(1 - |z|)} (f'(z)) = 0.
\]
We now turn to consider hyperbolic $F(p, q, s; \omega)$ type classes, which will be called $F^*(p, q, s; \omega)$. For $0 < p, s < \infty, -2 < q < \infty$, the hyperbolic class $F^*(p, q, s; \omega)$ consists of those functions $f \in B(\mathbb{D})$ for which
\[
\|f\|_{F^*_{p,q,s;\omega}} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (f^*(z))^p (1 - |z|^2)^q \frac{g^s(z, a)}{\omega(1 - |z|)} dA(z) < \infty.
\]
Moreover, we say that $f \in F^*(p, q, s)$ belongs to the class $F^*_0(p, q, s)$ if
\[
\lim_{|a| \to 1} \int_{\mathbb{D}} (f^*(z))^p (1 - |z|^2)^q \frac{g^s(z, a)}{\omega(1 - |z|)} dA(z) = 0.
\]
The usual general Besov-type spaces $F(p, q, s; \omega)$ (defined using the conventional derivative $f'$ instead of $f^*$) and their norms are denoted by the same symbols but with $f'$.

Yamashita was probably the first one considered systematically hyperbolic function classes. He introduced and studied hyperbolic Hardy, BMOA and Dirichlet classes in [21, 22, 23] and others. More recently, Smith studied inner functions in the hyperbolic little Bloch-class [18], and the hyperbolic counterparts of the $Q_p$ spaces were studied by Li in [12] and Li et. al. in [13]. Further, hyperbolic $Q_p$ classes and composition operators studied by Pérez-González et. al. in [15].

In this paper, we study the hyperbolic $\alpha$-Bloch classes $B^*_{\omega,\alpha}$ and the general hyperbolic $F(p, q, s; \omega)$ type classes. We will also give some results to characterize Lipschitz continuous and compact composition operators mapping from the hyperbolic $(\alpha, \omega)$-Bloch class $B^*_{\omega,\alpha}$ to $F^*(p, q, s; \omega)$ class by conditions depending on the symbol $\phi$ only.
Note that the general hyperbolic \( F(p, q, s; 1) \) type classes include the class of so-called \( Q_p^* \) classes and the class of (hyperbolic) Besov class \( B^*_p \). Thus, the results are generalizations of the recent results of Pérez-González, Rättyä and Taskinen [15].

For any holomorphic self-map \( \phi \) of \( \mathbb{D} \). The symbol \( \phi \) induces a linear composition operator \( C\phi(f) = f \circ \phi \) from \( \mathcal{H}(\mathbb{D}) \) or \( B(\mathbb{D}) \) into itself. The study of composition operator \( \phi \) acting on spaces of analytic functions has engaged many analysts for many years (see e.g. [2, 3, 10, 11, 13, 14, 19, 24] and others).

Recall that a linear operator \( T : X \to Y \) is said to be bounded if there exists a constant \( C > 0 \) such that \( \|T(f)\|_Y \leq C\|f\|_X \) for all maps \( f \in X \). By elementary functional analysis, a linear operator between normed spaces is bounded if and only if it is continuous, and the boundedness is trivially also equivalent to the Lipschitz-continuity. Moreover, \( T : X \to Y \) is said to be compact if it takes bounded sets in \( X \) to sets in \( Y \) which have compact closure. For Banach spaces \( X \) and \( Y \) contained in \( B(\mathbb{D}) \) or \( \mathcal{H}(\mathbb{D}) \), \( T : X \to Y \) is compact if and only if for each bounded sequence \( \{x_n\} \in X \), the sequence \( \{Tx_n\} \in Y \) contains a subsequence converging to a function \( f \in Y \).

**Definition 1.2.** A composition operator \( C\phi : B^*_{\alpha} \to F^*(p, q, s; \omega) \) is said to be bounded, if there is a positive constant \( C \) such that \( \|C\phi f\|_{F^*(p, q, s; \omega)} \leq C\|f\|_{B^*_{\alpha}} \) for all \( f \in B^*_{\alpha} \).

**Definition 1.3.** A composition operator \( C\phi : B^*_{\alpha} \to F^*(p, q, s; \omega) \) is said to be compact, if it maps any ball in \( B^*_{\alpha} \) onto a precompact set in \( F^*(p, q, s; \omega) \).

The following lemma follows by standard arguments similar to those outline in [19]. Hence we omit the proof.

**Lemma 1.1.** Assume \( \phi \) is a holomorphic mapping from \( \mathbb{D} \) into itself. Let \( 0 < p, s < \infty, -1 < q < \infty \) and \( 0 < \alpha < \infty \). Then \( C\phi : B^*_{\alpha} \to F^*(p, q, s; \omega) \) is compact if and only if for any bounded sequence \( \{f_n\}_{n \in \mathbb{N}} \in B^*_{\alpha} \) which converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( n \to \infty \), we have \( \lim_{n \to \infty} \|C\phi f_n\|_{F^*(p, q, s; \omega)} = 0 \).

The following lemma can be found in [20], Theorem 2.1.1.

**Lemma 1.2.** Let \( 0 < \alpha < \infty \). Then there are two holomorphic maps \( f, g : \mathbb{D} \to \mathbb{C} \) such that \( |f'(z)| + |g'(z)| \approx (1 - |z|^2)^{-\alpha}, \ z \in \mathbb{D} \).
2. Natural metrics

In this section, we introduce natural metrics on the hyperbolic \((\alpha; \omega)\)-Bloch classes \(B^*_{\omega,\alpha}\) and the classes \(F^*(p, q, s; \omega)\).

Let \(0 < p, s < \infty, -2 < q < \infty\) and \(0 < \alpha < 1\). First, we can find a natural metric in \(B^*_{\omega,\alpha}\) by defining

\[
d(f, g; B^*_{\omega,\alpha}) := d_{B^*_{\omega,\alpha}}(f, g) + \|f - g\|_{B_{\omega,\alpha}} + |f(0) - g(0)|,
\]

where

\[
d_{B^*_{\omega,\alpha}}(f, g) := \sup_{z \in \mathbb{D}} \left| \frac{f'(z)}{1 - |f(z)|^2} - \frac{g'(z)}{1 - |g(z)|^2} \right| \frac{(1 - |z|^2)^\alpha}{\omega(1 - |z|)}.
\]

for \(f, g \in B^*_{\omega,\alpha}\). The presence of the conventional \((\alpha; \omega)\)-Bloch-norm here perhaps unexpected. It is motivated as the example see [15] (see [15], Example in Section 7). It shows the phenomenon that, though trivially \(d_{B^*_{\omega,\alpha}}(f, 0) \geq \|f\|_{B_{\omega,\alpha}}\) for all \(f \in B^*_{\omega,\alpha}\), the same does no more hold for the differences of two functions: there does not even exist a constant \(C > 0\) such that

\[
\sup_{z \in \mathbb{D}} \left| \frac{f'(z)}{1 - |f(z)|^2} - \frac{g'(z)}{1 - |g(z)|^2} \right| \frac{(1 - |z|^2)^\alpha}{\omega(1 - |z|)} \geq C\|f - g\|_{B_{\omega,\alpha}}
\]

hold for all \(f, g \in B^*_{\omega,\alpha}, 0 < \alpha < 1\). For \(f, g \in F^*(p, q, s; \omega)\), define their distance by

\[
d(f, g; F^*(p, q, s; \omega)) := d_{F^*}(f, g) + \|f - g\|_{F(p, q, s; \omega)} + |f(0) - g(0)|,
\]

where

\[
d_{F^*}(f, g) := \left( \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \left| \frac{f'(z)}{1 - |f(z)|^2} - \frac{g'(z)}{1 - |g(z)|^2} \right|^p \frac{(1 - |z|^2)^q g^*(z, a) dA(z)}{\omega(1 - |z|)} \right)^{\frac{1}{p}}.
\]

The following characterization of complete metric space \(d(., .; B^*_{\omega,\alpha})\) can be proved as the corresponding result in [15].

**Proposition 2.1.** The class \(B^*_{\omega,\alpha}\) equipped with the metric \(d(., .; B^*_{\omega,\alpha})\) is a complete metric space. Moreover, \(B^*_{\omega,\alpha,0}\) is a closed (and therefore complete) subspace of \(B^*_{\omega,\alpha}\).

Now we are in a position to prove the following proposition.
Proposition 2.2. The class $F^*(p, q, s; \omega)$ equipped with the metric $d(., .; F^*(p, q, s; \omega))$ is a complete metric space. Moreover, $F_0^*(p, q, s; \omega)$ is a closed (and therefore complete) subspace of $F^*(p, q, s; \omega)$.

Proof. For $f, g, h \in F^*(p, q, s; \omega)$, we have

- $d(f, g; F^*(p, q, s; \omega)) \geq 0$,
- $d(f, f; F^*(p, q, s; \omega)) = 0$,
- $d(f, g; F^*(p, q, s; \omega)) = 0$ implies $f = g$.
- $d(f, g; F^*(p, q, s; \omega)) = d(g, f; F^*(p, q, s; \omega))$,
- $d(f, h; F^*(p, q, s; \omega)) \leq d(f, g; F^*(p, q, s; \omega)) + d(g, h; F^*(p, q, s; \omega))$.

Hence, $d$ is metric on $F^*(p, q, s; \omega)$. For the completeness proof, let $(f_n)_{n=0}^{\infty}$ be a Cauchy sequence in the metric space $F^*(p, q, s; \omega)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ such that $d(f_n, f_m) < \varepsilon$, for all $n, m > N$. Since $f_n \in B(\mathbb{D})$ such that $f_n$ converges to $f$ uniformly on compact subsets of $\mathbb{D}$. Let $m > N$ and $0 < r < 1$. In view of Fatou’s lemma, we find that

$$
\int_{D(0,r)} \left| \frac{f'(z)}{1 - |f(z)|^2} - \frac{f'_m(z)}{1 - |f_m(z)|^2} \right|^p \frac{(1 - |z|^2)^q g^*(z, a)}{\omega(1 - |z|)} dA(z)
$$

$$
= \int_{D(0,r)} \lim_{n \to \infty} \left| \frac{f'_n(z)}{1 - |f_n(z)|^2} - \frac{f'_m(z)}{1 - |f_m(z)|^2} \right|^p \frac{(1 - |z|^2)^q g^*(z, a)}{\omega(1 - |z|)} dA(z)
$$

$$
\leq \lim_{n \to \infty} \int_{D} \left| \frac{f'_n(z)}{1 - |f_n(z)|^2} - \frac{f'_m(z)}{1 - |f_m(z)|^2} \right|^p \frac{(1 - |z|^2)^q g^*(z, a)}{\omega(1 - |z|)} dA(z) \leq \varepsilon^p.
$$

Letting $r \to 1^-$, we obtain that

$$
\int_{D} (f^*(z))^p \frac{(1 - |z|^2)^q g^*(z, a)}{\omega(1 - |z|)} dA(z) \leq 2^p \varepsilon^p + 2^p \int_{D} (f^*_m(z))^p \frac{(1 - |z|^2)^q g^*(z, a)}{\omega(1 - |z|)} dA(z).
$$

(2)

This yields $\|f\|_{F^*(p, q, s; \omega)}^p \leq 2^p \varepsilon^p + 2^p \|f_m\|_{F^*(p, q, s; \omega)}^p$, thus $f \in F^*(p, q, s; \omega)$. We also find that $f_n \to f$ with respect to the metric of $F^*(p, q, s; \omega)$. The second part of the assertion follows by (1).

3. Lipschitz continuity
For \(0 < p, s < \infty, -2 < q < \infty\) and \(0 < \alpha < \infty\). We define the following notations:

\[
\Phi_{\phi}(p, q, s, a; \omega) = \int_{D} \frac{|\phi'(z)|^{p} (1 - |z|^{2})^{q} g^{s}(z, a)}{(1 - |\phi(z)|^{2})^{op} \omega(1 - |z|)} dA(z)
\]

and

\[
\Omega_{\phi, r}(p, q, s, a; \omega) = \int_{|\phi| \geq r} \frac{|\phi'(z)|^{p} (1 - |z|^{2})^{q} g^{s}(z, a)}{(1 - |\phi(z)|^{2})^{op} \omega(1 - |z|)} dA(z).
\]

**Theorem 3.1.** Let \(\omega : (0, 1] \rightarrow (0, \infty)\) and assume \(\phi\) is a holomorphic mapping from \(\mathbb{D}\) into itself. Let \(0 \leq p < \infty, 0 \leq s \leq 1, -1 < q < \infty\) and \(0 < \alpha \leq 1\). Then the following are equivalent:

(i) \(C_{\phi} : B_{\omega, \alpha}^{*} \rightarrow F^{*}(p, q, s; \omega)\) is bounded;

(ii) \(C_{\phi} : B_{\omega, \alpha}^{*} \rightarrow F^{*}(p, q, s; \omega)\) is Lipschitz continuous;

(iii) \(\sup_{a \in D} \Phi_{\phi}(p, q, s, a; \omega) < \infty\).

**Proof.** First, assume that (i) holds, then there exists a constant \(C\) such that

\[
\|C_{\phi}f\|_{F^{*}(p, q, s; \omega)} \leq C\|f\|_{B_{\omega, \alpha}^{*}}, \quad \text{for all } f \in B_{\omega, \alpha}^{*}.
\]

For given \(f \in B_{\omega, \alpha}^{*}\), the function \(f_{t}(z) = f(tz)\), where \(0 < t < 1\), belongs to \(B_{\omega, \alpha}^{*}\) with the property \(\|f_{t}\|_{B_{\omega, \alpha}^{*}} \leq \|f\|_{B_{\omega, \alpha}^{*}}\). Let \(f, g\) be the functions from Lemma 1.1, such that

\[
\omega(1 - |z|) \leq f^{*}(z) + g^{*}(z),
\]

for all \(z \in \mathbb{D}\). It follows that

\[
\frac{|\phi'(z)|}{(1 - |\phi(z)|)^{\alpha}} \leq (f \circ \phi)^{*}(z) + (g \circ \phi)^{*}(z).
\]

Thus, we have

\[
\int_{D} \frac{|t\phi'(z)|^{p} (1 - |z|^{2})^{q} g^{s}(z, a)}{(1 - |t\phi(z)|^{2})^{op} \omega(1 - |z|)} dA(z)
\]

\[
\leq 2^{p} \int_{D} \left[ (f \circ t\phi)^{*}(z) + (g \circ t\phi)^{*}(z) \right]^{p} \frac{(1 - |z|^{2})^{q} g^{s}(z, a)}{\omega(1 - |z|)} dA(z)
\]

\[
\leq 2^{p} \|C_{\phi}\|^{p} \left( \|f\|^{p}_{B_{\omega, \alpha}^{*}} + \|g\|^{p}_{B_{\omega, \alpha}^{*}} \right).
\]

This estimate together with the Fatou’s lemma implies (iii).
Conversely, assuming that (iii) holds and that \( f \in \mathcal{B}^*_{\omega,\alpha} \), we see that
\[
\sup_{a \in D} \int_D \left( (f \circ \phi)^*(z) \right)^p \frac{(1 - |z|^2)^q g^*(z, a)}{\omega(1 - |z|)} dA(z) \\
= \sup_{a \in D} \int_D (f^*(\phi(z)))^p |\phi'(z)|^p \frac{(1 - |z|^2)^q g^*(z, a)}{\omega(1 - |z|)} dA(z) \\
\leq \|f\|^p \sup_{a \in D} \int_D \frac{|\phi'(z)|^p (1 - |z|^2)^q g^*(z, a)}{(1 - |\phi(z)|^2)^{\alpha p}} \omega(1 - |z|) dA(z).
\]

Hence, it follows that (i) holds.

(ii) \iff (iii). Assume first that \( C_{\phi} : \mathcal{B}^*_{\omega,\alpha} \to F^*(p, q, s; \omega) \) is Lipschitz continuous, that is, there exists a positive constant \( C \) such that
\[
d(f \circ \phi, g \circ \phi; F^*(p, q, s; \omega)) \leq Cd(f, g; \mathcal{B}^*_{\omega,\alpha}), \quad \text{for all } f, g \in \mathcal{B}^*_{\omega,\alpha}.
\]
Taking \( g = 0 \), we find
\[
\|f \circ \phi\|_{F^*(p, q, s; \omega)} \leq C (\|f\|_{\mathcal{B}^*_{\omega,\alpha}} + \|f\|_{\mathcal{B}_{\omega,\alpha}} + |f(0)|), \quad \text{for all } f \in \mathcal{B}^*_{\omega,\alpha}.
\] (3)
The assertion (iii) for \( \alpha = 1 \) follows by choosing \( f(z) = z \) in (3). If \( 0 < \alpha < 1 \), then
\[
|f(z)| = \left| \int_0^z f'(s) ds + f(0) \right| \leq \|f\|_{\mathcal{B}_{\omega,\alpha}} \int_0^{|z|} \frac{dx}{(1 - x^2)^\alpha} + |f(0)| \\
\leq \frac{\|f\|_{\mathcal{B}_{\omega,\alpha}}}{(1 - \alpha)} + |f(0)|,
\]
which yields
\[
|f(\phi(0)) - g(\phi(0))| \leq \frac{\|f - g\|_{\mathcal{B}_{\omega,\alpha}}}{(1 - \alpha)} + |f(0) - g(0)|.
\]
Moreover, Lemma 1.1. implies the existence of \( f, g \in \mathcal{B}^*_{\omega,\alpha} \) such that
\[
(f'(z) + g'(z))(1 - |z|^2)^\alpha \geq C > 0, \quad \text{for all } z \in \mathbb{D}.
\] (4)
Combining (3) and (4), we obtain
\[
\|f\|_{\mathcal{B}_{\omega,\alpha}} + \|g\|_{\mathcal{B}_{\omega,\alpha}} + \|f\|_{\mathcal{B}_{\omega,\alpha}} + \|g\|_{\mathcal{B}_{\omega,\alpha}} + |f(0)| + |g(0)| \\
\geq C \int_0^{|z|} \frac{|\phi'(z)|^p (1 - |z|^2)^q g^*(z, a)}{(1 - |\phi(z)|^2)^{\alpha p}} \omega(1 - |z|) dA(z) \\
\geq C \Phi_{\phi}(\alpha, p, q, s, a; \omega),
\]
for which the assertion (iii) follows.
Assume now that (iii) is satisfied, we have

\[
d(f \circ \phi, g \circ \phi; F^*(p, q, s; \omega))
\]

\[
= d_{F^*}(f \circ \phi, g \circ \phi) + \|f \circ \phi - g \circ \phi\|_{F(p, q, s; \omega)} + |f(\phi(0)) - g(\phi(0))|
\]

\[
\leq d_{B^*_{\omega, \alpha}}(f, g)\left(\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{\left|\phi'(z)\right|^p}{(1 - |\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q g^*(z, a) \, dA(z)\right)^\frac{1}{p}
\]

\[
+ \|f - g\|_{B^*_{\omega, \alpha}}\left(\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{\left|\phi'(z)\right|^p}{(1 - |\phi(z)|^2)^{\alpha p}} (1 - |z|^2)^q g^*(z, a) \, dA(z)\right)^\frac{1}{p}
\]

\[
+ \frac{\|f - g\|_{B^*_{\omega, \alpha}}}{(1 - \alpha)} + |f(0) - g(0)|
\]

\[
\leq C'd(f, g; B^*_{\omega, \alpha}).
\]

Thus \(C_{\phi} : B^*_{\omega, \alpha} \to F^*(p, q, s; \omega)\) is Lipschitz continuous and the proof is completed.

**Remark 3.1.** Theorem 3.1 shows that \(C_{\phi} : B^*_{\omega, \alpha} \to F^*(p, q, s; \omega)\) is bounded if and only if it is Lipschitz-continuous, that is, if there exists a positive constant \(C\) such that

\[
d(f \circ \phi, g \circ \phi; F^*(p, q, s; \omega)) \leq Cd(f, g; B^*_{\omega, \alpha}), \quad \text{for all } f, g \in B^*_{\omega, \alpha}.
\]

By elementary functional analysis, a linear operator between normed spaces is bounded if and only if it is continuous, and the boundedness is trivially also equivalent to the Lipschitz-continuity. So, our result for composition operators in hyperbolic spaces is the correct and natural generalization of the linear operator theory.

The following observation is sometimes useful.

**Proposition 3.1.** Assume \(\phi\) is a holomorphic mapping from \(\mathbb{D}\) into itself. Let \(0 < p < \infty, -1 < q < \infty\) and \(0 < \alpha < \infty\). If \(C_{\phi} : B^*_{\omega, \alpha} \to F^*(p, q, s; \omega)\) is compact, it maps closed balls onto compact sets.

**Proof.** If \(B \subset B^*_{\omega, \alpha}\) is a closed ball and \(g \in F^*(p, q, s; \omega)\) belongs to the closure of \(C_{\phi}(B)\), we can find a sequence \((f_n)_{n=1}^\infty \subset B\) such that \(f_n \circ \phi\) converges to \(g \in F^*(p, q, s; \omega)\) as \(n \to \infty\). But \((f_n)_{n=1}^\infty\) is a normal family, hence it has a subsequence \((f_{n_j})_{j=1}^\infty\) converging uniformly on the compact subsets of \(\mathbb{D}\) to an analytic function \(f\). As in earlier arguments of Proposition 2.1 in [15], we get a positive estimate which shows that \(f\) must belong to the closed ball \(B\). On the other hand, also the sequence \((f_{n_j} \circ \phi)_{j=1}^\infty\) converges uniformly
on compact subsets to an analytic function, which is \( g \in F^*(p, q, s; \omega) \). We get \( g = f \circ \phi \), i.e. \( g \) belongs to \( C_\phi(B) \). Thus, this set is closed and also compact.

**Remark 3.2.** It is still an open problem to extend the results of this paper by using generalized hyperbolic derivative as defined in [1].

**Remark 3.3.** It is still an open problem to extend the results of this paper to the classes \( Q_K(p, q) \) and \( Q_{K, \omega}(p, q) \) of hyperbolic functions. For some studies on analytic or meromorphic \( Q_{K, \omega}(p, q) \) and \( Q_K(p, q) \) classes, we refer to [4, 5, 6, 7, 8, 9, 16, 17].

**References**


