COUPLED FIXED POINT THEOREM FOR WEAK COMPATIBLE MAPPINGS IN MENGER SPACES

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Abstract: In this paper, we prove coupled fixed point theorems for a pair of weakly compatible mappings under φ-contractive conditions in Menger spaces without appeal to continuity of mappings. We support our result by providing a suitable example. At the end, we give an application of our result.

Keywords: Weakly compatible maps; Menger space, t-norm of H-type.

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1. Introduction

In 1942 Menger [7] introduced the notion of a probabilistic metric space (PM-space) which is in fact, a generalization of metric space. The idea in probabilistic metric space is to associate a distribution function with a point pair, say (p, q), denoted by \( F(p, q, t) \) where \( t > 0 \) and interpret this function as the probability that distance between p and q is less than t, whereas in the metric space the distance function is a single positive number. Sehgal [9] initiated the study of fixed points in probabilistic metric spaces. The study of these spaces was expanded rapidly with the pioneering works of Schweizer-Sklar [11].

In 1991, Mishra[8] introduced the notion of compatible mappings in the
setting of probabilistic metric space. In 1996, Jungck [5] introduce the notion of weakly compatible mappings as follows:

Two self mappings S and T are said to be weakly compatible if they commute at their coincide points, i.e., \( Tu = Su \) for some \( u \in X \), then \( TSu = STu \).

Further, Singh and Jain [10] proved some results for weakly compatible in Menger spaces. Fang [3] defined \( \phi \)-contractive conditions and proved some fixed point theorems under \( \phi \)-contractions for compatible and weakly compatible maps in Menger PM-spaces using t-norm of H-type, introduced by Hadžić [4]. Recently, Bhaskar and Lakshmikantham [2], Lakshmikantham and Ćirić [6] gave some coupled fixed point theorems in partially ordered metric spaces.

Now, we prove a coupled fixed point theorem for a pair of weakly compatible maps satisfying \( \phi \)-contractive conditions in Menger PM-space with a continuous t-norm of H-type. We support our result by an example. At the end, we give an application of our result.

2. Preliminaries

First, recall that a real valued function \( f \) defined on the set of real numbers is known as a distribution function if it is non-decreasing, continuous and inf. \( f(x) = 0 \), sup. \( f(x) = 1 \). In what follows \( H(x) \) denotes the distribution function defined as follows:

\[
H(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
1 & \text{if } x > 0.
\end{cases}
\]

**Definition 2.1.** A probabilistic metric space (PM-space) is a pair \((X, F)\) where \( X \) is a set and \( F \) is a function defined on \( X \times X \) into the set of distribution functions such that if \( x, y \) and \( z \) are points of \( X \), then

(F-1) \( F(x, y; 0) = 0 \),

(F-2) \( F(x, y; t) = H(t) \) iff \( x = y \),

(F-3) \( F(x, y; t) = F(y, x; t) \),

(F-4) if \( F(x, y; s) = 1 \) and \( F(y, z; t) = 1 \), then \( F(x, z; s+t) = 1 \) for all \( x, y, z \in X \) and \( s, t \geq 0 \).

For each \( x \) and \( y \) in \( X \) and for each real number \( t \geq 0 \), \( F(x, y; t) \) is to be thought
of as the probability that the distance between $x$ and $y$ is less than $t$.

It is interesting to note that, if $(X, d)$ is a metric space, then the distribution function $F(x, y; t)$ defined by the relation $F(x, y; t) = H(t - d(x, y))$ induces a PM-space.

**Definition 2.2.** A t-norm $t$ is a 2-place function, $t : [0,1] \times [0,1] \to [0,1]$ satisfying the following:

(i) $t(0,0) = 0$,
(ii) $t(0,1) = 1$,
(iii) $t(a,b) = t(b,a)$,
(iv) if $a \leq c$, $b \leq d$, then $t(a,b) \leq t(c,d)$,
(v) $t(t(a,b),c) = t(a,t(b,c))$ for all $a, b, c$ in $[0,1]$.

**Definition 2.3.** A Menger PM-space is a triplet $(X, F, t)$ where $(X, F)$ is a PM-space and $t$ is a t-norm with the following condition:

(F-5) $F(x, z; s + t) \geq t(F(x, y; s), F(y, z; t))$, for all $x, y, z$ in $X$ and $s, t \geq 0$.

This inequality is known as Menger’s triangle inequality.

In our theory, we consider $(X, F, t)$ to be a Menger PM-space along with the following condition:

(F-6) $\lim_{t \to \infty} F(x, y, t) = 1$, for all $x, y$ in $X$.

**Definition 2.4[4].** Let $\sup_{0 < t < 1} \Delta(t, t) = 1$. A t-norm $\Delta$ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$\Delta^1(t) = t, \quad \Delta^{m+1}(t) = t \Delta(\Delta^m(t)), \quad m = 1, 2, \ldots, \quad t \in [0, 1].$$

The t-norm $\Delta_M = \min.$ is an example of t-norm of H-type.

**Remark 2.1.** $\Delta$ is a H-type t-norm iff for any $\lambda \in (0, 1)$, there exists $\delta(\lambda) \in (0, 1)$ such that $\Delta^m(t) > (1-\lambda)$ for all $m \in \mathbb{N}$, when $t > (1-\delta)$.

**Definition 2.5.** A sequence $\{x_n\}$ in a Menger PM space $(X, F, t)$ is said

(i) to converge to a point $x$ in $X$ if for every $\varepsilon > 0$ and $\lambda > 0$, there is an integer $n_0$ such that $F(x_n, x, \varepsilon) > 1 - \lambda$, for all $n \geq n_0$.

(ii) to be Cauchy if for each $\varepsilon > 0$ and $\lambda > 0$, there is an integer $n_0$ such that $F(x_n, x_m, \varepsilon) > 1 - \lambda$, for all $n, m \geq n_0$. 
[(iii) to be complete if every Cauchy sequence in it converges to a point of it.\]

**Definition 2.6[3].** Define $\Phi = \{ \phi : R^+ \to R^+ \}$, where $R^+ = [0, +\infty)$ and each $\phi \in \Phi$ satisfies the following conditions:

(\phi-1) $\phi$ is non-decreasing;

(\phi-2) $\phi$ is upper semicontinuous from the right;

(\phi-3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi(\phi^n(t))$, $n \in \mathbb{N}$. Clearly, if $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$.

**Definition 2.7[3].** An element $x \in X$ is called a common fixed point of the mappings

$f: X \times X \to X$ and $g: X \to X$ if

$$x = f(x, x) = g(x).$$

**Definition 2.8[6].** An element $(x, y) \in X \times X$ is called a

(i) coupled fixed point of the mapping $f: X \times X \to X$ if

$$f(x, y) = x, \quad f(y, x) = y.$$  

(ii) coupled coincidence point of the mappings $f: X \times X \to X$ and $g: X \to X$ if

$$f(x, y) = g(x), \quad f(y, x) = g(y).$$

(iii) common coupled fixed point of the mappings $f: X \times X \to X$ and $g: X \to X$ if

$$x = f(x, y) = g(x), \quad y = f(y, x) = g(y).$$

**Definition 2.9[3].** The mappings $f: X \times X \to X$ and $g: X \to X$ are called commutative if

$$gf(x, y) = f(gx, gy),$$

for all $x, y \in X$.

Abbas, Khan and Redenović [1] introduced the notion of $w$-compatible mappings as follows:

The mappings $f : X \times X \to X$ and $g : X \to X$ are called $w$-compatible if

$$g(f(x, y)) = f(gx, gy) \text{ whenever } g(x) = f(x, y) \text{ and } g(y) = f(y, x).$$

**Definition 2.10.** The maps $f: X \times X \to X$ and $g: X \to X$ are called weakly compatible if

$f(x, y) = g(x), f(y, x) = g(y)$ implies $gf(x, y) = f(gx, gy), gf(y, x) = f(gy, gx)$, for all $x, y$ in $X$. 
3. Main results

For convenience, we denote

\[(3.1) \quad [F(x, y, t)]^n = \frac{F(x, y, t) \ast F(x, y, t) \ast \ldots \ast F(x, y, t)}{n}, \quad \text{for all } n \in \mathbb{N}.\]

Now we prove our main result.

**Theorem 3.1.** Let \((X, F, \ast)\) be Menger PM-Space, \(*\) being continuous \(t\)–norm of H-type. Let \(f: X \times X \to X\) and \(g: X \to X\) be two mappings and there exists \(\phi \in \Phi\) such that

\[(3.2) \quad F(f(x, y), f(u, v), \phi(t)) \geq \psi[F(gx, gu, t) \ast F(gy, gv, t)], \quad \text{for all } x, y, u, v \in X \text{ and } t > 0,
\]

where \(\psi: [0, 1] \to [0, 1]\) is a continuous function such that \(\psi(t) \geq t\) for all \(t \in [0, 1]\).

Suppose that \(f(X \times X) \subseteq g(X)\), \(f\) and \(g\) are weakly compatible, range space of one of the maps \(f\) or \(g\) is complete. Then \(f\) and \(g\) have a coupled coincidence point.

Moreover, there exists a unique point \(x\) in \(X\) such that \(x = f(x, x) = g(x)\).

**Proof.**

Let \(x_0, y_0\) be two arbitrary points in \(X\). Since \(f(X \times X) \subseteq g(X)\), we can choose \(x_1, y_1\) in \(X\) such that \(g(x_1) = f(x_0, y_0), g(y_1) = f(y_0, x_0)\).

Continuing in this way we can construct two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that \(g(x_{n+1}) = f(x_n, y_n)\) and \(g(y_{n+1}) = f(y_n, x_n), \text{ for all } n \geq 0.\)

**Step 1.** We first show that \(\{gx_n\}\) and \(\{gy_n\}\) are Cauchy sequences.

Since \(*\) is a \(t\)-norm of H-type, for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that

\[(3.3) \quad \left(1 - \delta\right) \ast \left(1 - \delta\right) \ast \ldots \ast \left(1 - \delta\right) \geq (1 - \varepsilon), \quad \text{for all } p \in \mathbb{N}.\]

Since \(\lim_{t \to \infty} F(x, y, t) = 1\), for all \(x, y\) in \(X\), there exists \(t_0 > 0\) such that

\[F(gx_0, gx_1, t_0) \geq (1 - \delta) \quad \text{and} \quad F(gy_0, gy_1, t_0) \geq (1 - \delta).\]

Also, since \(\phi \in \Phi\), using condition (\(\phi\)-3), we have \(\sum_{n=1}^{\infty} \phi^n(t_0) < \infty\). Then for any \(t > 0\), there exists \(n_0 \in \mathbb{N}\) such that

\[(3.4) \quad t > \sum_{k=n_0}^{\infty} \phi^k(t_0).\]

Using condition (3.2), we have

\[F(gx_1, gx_2, \phi(t_0)) = F(f(x_0, y_0), f(x_1, y_1), \phi(t_0)) \geq \psi[F(gx_0, gx_1, t_0) \ast F(gy_0, gy_1, t_0)]\]
Therefore, \( F(gx_0, gx_1, t_0) \geq F(gx_0, gy_1, t_0) \).

\[
F(gy_1, gy_2, \phi(t_0)) = F(f(y_0, x_0), f(y_1, x_1), \phi(t_0))
\geq \psi [F(gy_0, gy_1, t_0) \ast F(gx_0, gx_1, t_0)]
\geq F(gy_0, gy_1, t_0) \ast F(gx_0, gx_1, t_0).
\]

Similarly, we can also get

\[
F(gx_2, gx_3, \phi^2(t_0)) = F(f(x_1, y_1), f(x_2, y_2), \phi^2(t_0))
\geq \psi [F(gx_1, gx_2, \phi(t_0)) \ast F(gy_1, gy_2, \phi(t_0))]
\geq F(gx_1, gx_2, \phi(t_0)) \ast F(gy_1, gy_2, \phi(t_0))
\geq [F(gx_0, gx_1, t_0)]^2 \ast [F(gy_0, gy_1, t_0)]^2.
\]

Continuing in this way, we can get

\[
F(gx_n, gx_{n+1}, \phi^n(t_0)) \geq [F(gx_0, gx_1, t_0)]^{2^{n-1}} \ast [F(gy_0, gy_1, t_0)]^{2^{n-1}}.
\]

\[
F(gy_0, gy_{n+1}, \phi^n(t_0)) \geq [F(gy_0, gy_1, t_0)]^{2^{n-1}} \ast [F(gx_0, gx_1, t_0)]^{2^{n-1}}.
\]

So, from (3.3) and (3.4), for \( m > n \geq n_0 \), we have

\[
F(gx_n, gx_m, t)
\geq F(gx_n, gx_m, \sum_{k=n_0}^{\infty} \phi^k(t_0))
\geq F(gx_n, gx_m, \sum_{k=n_0}^{\infty} \phi^k(t_0))
\geq F(gx_n, gx_{n+1}, \phi^n(t_0)) \ast F(gx_{n+1}, gx_{n+2}, \phi^{n+1}(t_0)) \ast \ldots \ast F(gx_{m-1}, gx_m, \phi^{m-1}(t_0))
\geq \left[ [F(gx_0, gx_1, t_0)]^{2^{n-1}} \ast [F(gy_0, gy_1, t_0)]^{2^{n-1}} \right] \ast \ldots \ast \left[ [F(gx_0, gx_1, t_0)]^{2^{m-1}} \ast [F(gy_0, gy_1, t_0)]^{2^{m-1}} \right]
\ast \ldots \ast \left[ [F(gx_0, gx_1, t_0)]^{2^{m-2}} \ast [F(gy_0, gy_1, t_0)]^{2^{m-2}} \right]
= [F(gx_0, gx_1, t_0)]^{2^{n-1}(2^{m-n-1})} \ast [F(gy_0, gy_1, t_0)]^{2^{n-1}(2^{m-n-1})}
\geq (1 - \delta) \ast (1 - \delta) \ast \ldots \ast (1 - \delta) \geq (1 - \epsilon), \text{which implies that}
\]

\[
F(gx_n, gx_m, t) \geq (1 - \epsilon), \text{for all } m, n \in N \text{ with } m > n \geq n_0 \text{ and } t > 0.
\]

Therefore, \( \{gx_n\} \) is a Cauchy sequence. Similarly, we can get that \( \{gy_n\} \) is a Cauchy
sequence.

**Step 2.** To show that f and g have a coupled coincidence point.

Without loss of generality, one can assume that g(X) is complete, then there exists points x, y in g(X) so that \( \lim_{n \to \infty} g(x_{n+1}) = x, \lim_{n \to \infty} g(y_{n+1}) = y. \)

For x, y \( \in \) g(X) implies the existence of p, q in X such that g(p) = x, g(q) = y and hence \( \lim_{n \to \infty} g(x_{n+1}) = \lim_{n \to \infty} f(x_n, y_n) = g(p) = x, \)
\( \lim_{n \to \infty} g(y_{n+1}) = \lim_{n \to \infty} f(y_n, x_n) = g(q) = y. \)

From (3.2), we have
\[
F(f(x_n, y_n), f(p, q), \phi(t)) \geq \psi[F(gx_n, g(p), t) * F(gy_n, g(q), t)] \\
\geq F(gx_n, g(p), t) * F(gy_n, g(q), t).
\]

Taking limit as \( n \to \infty \), we get
\[ F(g(p), f(p, q), \phi(t)) = 1 \] that is, f(p, q) = g(p) = x.

Similarly, f(q, p) = g(q) = y.

But f and g are weakly compatible, so that f(p, q) = g(p) = x and f(q, p) = g(q) = y implies gf(p, q) = f(g(p), g(q)) and gf(q, p) = f(g(q), g(p)), that is g(x) = f(x, y) and g(y) = f(y, x).

Hence f and g have a coupled coincidence point.

**Step 3.** To show that g(x) = x and g(y) = y.

Since * is a \( t \)-norm of H-type, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
\left(1 - \delta\right) \ast \left(1 - \delta\right) \ast \ldots \ast \left(1 - \delta\right) \geq (1 - \epsilon), \quad \text{for all } p \in \mathbb{N}.
\]

Since \( \lim_{t \to \infty} F(x, y, t) = 1 \), for all x, y in X, there exists \( t_0 > 0 \) such that
\[ F(gx, x, t_0) \geq (1 - \delta) \text{ and } F(gy, y, t_0) \geq (1 - \delta). \]

Also, since \( \phi \in \Phi \), using condition \( (\phi-3) \), we have \( \sum_{n=1}^{\infty} \phi^n(t_0) < \infty \).

Then for any \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that
\[ t > \sum_{k=n_0}^{\infty} \phi^k(t_0). \]

From (3.2), we have
\[
F(gx, x, \phi(t_0)) = F(f(x, y), f(p, q), \phi(t_0)) \\
\geq \psi[F(gx, gp, t_0) * F(gy, gq, t_0)] \\
\geq F(gx, gp, t_0) * F(gy, gq, t_0)
\]
\[ F(gx, x, t_0) \] 

Similarly, \[ F(gy, y, \phi(t_0)) \geq F(gy, y, t_0) * F(gx, x, t_0). \]

Continuing in a same way, we have for all \( n \in \mathbb{N}, \)

\[ F(gx, x, \phi^n(t_0)) \geq [F(gx, x, t_0)]^{2^{n-1}} * [F(gy, y, t_0)]^{2^{n-1}}. \]

Thus, we have

\[ F(gx, x, t) \geq F(gx, x, \sum_{k=n_0}^{\infty} \phi^k(t_0)) \]

\[ \geq F(gx, x, \phi^{n_0}(t_0)) \]

\[ \geq [F(gx, x, t_0)]^{2^{n_0-1}} * [F(gy, y, t_0)]^{2^{n_0-1}} \]

\[ \geq \left(1 - \delta\right) * \left(1 - \delta\right) * \ldots * \left(1 - \delta\right) \]

\[ \geq (1 - \epsilon). \]

So, for any \( \epsilon > 0, \) we have \( F(gx, y, t) \geq (1 - \epsilon), \) for all \( t > 0. \)

This implies \( g(x) = x. \) Similarly, \( g(y) = y. \)

**Step 4.** Next we shall show that \( x = y. \)

Since \( * \) is a \( t \)-norm of \( H \)-type, for any \( \epsilon > 0, \) there exists \( \delta > 0 \) such that

\[ \left(1 - \delta\right) * \left(1 - \delta\right) * \ldots * \left(1 - \delta\right) \geq (1 - \epsilon), \]

for all \( p \in \mathbb{N}. \)

Since \( \lim_{t \to \infty} F(x, y, t) = 1, \) for all \( x, y \) in \( X, \) there exists \( t_0 > 0 \) such that

\[ F(x, y, t_0) \geq (1 - \delta). \]

Since \( \phi \in \Phi, \) using condition (\( \phi \)-3), we have \( \sum_{n=1}^{\infty} \phi^n(t_0) < \infty. \) Then for any \( t > 0, \)

there exists \( n_0 \in \mathbb{N} \) such that

\[ t > \sum_{k=n_0}^{\infty} \phi^k(t_0). \]

From (3.2), we have

\[ F(x, y, \phi(t_0)) = F(f(p, q), f(q, p), \phi(t_0)) \]

\[ \geq \psi[F(gp, gq, t_0) * F(gq, gp, t_0)] \]

\[ \geq F(gp, gq, t_0) * F(gq, gp, t_0) \]

\[ = [F(x, y, t_0)]^2. \]

Continuing likewise, we have for all \( n \in \mathbb{N}, \)

\[ F(x, y, \phi^{n_0}(t_0)) \geq [F(x, y, t_0)]^{2^{n_0}}. \]

Thus, we have
F(x, y, t) \geq F(x, y, \sum_{k=n_0}^{\infty} \phi^k(t_0)) \\
\geq F(x, y, \phi^{n_0}(t_0)) \\
\geq [F(x, y, t_0)]^{2n_0} \\
\geq (1 - \delta) * (1 - \delta) * ... * (1 - \delta) \geq (1 - \epsilon), \text{ which implies that } x = y.

Thus, we have proved that f and g have a common fixed point x in X.

Step 5. We now prove the uniqueness of x.

Let z be any point in X such that z \neq x with g(z) = z = f(z, z).

Since * is a t-norm of H-type, for any \epsilon > 0, there exists \delta > 0 such that

\[
\prod_{p=1}^{\infty} (1 - \delta) \geq (1 - \epsilon), \text{ for all } p \in \mathbb{N}.
\]

Since \lim_{t \to \infty} F(x, y, t) = 1, for all x, y in X, there exists \( t_0 > 0 \) such that

F(x, z, t_0) \geq (1 - \delta).

Also, since \( \phi \in \Phi \), using condition (\phi-3), we have \( \sum_{n=1}^{\infty} \phi^n(t_0) < \infty \). Then for any \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that

\[
t > \sum_{k=n_0}^{\infty} \phi^k(t_0).
\]

Using condition (3.2), we have

\[
F(x, z, \phi(t_0)) = F(f(x, x), f(z, z), \phi(t_0)) \\
\geq \psi[F(g(x), g(z), t_0) \ast F(g(x), g(z), t_0)] \\
\geq F(g(x), g(z), t_0) \ast F(g(x), g(z), t_0) \\
= F(x, z, t_0) \ast F(x, z, t_0) \\
= [F(x, z, t_0)]^2.
\]

Thus, we have

\[
F(x, z, t) \geq F(x, z, \sum_{k=n_0}^{\infty} \phi^k(t_0)) \\
\geq F(x, z, \phi^{n_0}(t_0)) \\
\geq ([F(x, z, t_0)]^{2n_0})^2 \\
= ([F(x, z, t_0)]^{2n_0})^2 \\
\geq (1 - \delta) * (1 - \delta) * ... * (1 - \delta) \geq (1 - \epsilon), \text{ which implies that } x = y.
\]
Hence, f and g have a unique common fixed point in X.

Next, we give an example in support of the Theorem 3.1.

**Example 3.1.** Let $X = [-2, 2)$, $a * b = ab$ for all $a, b \in [0, 1]$ and $\varphi(t) = \frac{t}{t+1}$. Then $(X, F, *)$ is a Menger space, where

$$F(x, y, t) = [\varphi(t)]^{|x-y|},$$

for all $x, y$ in $X$ and $t > 0$.

Let $\psi(t) = t$, $\phi(t) = \frac{t}{2}$, $g(x) = x$ and the mapping $f : X \times X \to X$ be defined by $f(x, y) = \frac{x^2}{16} + \frac{y^2}{16} - 2$.

It is easy to check that $f(X \times X) \subseteq X = g(X)$. Further, $f(X \times X)$ is complete and the pair $(f, g)$ is weakly compatible. We now check the condition (3.2),

$$F(f(x, y), f(u, v), \phi(t)) = \frac{x^2 + y^2}{16} - 2.$$
It follows immediately from Theorem 3.1.

Next we give an application of Theorem 3.1.

4. An Application

Theorem 4.1. Let \((X, F, \ast)\) be a Menger PM-space, \(\ast\) being continuous t-norm defined by \(a \ast b = \min\{a, b\}\) for all \(a, b\) in \(X\). Let \(M, N\) be weakly compatible self maps on \(X\) satisfying the following conditions:

(4.1) \(M(X) \subseteq N(X)\),

(4.2) there exists \(\phi \in \Phi\) such that

\[
F(Mx, My, \phi(t)) \geq \psi[F(Nx, Ny, t)]
\]

for all \(x, y\) in \(X\) and \(t > 0\), where \(\psi: [0, 1] \to [0, 1]\) is continuous and \(\psi(t) \geq t\) for all \(t \in [0, 1]\).

If range space of any one of the maps \(M\) or \(N\) is complete, then \(M\) and \(N\) have a unique common fixed point in \(X\).

Proof.

By taking \(f(x, y) = M(x)\) and \(g(x) = N(x)\) for all \(x, y \in X\) in theorem (3.1), we get the desired result.

Taking \(\phi(t) = kt, k \in (0, 1)\) and \(\psi(t) = t\) we have the following:

Corollary 4.2. Let \((X, F, \ast)\) be a Menger PM-space, \(\ast\) being continuous t-norm defined by \(a \ast b = \min\{a, b\}\) for all \(a, b\) in \(X\). Let \(M, N\) be weakly compatible self maps on \(X\) satisfying (4.1) and the following condition:

(4.3) there exists \(k \in (0, 1)\) such that

\[
F(Mx, My, kt) \geq F(Nx, Ny, t)
\]

for all \(x, y\) in \(X\) and \(t > 0\).

If range space of any one of the maps \(M\) or \(N\) is complete, then \(M\) and \(N\) have a unique common fixed point in \(X\).

Taking \(N = I_X\) (the identity map on \(X\)) in Corollary 4.2, we have the following:

Corollary 4.3. Let \((X, F, \ast)\) be a Menger PM-space, \(\ast\) being continuous t-norm defined by \(a \ast b = \min\{a, b\}\) for all \(a, b\) in \(X\). Let \(M, N\) be weakly compatible self maps on \(X\) satisfying (4.1) and the following condition:

(4.4) there exists \(k \in (0, 1)\) such that

\[
F(Mx, My, kt) \geq F(x, y, t)
\]

for all \(x, y\) in \(X\) and \(t > 0\).

If range space of the map \(M\) is complete, then \(M\) and \(N\) have a unique common fixed point in \(X\).
point in X.

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