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FIXED POINT THEOREM USING SEMI COMPATIBLE AND SUB SEQUENTIALLY CONTINUOUS MAPPINGS IN MENGER SPACE

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Abstract: In this paper, we proved a common fixed point theorem in complete Menger space by using the concept of semi compatible and sub sequentially continuous mappings, which generalizes the result proved by Preeti Malviya et.al. Further our result is supported with a suitable example.

Keywords: common fixed point; semi compatible mappings; sub sequentially continuous mappings; occasionally weakly compatible mappings; Menger space.

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1. INTRODUCTION

Menger [1] initiated the theory of probabilistic metric (PM) space and the conversion of probabilistic notion into geometry was one of the great efforts. Mishra [2] introduced the notion

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of compatible mappings in Menger space Altumn Turkoglu [3] generated two common fixed point theorems of complete PM-space by utilizing the implicit conditions. Schweizer and Sklar [4] contributed for the enrichment of PM-space in fixed point theory. Sehgal and Bharucha-Reid [5] extended the concept of contraction mapping to the setting of the Menger space. They proved a generalization of classical Banach contraction on complete Menger space. V.H. Badshah, Suman Jain and Subhash Madloi [12] proved fixed point theorem using semi compatible in Menger probabilistic metric space. [Bouhdjera and Gedet Thobe [9] introduced two notions sub sequentially continuous and sub- compatibility which are weaker forms than reciprocal-continuity.

2. PRELIMINARIES

Definition 2.1 [6] A mapping $t: [0,1]\, X\, [0,1] \rightarrow [0,1]$ is called continuously $t-norm$ if it has the following properties

(a) t *is abelian and associative*

(b) $t\,(\gamma, 1) = \gamma,\ \forall\, \gamma \in [0,1]$

(c) $t\,(\gamma, \omega) \leq t\,(\alpha, \vartheta)$ *for* $\gamma \leq \alpha$ *and* $\omega \leq \vartheta\ \forall\, \gamma, \alpha, \vartheta, \omega \in [0,1]$.

Definition 2.2[6] A mapping $F : R \rightarrow R$ said to a distribution if it is non-decreasing left continuous with $inf\,\{F(t) : t \in R\} = 0$ and Sup $\{F(t) : t \in R\} = 1$. We denote L as the set of all distribution functions.

Definition. 2.3 [6] A PM- space is jointly ordered pair (X, F) having a set $X \neq \emptyset$ and the function $F : X \times X \rightarrow L$ where L is the collection of all distribution functions and the value of F at $(u, v) \in X \times X$ is represented by $F_{u,v}$ the function $F_{u,v}$ obeys the following conditions

(P_1) $F_{u,\ v}(\alpha) = 1\ \forall\ \alpha > 0$ if and only if u = v

(P_2) $F_{u,\ v}(0) = 0$

(P_3) $F_{u,\ v}(\alpha) = F_{v,\ u}(\alpha)$.

(P_4) If $F_{u,\ v}(\alpha) = 1$ and $F_{v,\ w}(\beta) = 1 \rightarrow F_{u,\ w}(\alpha + \beta) = 1\ \forall$ u, v, w in X, $\alpha, \beta > 0$.

Definition 2.4 [6] A Menger space is a triplet (X, F, t) where (X, F) is a probabilistic metric space and t-norm with the condition

$F_{u,\ w}(\alpha + \beta) \geq$ t ($F_{u,\ v}(\alpha)$, $F_{v,\ w}(\beta)$) for all x, y, z $\in$ X and $\alpha,\ \beta > 0$.

Remark 2.5 [11] Every metric space is realized as probabilistic metric space by treating $F: X \times X \rightarrow L$ defined by $F_{u,v}(\alpha) = \mathrm{H}(\alpha - d(u,v))$ for all u,v $\in X$, where H is a Heaviside function defined by

$$\mathrm{H}(\alpha) = \begin{cases} 1, & if\ \alpha \geq 0, \\ 0, & if\ \alpha < 0. \end{cases}$$

Example 2.6 Consider $X = [0,\ \infty)$ and d is the metric on X for each $t \in [0,1]$.

Define

$$F_{u,\ v}(\mathrm{t}) = \begin{cases} \frac{t}{t+|\ \alpha-\beta\ |}, & t > 0 \\ 0 & ,\ t = 0 \end{cases} \qquad \forall\ \alpha, \beta \text{ in X and } t > 0.$$

Then (X, F, t) is a Menger space.

Definition 2.7 [6] A sequence $\langle x_n \rangle$ in Menger space (X, F, t) is said to converge to β in X if and only if fo;r each $\epsilon > 0$ and $t > 0$ there is a positive integer $N(\epsilon) \in N \ni$ (X, F, t) $F_{x_n,\ \beta}(\epsilon) > 1 - t\ \forall\, n \geq N(\epsilon)$.

Definition 2.8 [6] A PM- space (X, F, t) is complete if every Cauchy sequence in X is convergent.

Definition 2.9 [4] Self mappings P and S of a Menger space (X, F, t) are called compatible if $F_{PSx_n,\ SPx_n}(\beta) \rightarrow 1\ \forall\ \beta > 0$ whenever a sequence $\langle x_n \rangle$ in X $\ni$ $Px_{n,}\ Sx_n \rightarrow z$ for some z in X as n $\rightarrow \infty$.

Example 2.10 Consider $X = [0,\ \infty)$ and d is the metric on X and for each $t \in [0,1]$ Define

$$F_{u,\ v}(\mathrm{t}) = \begin{cases} \frac{t}{t+|\ \alpha-\beta\ |}, & t > 0 \\ 0 & ,\ t = 0 \end{cases} \qquad \forall\ \alpha, \beta \text{ in X and } t > 0.$$

Define $P,\ S: X \rightarrow X\ as\ \ P(x) = x^2\ \ \forall\ x$ in X and $S(x) = \begin{cases} 3x, & x \in [0,\ 8] \\ 81, & x \in [1,\ \infty) \end{cases}$

Consider a sequence $x_n = 3 + \frac{7}{n} \rightarrow 3$ for n ≥ 1. Then

$Px_n = P(3 + \frac{7}{n}) = (3 + \frac{7}{n})^2 \to 9$ and

$Sx_n = S(3 + \frac{7}{n}) = 3(3 + \frac{7}{n}) = 9 + \frac{21}{n} \to 9 \; as\, n \to \infty.$

Now $PSx_n = PS(3 + \frac{7}{n}) = P(9 + \frac{21}{n}) = (9 + \frac{21}{n})^2 \to 81$ and

$SPx_n = SP(3 + \frac{7}{n}) = S(3 + \frac{7}{n})2 = 81 \to 81 \; as\, n \to \infty.$

This gives $F_{PSx_n, SPx_n}(\beta) \to 1 \; for\ all\ \beta > 0 \; as\, n \to \infty.$

Thus, the mappings P, S are compatible.

Definition 2.11 [7] Self mappings P and S of a Menger- space (X, F, t) are termed as weakly compatible $if \; Sx = Px \Rightarrow SPx = PSx \; for\ all\ x \in X.$

Example 2.12 Consider $X = [0, \infty)$ and d is the metric on X and for each $t \in [0,1]$. Define

$$F_{u,\ v}(t) = \begin{cases} \frac{t}{t+|\ \alpha-\beta\ |}, & t > 0 \\ 0 & , t = 0 \end{cases} \quad \forall\ \alpha, \beta \text{ in X and } \quad t > 0.$$

Define $P,\ S : X \to X \; as \; P(x) = x^2$ and $S(x)$= x $\forall\ x$ in X.

We observe that coincidence points for the pair (P, S) are 0,1 and at x = 0, P(0)=S(0) implies $PS(0) = SP(0)$ at x = 1, P(1) = S(1) implies $PS(1) = SP(1)$.

Therefore, mappings are weakly compatible.

Definition 2.13 [6] Self mappings P and S of a Menger space (X, F, t) are called reciprocally continuous if PS$x_n \to$ Pz and SP$x_n \to$ Sz whenever a sequence $\langle x_n \rangle$ in X $\ni Px_n, Sx_n \to z$ for some z in X as n $\to \infty$.

Example 2.14 Consider $X = R$, d is the usual distance metric on X for $each\, t \in [0,1]$.

Define

$$F_{u,\ v}(t) = \begin{cases} \frac{t}{t+|\ \alpha-\beta\ |}, & t > 0 \\ 0 & , t = 0 \end{cases} \quad for\ all\ \alpha, \beta\ in\ X\ and\ t > 0.$$

Define $P, S : X \to X \; as \; P(x) = \begin{cases} 11x, & x < 11 \\ 13, & x \geq 11 \end{cases}$

$and \;\; S(x) = \begin{cases} 5x + 6, & x < 11 \\ 11, & x \geq 11. \end{cases}$

Consider a sequence $x_n = 1 + \frac{11}{n}$ for n ≥ 1 then

$Px_n = P\,(1 + \frac{11}{n}) = 11(1 + \frac{11}{n}) \to 11$ and

$Sx_n = S\left(1 + \frac{11}{n}\right) = 5\left(1 + \frac{11}{n}\right) + 6 = 11 + \frac{55}{n} \to 11 \ as\, n \to \infty.$

Now $PSx_n = PS\,(1 + \frac{11}{n}) = P\,(11 + \frac{55}{n}) = 13 \to 13 = P\,(11)$ and

$SPx_n = SP\,(3 + \frac{1}{n}) = S\,(11 + \frac{121}{n}) = 11 \to 11 = S\,(11).$

Then PSx_n → Pz and SPx_n → Sz as n → ∞.

Thus, P and S are reciprocally continuous but not continuous.

Definition 2.15 [6] Self mappings P and S of a Menger space (X, F, t) are said to semi compatible if $F_{PSx_n,Sz}\ (\beta) \to 1$ for all $\beta > 0$ whenever $\langle x_n \rangle$ in X $\ni Px_n, Sx_n \to z$ for some z in X as n → ∞.

Example 2.16 Consider X = $[0, 1000)$, d is the usual distance metric on X for $each\, t \in [0, 1]$.

Define

$$F_{u,\ v}(t) = \begin{cases} \frac{t}{t+|\ \alpha-\beta\ |}, & t > 0 \\ 0 & , t = 0 \end{cases} \quad for\ all\ \alpha, \beta\ in\ X\ and\ t > 0.$$

Define $P, S : X \to X\ as\ P\,(x) = x^3$ $\forall\ x$ in X and

$$S(x) = \begin{cases} 2x^2, & x < 8 \\ 512, & x \geq 8. \end{cases}$$

Consider sequence $x_n = 2 - \frac{1}{2n}$ for n ≥ 1. Then

$Px_n = P\,(2 - \frac{1}{2n}) = (2 - \frac{1}{2n})3 \to 8$ and

$Sx_n = S\,(2 - \frac{1}{2n}) = 2\,(2 - \frac{1}{2n})2 \to 8\ as\, n \to \infty.$

Now $PSx_n = PS\,(2 - \frac{1}{2n}) = P\,(\,2\,(\,2 - \frac{1}{2n})2\,) = (\,2\,(\,2 - \frac{1}{2n})2\,)3 \to 512 = S(8)$ and

$SPx_n = SP\,(2 - \frac{1}{2n}) = S\,((2 - \frac{1}{2n})3) = 2((2 - \frac{1}{2n})3\,)2 \to 128$.

Then $F_{PSx_n,Sz}\ (\beta) \to 1$ and Then $F_{PSx_n,SPx_n}(\beta)\ not\ tends\ to\ 1$ for all β as n → ∞.

Hence the mappings P and S are semi compatible but not compatible.

Definition 2.17 [8] Self mappings P and S of a Menger space (X, F, t) are termed as occasionally weakly compatible if and only if there exists for some x in X $\ni$ $Px = Sx$ implies $PSx = SPx$.

Clearly occasionally weakly compatible maps are weakly compatible mappings however the converse is not true in general.

We can justify with an example.

Example 2.18. Consider $X = [0,\ 4]$ and d is the metric on X and for each $t \in [0, 1]$

Define

$$F_{u,\ v}(t) = \begin{cases} \frac{t}{t+|\ \alpha-\beta\ |}, & t > 0 \\ 0 & , t = 0 \end{cases} \quad \forall\ \alpha, \beta \text{ in X \& } t > 0$$

Define the mappings P $, S : X \rightarrow X\ as\ \ P(x) = x^2$, x $\in [0,\ 4]$ and

$$S(x) = \begin{cases} x, & x \in [0,\ 1] \\ x + 2, & x \in [1,\ 4] \end{cases}.$$

We observe that coincidence points for the pair (P, S) are 0, 1 and 2. At $x = 2$, $P(2) = S(2)$ but $PS(2) \neq SP(2)$ at $x = 0$, $P(0) = S(0)$ and $PS(0) = SP(0)$.

Which shows that the mappings P, S are occasionally weakly compatible but are not weakly compatible.

Definition 2.19[9] Self-mappings P and S of a Menger space (X, F, t) are termed as sub sequentially continuous if and only if $Px_{n,} Sx_n \rightarrow z$ for some z in X and which satisfy

$F_{PSx_n, pz}\,(\alpha) \rightarrow\ 1\ and\ F_{SPx_n, Sz}(\alpha) \rightarrow\ 1\ \ for\ all\ \alpha\ >\ 0\ \ as\ n\ \rightarrow\ \infty.$

Remark 2.20.If P and S are continuous or reciprocally continuous mappings then they are sub sequentially continuous mappings but not conversely.

We justify this with an example.

Example 2.21 Consider X $= [0, 19)$, d is the usual distance metric on X for $each\ t\ \in\ [0, 1]$.

Define

$F_{u,\ v}(\mathrm{t}) = \begin{cases} \frac{t}{t+|\ \alpha-\beta\ |}, & t > 0 \\ 0 & , t = 0 \end{cases}$ $\quad for\ all\ \alpha, \beta\ in\ X\ and\ t\ >\ 0.$

Define $P, S : X \rightarrow X\ as\ P\ (x) = \begin{cases} 3 + x, & x \leq 3 \\ x, & x > 3 \end{cases}$

$and\ \ S(x) = \begin{cases} 3 - x, & x\ < 3 \\ 2x - 3 & x\ \geq 3 \end{cases}$.

Consider a sequence $x_n = 3 + \frac{1}{n}$ for n ≥ 1 then

$Px_n = P\ (3 + \frac{1}{n}) = 3 + \frac{1}{n} \rightarrow 3$ and

$Sx_n = S\ (3 + \frac{1}{n}) = 2\ (3 + \frac{1}{n}) - 3 = 3 + \frac{2}{n} \rightarrow 3\ \ as\ n \rightarrow \infty.$

Now $PSx_n = PS\ (\ 3 + \frac{1}{n}) = P\ (3 + \frac{2}{n}) = 3 + \frac{2}{n} \rightarrow 3 \neq 6 = P\ (\ 3\ )$ and

$SPx_n = SP\ (\ 3 + \frac{1}{n}) = S\ (3 + \frac{1}{n}) = 3 + \frac{2}{n} \rightarrow 3 = 3 = S\ (\ 3\ ).$

Thus, P and S are not reciprocally continuous.

However, for a sequence $x_n = \frac{1}{n}$ for n ≥ 1, then

$Px_n = 3 + \frac{1}{n} \rightarrow 3$ and

$Sx_n == 3 - \frac{1}{n} \rightarrow 3\ \ as\ n \rightarrow \infty.$

Now $PSx_n = P\ (3 - \frac{1}{n}) = 3 + 3 - \frac{1}{n}) = 6 - \frac{1}{n} \rightarrow 6 = 6 = P\ (3)$ and

$SPx_n = S\ (3 + \frac{1}{n}) = 2\ (3 + \frac{1}{n}) - 3 = 3 + \frac{2}{n} \rightarrow 3 = 3 = S(3).$

Therefore, P and S are sub-sequentially continuous but neither continuous nor reciprocally continuous.

Lemma 2.1 [2] "Let $(X, F, t)\ be\ a$ Menger-space with continuous t-norm if we find a constant $q \in (0, 1)\ such\ that\ F_{u,v}(qt) \geq F_{u,v}(t)\ for\ all\ u, v\ in\ X$ and $t > 0$ then u=v."

Lemma 2.2 [2] " Let $(X, F, t)\ be\ a$ Menger-space with continuous t-norm t(ω,ω) $\geq \omega$ for all $\omega \in [0,\ 1]$, If there exists a constant $\theta \in (0, 1)$ such that $F_{u_n,\ u_{n+1}}(\theta t) \geq F_{u_{n-1},\ u_n}(t)$, n = 1,2,3… then $\langle u_n \rangle$ is a Cauchy sequence in X. "

The following theorem was proved by Preeti Malviya, Vandana Gupta and V.H. Badshah in [6].

Theorem (A) Let P, Q, S and T be self-mappings on a complete Menger space (X, F, t) with continuous t-norm t (c, c) $\geq$ c for c $\in$ (0, 1) satisfying

(A_1) $P(X) \subseteq T(X)$ and $Q(X) \subseteq S(X)$

(A_2) (Q, T) is weakly compatible

(A_3) $for\ all\ \alpha, \beta\ in\ X\ and\ h > 1$

$$F_{P\alpha.\ Q\beta}\ (\mathrm{hx}) \geq \mathrm{Min}\ \{\ F_{S\alpha.\ T\beta}\ (\mathrm{x}), \{\ F_{S\alpha.\ P\alpha}\ (\mathrm{x})\ .\ F_{Q\beta.\ T\beta}\ (\mathrm{x})\ \},\ F_{P\alpha.\ S\alpha}\ (\mathrm{x})\ \}.$$

(A_4) If (P, S) is semi-compatible pair of reciprocally continuous mappings

then P, Q, S and T have unique common fixed point.

Now we improve the above result in the following way.

3. MAIN RESULT

Theorem 3.1 Let P,Q,S and T be self-mappings on a complete Menger space (X, F, t) with continuous t-norm t (c, c) $\geq$ c for c $\in$ (0, 1) satisfying

(B_1) $P(X) \subseteq T(X), Q(X) \subseteq S(X)$

(B_2) (Q, T) is occasionally weakly compatible mappings

(B_3) $F_{P\alpha.\ Q\beta}$ (hx) $\geq$ Min { $F_{S\alpha.\ T\beta}$ (x),{ $F_{S\alpha.\ P\alpha}$ (x) . $F_{Q\beta.\ T\beta}$ (x) }, $F_{P\alpha.\ S\alpha}$ (x) }

$for\ all\ \alpha, \beta\ in\ X\ and\ h > 1.$

(B_4) If the pair (P, S) is semi compatible and sub sequentially continuous

then P, Q, S and T have unique common fixed point.

Proof:

From (B_1) we can construct a sequence $\langle y_n \rangle$ for n $\geq$ 1 such that

$\langle y_{2n} \rangle$ = Px_{2n}= Tx_{2n+1}, $\langle y_{2n+1} \rangle$ = Qx_{2n+1} = Sx_{2n+2}.

We prove $\langle y_n \rangle$ is Cauchy sequence.

In (B_3) take $\alpha = x_{2n+1}$, $\beta = x_{2n+2}$ we get

$$F_{Px_{2n+1},\ Q\,x_{2n+2}}\ (\mathrm{hx}) \geq \mathrm{Min}\ \{F_{S\,x_{2n+1}\,.\ T\,x_{2n+2}}\ (\mathrm{x}), \{\ F_{S\,x_{2n+1}\,.\ P\,x_{2n+1}}\ (\mathrm{x})\ .\ F_{Qx_{2n+2}\ .\ T\,x_{2n+2}}\ (\mathrm{x})\},$$

$$F_{P\,x_{2n+1}\,.\ S\,x_{2n+1}}\ (\mathrm{x})\}.$$

This gives

$F_{y_{2n+1}.\ y_{2n+2}}$ (hx) $\geq$ Min $\{F_{y_{2n},\ y_{2n+1}}$ (x),$\{$ $F_{y_{2n},\ y_{2n+1}}$ (x) . $F_{y_{2n+2},\ y_{2n+1}}$(x)$\}$, $F_{y_{2n+1},\ y_{2n}}$(x)$\}$.

This results

$F_{y_{2n+1}.\ y_{2n+2}}$ (hx) $\geq$ $F_{y_{2n},y_{2n+1}}$ (x) .

Similarly

$F_{y_{2n+2}.\ y_{2n+3}}$ (hx) $\geq$ $F_{y_{2n+1},y_{2n+2}}$ (x).

In general we have $F_{y_{n+1}.\ y_n}$ (hx) $\geq$ $F_{y_n,\ y_{n-1}}$ (x) for n = 1, 2, 3….

By Lemma 2.2 $\langle y_n \rangle$ is cauchy sequence in complete Menger space X so that it converges to some z in X.

Consequently each sub sequence of it converges to z.

Since the pair (P, S) is semi-compatible then $\lim_{n \to \infty} F_{PSx_{2n}.SZ}$(t) = 1.

Also the pair (P, S) is sub sequentially continuous then

$\lim_{n \to \infty} F_{PSx_{2n}.\ PZ}$(t) = 1 and $\lim_{n \to \infty} F_{SPx_{2n}.\ SZ}$ (t) = 1.

This implies Pz =Sz.

Now we claim z = Pz.

Take α = z, $\beta = y_n$ in (B_3) we get

$F_{Pz,\ Qy_n}$ (hx) $\geq$ Min $\{$ $F_{Sz\ .\ Ty_n}$ (x),$\{$ $F_{Sz\ .\ Pz}$ (x) . $F_{Qy_n.\ Ty_n}$ (x) $\}$, $F_{Pz\ .\ Sz}$ (x) $\}$

letting n $\to$ ∞

this gives

$F_{Pz.\ z}$ (hx) $\geq$ Min $\{$ $F_{Pz,z}$ (x),$\{$ $F_{Pz,\ Pz}$ (x) . $F_{z.\ z}$ (x) $\}$, $F_{Pz\ ,\ Pz}$ (x) $\}$.

This results

$F_{Pz,\ z}$ (hx) $\geq$ $F_{Pz,\ z}$ (x).

By Lemma 2.1, we get z = Pz.

This gives z = Pz = Sz.

Now z = Pz $\in$ $P(X) \subseteq T(X)$ by (B_1) implies there exists some u $\in$ X such that z = Pz = T u.

Now we claim z = Qu.

Take α = z , y = u in (B_3) we get

$F_{Pz,\ Qu}$ (hx) $\geq$ Min {$F_{Sz\ .\ Tu}$ (x),{ $F_{Sz\ .\ Pz}$ (x) . $F_{Qu.\ Tu}$ (x) }, $F_{Pz\ .\ Sz}$ (x) }

using z = Pz = Sz = T u

$F_{z\ .\ Qu}$ (hx) $\geq$ Min [$F_{z,\ z}$ (x), {$F_{z\ ,\ z}$ (x) . $F_{Qu.\ z}$ (x) }, $F_{z,\ z}$ (x) }.

This gives

$F_{z\ .Qu}$ (hx) $\geq$ $F_{z\ .Qu}$ (x).

By Lemma 2.1, we get, z = Qu

this gives $z = Qu = Tu$.

Again since the pair (Q, T) is occasionally weakly compatible such that $Qu = Tu$ implies $QTu = TQu$.

This results $Qz = Tz$.

Now we claim z = Qz.

Take α = z , y = z in (B_3) we get

$F_{Pz,\ Qz}$ (hx) $\geq Min$ { $F_{Sz\ .\ Tz}$ (x),{ $F_{Sz\ .\ Pz}$ (x) . $F_{Qz.\ Tz}$ (x) }, $F_{Pz,\ Sz}$ (x)}

using z = Pz = Sz and Qz = Tz

$F_{z,\ Qz}$ (hx) $\geq Min$ { $\boldsymbol{F}_{z\ ,\ z}$ (x), { $F_{z\ ,\ z}$ (x) . $F_{Qz,.\ Qz}$ (x) }, $F_{z,\ z}$ (x) }

this gives

$F_{z\ .\ Qz}$ (hx) $\geq$ $F_{z\ .\ z}$ (x).

By Lemma 2.1 we get $z = Qz$.

This implies $z = Pz = Sz = Qz = Tz$.

This gives z is a common fixed point for the mappings P, Q, S and T.

Uniqueness:

If possible z_1 is another fixed point then z_1= Pz_1 = Sz_1 = Qz_1 = Tz_1.

Put α = z, $\beta = z_1$ in (B_3) we have

$F_{Pz,\ Qz_1}$ (hx) $\geq$ Min { $F_{Sz,\ Tz}$ (x),{ $F_{Sz,\ Pz}$ (x) . $F_{Qz_1.\ Tz_1}$ (x) }, $F_{Pz\ .\ Sz}$ (x) }

this gives

$F_{z\,.\,z_1}$ (hx) $\geq Min$ { $F_{z\,.\,z_1}$ (x), { $F_{z\,.z}$ (x) . $F_{z_1.\,z_1}$ (x) }, $F_{z\,,\,z}$ (x) }

this results

$F_{z.\,z_1}$ (hx) $\geq$ $F_{z.\,z_1}$ (x).

By Lemma 2.1 we get $z = z_1$.

Therefore, z is the unique common fixed point for the four mappings P, Q, S and T.

We provide a supporting example to justify our theorem.

Example 3.2. Consider X $= [1, 5]$ d is distance metric on X $each\ t \in [0,\ 1]$

Define $F_{u,v}$(t) $= \begin{cases} \frac{t}{t+|\ \alpha-\beta\ |}, & t > 0 \\ 0 & , t = 0 \end{cases}$ $for\ all\ \alpha, \beta\ in\ X\ ,\ t > 0.$

Define $P, Q, S\ and\ T : X \rightarrow X\ as$

$$P\ (x) = Q\ (x) = \begin{cases} x, & 0 \leq x < \frac{3}{2} \\ 3, & \frac{3}{2} \leq x \leq 3 \\ 0, & 3 < x \leq 5 \end{cases} \quad and$$

$$S(x) = T(x) = \begin{cases} 3 - x, & 0 \leq x < \frac{3}{2} \\ 3, & \frac{3}{2} \leq x \leq 3 \\ 5 - x, & 3 < x \leq 5 \end{cases} .$$

Now $P(X) = Q(X) = [0, \frac{3}{2}]\ U\ [3]$ and $S\ (X) = T\ (X) = [0, 3]$ so that

$P(X) \subseteq T(X),\ Q(X) \subseteq S(X).$

Consider a sequence $x_n = \frac{3}{2} + \frac{1}{n}$ for n ≥ 1 then

$Px_n = P\left(\frac{3}{2} + \frac{1}{n}\right) = 3 \rightarrow 3$ and

$Sx_n = S\left(\frac{3}{2} + \frac{1}{n}\right) = 3 \rightarrow 3$ as n $\rightarrow \infty$.

$PSx_n = PS\left(\frac{3}{2} + \frac{1}{n}\right) = P\ (3) = 3 \rightarrow 3,\ P\ (3) = 3$ and also

$SPx_n = SP\left(\frac{3}{2} + \frac{1}{n}\right) = S\ (3) = 3 \rightarrow 3,\ S\ (3) = 3$ as n $\rightarrow \infty$.

Since $PSx_n \rightarrow 3,\ S\ (3) = 3.$

This gives the pair (P, S) is semi compatible, and the pair (P, S) is sub sequentially continuous since $PSx_n \rightarrow Pz, SPx_n \rightarrow Sz.$

Further consider a sequence $x_n = \frac{3}{2} - \frac{1}{n}$ for n ≥ 1 then

$Px_n = P\left(\frac{3}{2} - \frac{1}{n}\right) = \left(\frac{3}{2} - \frac{1}{n}\right) \to \frac{3}{2}$ and

$Sx_n = S\left(\frac{3}{2} - \frac{1}{n}\right) = \left(3 - [\frac{3}{2} - \frac{1}{n}]\right) = \left(\frac{3}{2} + \frac{1}{n}\right) \to \frac{3}{2}$ as n → ∞.

Now $PSx_n = PS\left(\frac{3}{2} - \frac{1}{n}\right) = P\left(\frac{3}{2} + \frac{1}{n}\right) = 3 \to 3,\ P\left(\frac{3}{2}\right) = 3$ and also

$SPx_n = SP\left(\frac{3}{2} - \frac{1}{n}\right) = S\left(\frac{3}{2} - \frac{1}{n}\right) = \left(3 - [\frac{3}{2} - \frac{1}{n}]\right) = \left(\frac{3}{2} + \frac{1}{n}\right) \to \frac{3}{2},\ S\left(\frac{3}{2}\right) = 3$

as n → ∞.

Since $PSx_n \to 3, SPx_n \to \frac{3}{2}$ as n → ∞ so that (P, S) is not compatible.

Again since $PSx_n \to Pz$ but $SPx_n doesnot\ tends\ to\ Sz$ as n → ∞ so that (P, S) is not reciprocally continuous.

Since x = 5 is coincidence point for the pair (P, S) giving that

$P(5) = 0 = S(5)$ results $PS(5) = P(0) = 0\ and\ SP(5) = S(0) = 3$ so that

$PS(5) \neq SP(5)$.

Thus, the mappings are not weakly compatible.

Further $at\ x = 3\ P(3) = S(3)$, this implies $SP(3) = PS(3)$.

Therefore, the pair (P, S) is occasionally weakly compatible.

Thus the pairs $(P, S), (Q, T)$ are semi compatible and sub sequentially continuous mappings and $P(3) = S(3) = Q(3) = T(3) = 3$.

Thus, 3 is the unique common fixed point for the mappings P, Q, S and T.

CONCLUSION

We proved Theorem (3.1) by using the concepts of

(a) occasionally weakly compatible mappings instead of weakly compatible mappings and

(b) sub sequentially continuous mappings instead of reciprocally continuous mappings.

These are obviously weaker conditions compared to those conditions of Theorem (A). Thus, we assert that our result is an improvisation of Theorem (A).

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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