THE MULTIPLICATIVE SECOND HYPER ZAGREB INDEX OF SOME GRAPH OPERATIONS

M. ARUVI1,∗, J. MARIA JOSEPH2, E. RAMGANESHP

1Department of Mathematics, University College of Engineering-BIT Campus, Anna University, Tiruchirappalli, India
2Department of Mathematics, St. Joseph’s College, Affiliated to Bharathidasan University, Tiruchirappalli, India
3Department of Educational Technology, Bharathidasan University, Tiruchirappalli, India

Copyright © 2020 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this paper, the multiplicative second hyper Zagreb index is presented and the sharp upper bound for this index of various graph operations for example, join, composition, cartesian and corona products of graphs are derived. And we prove that the sharp upper bound is tight.

Keywords: topological indices; graph operations.

2010 AMS Subject Classification: 05C07, 05C09, 05C76.

1. INTRODUCTION

All graphs observed here are simple, connected and finite. Let \( V(G), E(G) \) and \( d_G(w) \) indicate the vertex set, the edge set and the degree of a vertex of a graph \( G \) respectively. A graph with \( p \) vertices and \( q \) edges is known as a \((p,q)\) graph. We encourage the readers to see[5] for basic definitions and notations of a graph.

∗Corresponding author
E-mail address: aruvim.aut@gmail.com
Received September 4, 2020
A topological index is a numerical parameter which is mathematically attained from the graph structure.

Gutman et al.[2] introduced the first and second Zagreb indices of a graph \( G \) as follows:

\[
M_1(G) = \sum_{wz \in E(G)} (d_G(w) + d_G(z)) = \sum_{w \in V(G)} d_G^2(w) \quad \text{and} \quad M_2(G) = \sum_{wz \in E(G)} d_G(w)d_G(z)
\]

Shirdel et al. in [7] found Hyper-Zagreb index \( HM(G) \) which is established as

\[
HM(G) = \sum_{wz \in E(G)} [d_G(w) + d_G(z)]^2.
\]

Also, they have computed the hyper - Zagreb index of the cartesian product, composition, join and disjunction of graphs.

A forgotten topological index \( F \)-index [4] is defined for a graph \( G \) as

\[
F(G) = \sum_{w \in V(G)} d_G^3(w) = \sum_{wz \in E(G)} [d_G^2(w) + d_G^2(z)]
\]

Farahani et al. [3] defined the second hyper Zagerb as

\[
HM_2(G) = \sum_{wz \in E(G)} [d_G(w)d_G(z)]^2.
\]

Here we introduce a second forgotten topological index \( F_2 \) which is defined for a graph \( G \) as

\[
F_2(G) = \sum_{w \in V(G)} d_G^4(w).
\]

V.R.Kulli [6] introduced the first and second Gourava indices and defined as

\[
GO_1(G) = \sum_{wz \in E(G)} (d_G(w) + d_G(z)) + (d_G(w)d_G(z))
\]

and

\[
GO_2(G) = \sum_{wz \in E(G)} d_G(w)d_G(z)(d_G(w) + d_G(z))
\]

Todeschine et al. [9,10] presented the multiplicative variants of ordinary Zagreb indices, which are defined as follows:
\[ \prod_1 = \prod_{v \in V(G)} d_G(w)^2 = \prod_{w \in E(G)} [d_G(w) + d_G(z)] \]

and \[ \prod_2 = \prod_{v \in V(G)} d_G(w)d_G(z) \]

Recently, Akbar [1] has introduced the multiplicative hyper Zagreb index, denoted by

\[ \prod HM(G) = \prod_{uv \in E(G)} (d_G(u) + d_G(v))^2 \]

Also in this paper, the upper bounds on the multiplicative hyper Zagreb index of the cartesian, corona product, composition, join and disjunction of graphs.

In this paper, we introduce a new graph invariant namely multiplicative second hyper Zagreb index, denoted by

\[ \prod HM_2(G) = \prod_{uv \in E(G)} (d_G(u)d_G(v))^2 \]

In this paper, we compute the sharp upper bound for the multiplicative second hyper Zagreb index of the graph operations for example, join, composition, cartesian and corona products and prove that our bound is tight.

2. PRELIMINARIES

Lemma 2.1. [5, 8]

(a) \[ d_{G_1 + G_2}(w) = \begin{cases} d_{G_1}(w) + V(G_2), & w \in V(G_2) \\ d_{G_2}(w) + V(G_1), & w \in V(G_2) \end{cases} \]

(b) \[ d_{G_1[G_2]}(w, z) = V(G_2)d_{G_1}(w) + d_{G_2}(z) \]

(c) \[ d_{G_1 \square G_2}((w_i, z_j)) = d_{G_1}(w_i) + d_{G_2}(z_j), \text{ where } (w_i, z_j) \in V(G_1 \square G_2). \]

(d)

\[ d_{G_1 \circ G_2}(w) = \begin{cases} d_{G_1}(w) + p_2 & \text{if } w \in V(G_1) \\ d_{G_1}(w) + p_2 & \text{if } w \in V(G_2, i) \text{ for some } 0 \leq i \leq p_1 - 1, \end{cases} \]

where \( u \in V(G_1 \circ G_2) \) \( G_2, i \) is the \( i \)-th copy of the graph \( G_2 \) in \( G_1 \circ G_2 \).

Lemma 2.2 (Arithmetic geometric Inequality). Let \( y_1, y_2, \ldots, y_n \) be non-negative numbers. Then

\[ \frac{y_1 + y_2 + \cdots + y_n}{n} \geq \sqrt[n]{y_1 y_2 \cdots y_n} \]
3. **The Multiplicative Second Hyper Zagreb Index of Join of Graphs**

**Theorem 3.1.** Let $G_i, i = 1, 2$ be a $(p_i, q_i)$-graph. Then

$$\prod H_{M2}(G_1 + G_2) \leq \left[ \frac{HM_2(G_1) + p_2^2 HM(G_1) + p_2^2 q_1 + 2p_2 GO_2(G_1)}{q_1} + 2p_2^2 M_2(G_1) + 2p_2^2 M_1(G_1) \right]^{q_1}$$

$$\times \left[ \frac{HM_2(G_2) + p_1^2 HM(G_2) + p_1^2 q_2 + 2p_1 GO_2(G_2) + 2p_1^2 M_2(G_2) + 2p_1^2 M_1(G_2)}{q_2} \right]^{q_2}$$

$$\times \left[ \frac{M_1(G_1)M_1(G_2) + p_1^2 p_2 M_1(G_1) + p_1 p_2^2 M_1(G_2) + 4p_1 q_2 M_1(G_1) + p_1^3 p_2^3 + p_2 q_1 M_1(G_2) + 4p_1^3 p_2^2 q_2 + 4p_1^2 p_2^2 q_1 + 16p_1 p_2 q_1 q_2}{p_1 p_2} \right]^{p_1 p_2}$$

**Proof:** From the definition of the second hyper Zagreb index,

$$\prod H_{M2}(G_1 + G_2) = \prod_{wz \in E(G_1+G_2)} [d_{G_1+G_2}^2(w) d_{G_1+G_2}^2(z)]$$

$$= \prod_{wz \in E(G_1)} [d_{G_1}^2(w) d_{G_1}^2(z)]$$

$$\times \prod_{wz \in E(G_2)} [d_{G_2}^2(w) d_{G_2}^2(z)]$$

$$\times \prod_{w \in V(G_1)} \prod_{z \in V(G_2)} [d_{G_1+G_2}^2(w) d_{G_1+G_2}^2(z)]$$

$$= A \times B \times C$$
where $A, B$ and $C$ indicate the products of the above terms in order.

Now we calculate $A$.

$$A = \prod_{wz \in E(G_1)} [d_{G_1+G_2}(w)d_{G_1+G_2}(z)]$$

$$= \prod_{wz \in E(G_1)} [(d_{G_1}(w) + p_2)^2(d_{G_1}(z) + p_2)^2]$$

$$\leq \left[ \sum_{wz \in E(G_1)} [(d_{G_1}(w) + p_2)^2(d_{G_1}(z) + p_2)^2] \right]^{q_1}$$

$$= \left[ \sum_{wz \in E(G_1)} [d_{G_1}^2(w) + p_2^2 + 2p_2d_{G_1}(w)] [d_{G_1}^2(z) + p_2^2 + 2p_2d_{G_1}(z)] \right]^{q_1}$$

$$= \left[ HM_2(G_1) + p_2^2HM(G_1) + p_2^4q_1 + 2p_2GO_2(G_1) + 2p_2^2M_2(G_1) + 2p_2^3M_1(G_1) \right]^{q_1}$$

Next we calculate $B$.

$$B = \prod_{wz \in E(G_2)} [d_{G_1+G_2}(w)d_{G_1+G_2}(z)]$$

$$= \prod_{wz \in E(G_2)} [(d_{G_2}(w) + p_1)^2(d_{G_2}(z) + p_1)^2]$$

$$\leq \left[ \sum_{wz \in E(G_2)} [(d_{G_2}(w) + p_1)^2(d_{G_2}(z) + p_1)^2] \right]^{q_2}$$

$$= \left[ \sum_{wz \in E(G_2)} [d_{G_2}^2(w) + p_1^2 + 2p_1d_{G_2}(w)] [d_{G_2}^2(z) + p_1^2 + 2p_1d_{G_2}(z)] \right]^{q_2}$$
Finally, we compute $C$.

\[
C = \prod_{w \in V(G_1)} \prod_{z \in V(G_2)} \left[ d_{G_1 + G_2}^2(w) + d_{G_1 + G_2}^2(z) \right]^{p_1 p_2}\]

\[
= \prod_{w \in V(G_1)} \prod_{z \in V(G_2)} \left[ (d_{G_1}(w) + p_2)^2 (d_{G_2}(z) + p_1)^2 \right]^{p_1 p_2}\]

\[
\leq \left[ \frac{\sum_{w \in V(G_1)} \sum_{z \in V(G_2)} \left[ (d_{G_1}(w) + p_2)^2 (d_{G_2}(z) + p_1)^2 \right]^{p_1 p_2}}{p_1 p_2} \right]^{p_1 p_2}\]

\[
= \left[ \frac{M_1(G_1)M_1(G_2) + p_1^2 p_2 M_1(G_1) + p_1 p_2^2 M_1(G_2) + 4 p_1 q_2 M_1(G_1) + p_1^3 p_2^3 + p_2 q_1 M_1(G_2) + 4 p_1^2 p_2^2 q_2 + 4 p_1^2 p_2^2 q_1 + 16 p_1 p_2 q_1 q_2}{p_1 p_2} \right]^{p_1 p_2}\]

Now using $A$, $B$ and $C$ we get the desired result. \(
\square
\)

**Lemma 3.2.** Let $G_i, (i = 1, 2)$ be two regular graphs of degree $r_i$.

Let $G_i, (i = 1, 2)$ be a $(p_i, q_i)$ - graph. Then

\[
\prod HM_2(G_1 + G_2) = (r_1 + p_2)^{4 q_1} \times (r_2 + p_1)^{4 q_2} \times [(r_1 + p_2)^2 (r_2 + p_1)^2]^{p_1 p_2}
\]

**Proof:**

\[
\prod HM_2(G_1 + G_2) = \prod_{w \in E(G_1 + G_2)} \left[ d_{G_1 + G_2}^2(w) d_{G_1 + G_2}^2(z) \right]
\]
THE MULTIPLICATIVE SECOND HYPER ZAGREB INDEX OF SOME GRAPH OPERATIONS 2835

\begin{align*}
&= \prod_{wz \in E(G_1)} \left[ d_{G_1+G_2}^2(w) d_{G_1+G_2}^2(z) \right] \\
&\times \prod_{wz \in E(G_2)} \left[ d_{G_1+G_2}^2(w) d_{G_1+G_2}^2(z) \right] \\
&\times \prod_{w \in V(G_1)} \prod_{z \in V(G_2)} \left[ d_{G_1+G_2}^2(w) d_{G_1+G_2}^2(z) \right] \\
&= \prod_{wz \in E(G_1)} (r_1 + p_2)^2(r_1 + p_2)^2 \prod_{wz \in E(G_2)} (r_2 + p_1)^2(r_2 + p_1)^2 \\
&\prod_{wz \in E(G_1)} \prod_{wz \in E(G_2)} (r_1 + p_2)^2(r_2 + p_1)^2 \\
&= (r_1 + p_2)^{4q_1} \times (r_2 + p_1)^{4q_2} \\
&\times [(r_1 + p_2)^2(r_2 + p_1)^2]^{p_1p_2}
\end{align*}

(1)

Remark 3.3. We find the upper bound of Lemma 3.2 when \( G \) is a regular graph of degree \( r \) with \( p \) vertices and \( q \) edges. Here
\[
q = \frac{pr}{2}, M_1(G) = pr^2, M_2(G) = qr^2, F(G) = 2qr^2, F_2(G) = 2qr^3,
\]
\[
HM(G) = 4qr^2HM_2(G) = qr^4, GO_2(G) = 2qr^3
\]

Corollary 3.4. Let \( G_i, (i = 1, 2) \) be two regular graphs of degree \( r_i \).
Let \( G_i, (i = 1, 2) \) be a \((p_i, q_i)\) - graph. Then
\[
\prod_{i} HM_2(G_1 + G_2) \leq (r_1 + p_2)^{4q_1} \times (r_2 + p_1)^{4q_2} \\
\times [(r_1 + p_2)^2(r_2 + p_1)^2]^{p_1p_2}
\]

(2)

From (1) and (2) the bound is tight.
4. The Multiplicative Second Hyper Zagreb Index of Composition of Graphs

Theorem 4.1. Let $G_i, i = 1, 2$ be a $(p_i, q_i)$-graph. Then

$$\prod HM_2(G_1[G_2]) \leq \begin{bmatrix} p_2^4 q_2 F_2(G_1) + p_2^3 M_1(G_1) HM(G_2) + p_1 HM_2(G_2) \\ + 2p_2^3 F(G_1) M_1(G_2) + 2p_2^2 M_1(G_1) M_2(G_2) + 4p_2 q_1 GO_2(G_2) \end{bmatrix}_{p_1 q_2}$$

$$\times \begin{bmatrix} p_2^6 HM_2(G_1) + p_2^3 M_1(G_2) F(G_1) + q_1 (M_1(G_2))^2 + 4p_2^4 q_2 GO_2(G_2) \\ + 16p_2^2 q_2 M_2(G_1) + 4p_2 q_2 M_1(G_1) M_1(G_2) \end{bmatrix}_{q_1 p_2^2}$$

Proof:

$$\prod HM_2(G_1[G_2]) = \prod_{(w,k)\in E(G_1[G_2])} \left[ d_{G_1[G_2]}^2(w,k) d_{G_1[G_2]}^2(z,l) \right]$$

$$= \prod_{w\in V(G_1) \times l\in E(G_2)} \prod_{k\in V(G_2)} \prod_{l\in E(G_2)} \prod_{wz\in E(G_1)} \left[ d_{G_1[G_2]}^2(w,k) d_{G_1[G_2]}^2(z,l) \right]$$

$$= A \times B,$$

where $A$ and $B$ indicate the products of the above terms in order.

Now we compute $A$.

$$A = \prod_{w\in V(G_1) \times kl\in E(G_2)} \left[ d_{G_1[G_1]}^2(w,k) d_{G_1[G_2]}^2(w,l) \right]$$

$$= \prod_{w\in V(G_1) \times kl\in E(G_2)} \left[ [p_2 d_{G_1}(w) + d_{G_2}(k)]^2 [p_2 d_{G_1}(w) + d_{G_2}(l)]^2 \right]$$

$$\leq \begin{bmatrix} \sum_{w\in V(G_1) \times kl\in E(G_2)} \left[ [p_2 d_{G_1}(w) + d_{G_2}(k)]^2 [p_2 d_{G_1}(w) + d_{G_2}(k)]^2 \right] \end{bmatrix}_{p_1 q_2}$$
Using $A$ and $B$, we get the required result. \qed
Lemma 4.2. Let $G_i, i = 1, 2$ be two regular graphs of degree $r_i$ and let $G_i, i = 1, 2$ be a $(p_i, q_i)$-graph. Then $\prod HM_2 (G_1 [G_2]) = (p_2 r_1 + r_2)^4 (p_1 q_2 + p_2 q_1)$.

Proof:

$$\prod HM_2 (G_1 [G_2]) = \prod_{w \in V(G_1)} \prod_{kl \in E(G_2)} \left[ d_{G_1 [G_2]}^2 (w, k) d_{G_1 [G_2]}^2 (w, l) \right]$$

$$\times \prod_{k \in V(G_2)} \prod_{l \in V(G_2)} \prod_{wz \in E(G_1)} \left[ d_{G_1 [G_2]}^2 (w, k) d_{G_1 [G_2]}^2 (z, l) \right]$$

$$= \prod_{w \in V(G_1)} \prod_{kl \in E(G_2)} (p_2 r_1 + r_2)^2 (p_2 r_1 + r_2)^2$$

$$\times \prod_{k \in V(G_2)} \prod_{l \in V(G_2)} \prod_{wz \in E(G_1)} (p_2 r_1 + r_2)^2 (p_2 r_1 + r_2)^2$$

$$= (p_2 r_1 + r_2)^4 p_1 q_2 \times (p_2 r_1 + r_2)^4 p_2 q_1$$

$$= (p_2 r_1 + r_2)^4 (p_1 q_2 + p_2 q_1)$$

(3)

$\square$

Corollary 4.3. Let $G_i, (i = 1, 2)$ be two regular graphs of degree $r_i$.

Let $G_i, (i = 1, 2)$ be a $(p_i, q_i)$-graph. Then

$$\prod HM_2 (G_1 [G_2]) \leq \left[ q_2 F_2 (G_1) + 2F (G_1) M_1 (G_2) + 4M_1 (G_1) M_2 (G_2) + M_1 (G_1) F (G_2) + p_1 HM_2 (G_2) + 4q_1 GO_2 (G_2) \right]^{p_1 q_2}$$

(4)

From (3) and (4) our bound is tight.

5. The Multiplicative Second Hyper Zagreb Index of Cartesian Product of Graphs

Theorem 5.1. Let $G_i, i = 1, 2$ be a $(p_i, q_i)$-graph. Then

$$\prod HM_2 (G_1 \Box G_2) \leq \left[ q_2 F_2 (G_1) + 2F (G_1) M_1 (G_2) + 4M_1 (G_1) M_2 (G_2) + 4q_1 GO_2 (G_2) \right]^{p_1 q_2}$$
\[
\prod \left( \frac{p_1 F_2(G_2) + 2F(G_2)M_1(G_1) + 4M_1(G_2)M_2(G_1)}{p_2q_1} \right) + \frac{M_1(G_2)F(G_1) + p_2HM_2(G_1) + 4q_2GO_2(G_1)}{p_2q_1}
\]

**Proof:**

\[
\prod HM_2(G_1 \Box G_2) \leq \prod_{(w,k)(z,l) \in E(G_1 \Box G_2)} [d_{G_1 \Box G_2}^2(w,k)d_{G_1 \Box G_2}^2(z,l)]
\]

\[
= \prod_{w \in V(G_1)} \prod_{k \in E(G_2)} [d_{G_1 \Box G_2}^2(w,k)d_{G_1 \Box G_2}^2(z,l)]
\times \prod_{k \in V(G_2)} \prod_{w \in E(G_1)} [d_{G_1 \Box G_2}^2(w,k)d_{G_1 \Box G_2}^2(z,l)]
\]

\[
= A \times B
\]

where \(A\) and \(B\) indicate the products of the above terms in order.

Now we calculate \(A\).

\[
A = \prod_{w \in V(G_1)} \prod_{k \in E(G_2)} [d_{G_1 \Box G_2}^2(w,k)d_{G_1 \Box G_2}^2(z,l)]
\]

\[
= \prod_{w \in V(G_1)} \prod_{k \in E(G_2)} [(d_{G_1}^2(w) + d_{G_2}^2(k))^2(d_{G_1}^2(w) + d_{G_2}^2(l))^2]
\]

\[
\leq \left[ \sum_{w \in V(G_1)} \sum_{k \in E(G_2)} \left[ d_{G_1}^2(k) + d_{G_2}^2(w) + 2d_{G_1}(w)d_{G_2}(k) \right] \right]^{p_1q_2}
\]

\[
= \left[ \sum_{w \in V(G_1)} \sum_{k \in E(G_2)} \left[ d_{G_1}^2(k) + d_{G_2}^2(l) + 2d_{G_1}(w)d_{G_2}(l) \right] \right]^{p_1q_2}
\]

\[
= \left[ \sum_{w \in V(G_1)} \sum_{k \in E(G_2)} \left[ d_{G_1}^2(k) + d_{G_2}^2(l) + 2d_{G_1}(w)d_{G_2}(l) \right] \right]^{p_1q_2}
\]
Lemma 5.2. Let $G_i, i = 1, 2$ be two regular graphs of degree $r_i$ and let $G_i := 1, 2$ be a $(p_i, q_i)$-graph. Then $\prod HM_2(G_1 \square G_2) = (r_1 + r_2)^{4(p_1q_2 + p_2q_1)}$

Proof:

$$\prod HM_2(G_1 \square G_2) = \prod_{w \in V(G_1)} \prod_{k \in E(G_2)} \left[ d_{G_1 \square G_2}^2(w, k) d_{G_1 \square G_2}^2(w, l) \right]$$

$$\times \prod_{k \in V(G_2)} \prod_{w \in E(G_1)} \left[ d_{G_1 \square G_2}^2(w, k) + d_{G_1 \square G_2}^2(z, k) \right]$$

Now we compute $B$.

$$B = \prod_{k \in V(G_2)} \prod_{w \in E(G_1)} \left[ d_{G_1 \square G_2}^2(w, k) d_{G_1 \square G_2}^2(z, l) \right]$$

$$= \prod_{k \in V(G_2)} \prod_{w \in E(G_1)} \left[ (d_{G_1}(w) + d_{G_2}(k))^2 (d_{G_1}(z) + d_{G_2}(k))^2 \right]$$

$$\leq \prod_{k \in V(G_2)} \prod_{w \in E(G_1)} \left[ \sum_{k \in V(G_2)} \sum_{w \in E(G_1)} (d_{G_1}(w) + d_{G_2}(k))^2 (d_{G_1}(z) + d_{G_2}(k))^2 \right]$$

$$\leq \left[ q_1F_2(G_2) + 2F(G_2)M_1(G_1) + 4M_1(G_2)M_2(G_1) + M_1(G_2)F(G_1) + p_1HM_2(G_2) + 4q_1GO_2(G_2) \right]$$

$$\leq \left[ q_1F_2(G_2) + 2F(G_2)M_1(G_1) + 4M_1(G_2)M_2(G_1) + M_1(G_2)F(G_1) \right]$$

Using $A$ and $B$ we get the desired result. \qed
\[
\prod_{w \in V(G_1) \cap E(G_2)} (r_1 + r_2)^2 (r_1 + r_2)^2 \\
\times \prod_{k \in V(G_2), w \in E(G_2)} (r_1 + r_2)^2 (r_1 + r_2)^2
\]
\[
= (r_1 + r_2)^4 p_1 q_2 \times (r_1 + r_2)^4 p_2 q_1 \\
= (r_1 + r_2)^{4(p_1 q_2 + p_2 q_1)}
\]

(5)

Corollary 5.3. Let \(G_i, (i = 1, 2)\) be two regular graphs of degree \(r_i\).
Let \(G_i, (i = 1, 2)\) be a \((p_i, q_i)\)-graph. Then

\[
\prod_{w \in V(G_1) \cap E(G_2)} HM_2(G_1 \square G_2) \leq (r_1 + r_2)^{4(p_1 q_2 + p_2 q_1)}
\]

(6)

From (5) and (6) the bound is tight.

6. The Multiplicative Second Hyper Zagreb Index of Corona Product of Graphs

Theorem 6.1. Let \(G_i, i = 1, 2\) be a \((p_i, q_i)\)-graph. Then
Proof:

\[
\prod HM_2(G_1 \odot G_2) = \prod_{wz \in E(G_1)} ((d_{G_1}(w) + p_2)^2(d_{G_1}(z) + p_2)^2) \\
\times \prod_{w \in V(G_1)} \prod_{kl \in E(G_2)} ((d_{G_2}(k) + 1)^2(d_{G_2}(l) + 1)^2) \\
\times \prod_{w \in V(G_1)} \prod_{k \in V(G_2)} ((d_{G_1}(w) + p_2)^2(d_{G_2}(k) + 1)^2)
\]

\[
= A \times B \times C
\]

where \(A, B\) and \(C\) are the products of the about terms in order.

Now calculate \(A\),

\[
A = \prod_{wz \in E(G_1)} ((d_{G_1}(w) + p_2)^2(d_{G_1}(z) + p_2)^2)
\]

\[
= \left[ \sum_{wz \in E(G_1)} (d_{G_1}(w) + p_2)^2(d_{G_1}(z) + p_2)^2 \right]^{q_1}
\]

\[
= \left[ \sum_{w \in E(G_1)} [d_{G_1}^2(w) + p_2^2 + 2p_2d_{G_1}(w)][d_{G_1}^2(z) + p_2^2 + 2p_2d_{G_1}(z)] \right]^{q_1}
\]

\[
= \left[ \left( p_2^4 q_1 + 2p_2^3M_1(G_1) + 4p_2^2M_2(G_1) + p_2^2F(G_1) + HM_2(G_1) + 2p_2GO_2(G_1) \right) \right]^{q_1}
\]

Next compute \(B\),

\[
B = \prod_{w \in V(G_1)} \prod_{kl \in E(G_2)} ((d_{G_2}(k) + 1)^2(d_{G_2}(l) + 1)^2)
\]

\[
= \left[ \sum_{w \in V(G_1)} \sum_{kl \in E(G_2)} (d_{G_2}(k) + 1)^2(d_{G_2}(l) + 1)^2 \right]^{p_1q_2}
\]

\[
= \left[ \sum_{w \in V(G_1)} \sum_{kl \in E(G_2)} (d_{G_2}(k) + 1)^2(d_{G_2}(l) + 1)^2 \right]^{p_1q_2}
\]
Proof:

Let $G_1, G_2$ be two regular graph of degree $r_i$, and let $G_i, i = 1, 2$ be a $(p_i, q_i)$-graph. Then

$$\prod_i HM_2(G_1 \odot G_2) = (r_1 + p_2)^{4q_1} \times (r_2 + 1)^{4p_1q_2} \times ((r_1 + p_2)^2(r_2 + 1)^2)^{p_1p_2}$$

Proof:

$$\prod_i HM_2(G_1 \odot G_2) = \prod_i \left( (d_{G_1}(w) + p_2)^2(d_{G_2}(k) + 1)^2 \right)^{p_1p_2}$$

Finally, compute $C$

$$C = \prod_{w \in V(G_1)} \prod_{k \in V(G_2)} (d_{G_1}(w) + p_2)^2(d_{G_2}(k) + 1)^2$$

Now multiplying $A, B$ and $C$ we get the required result. \qed
\( \times \prod_{w \in V(G_1) \cap E(G_2)} (d_{G_1}(k) + 1)^2 (d_{G_2}(l) + 1)^2 \)

\( \times \prod_{w \in V(G_1)} \prod_{k \in V(G_2)} ((d_{G_1}(w) + p_2)^2 (d_{G_2}(k) + 1)^2) \)

\( = \prod_{w \in E(G_1)} (r_1 + p_2)^2 (r_1 + p_2)^2 \)

\( \times \prod_{u \in V(G_1) \cap E(G_2)} \prod_{k \in V(G_2)} (r_2 + 1)^2 (r_2 + 1)^2 \)

\( \times \prod_{w \in V(G_1) \cap V(G_2)} (r_1 + p_2)^2 (r_2 + 1)^2 \)

\( = (r_1 + p_2)^{4q_1} \times (r_2 + 1)^{4p_1q_2} \times ((r_1 + p_2)^2 (r_2 + 1)^2)^{p_1p_2} \)

(7)

\[ \square \]

**Corollary 6.3.** Let \( G_i \), \( i = 1, 2 \), be two regular graphs of degree \( r_i \).

Let \( G_i \), \( i = 1, 2 \), be a \((p_i, q_i)\) - graph. Then

\[ \prod_{w \in V(G_1) \cap E(G_2)} (d_{G_1}(k) + 1)^2 (d_{G_2}(l) + 1)^2 \]

\( \times \prod_{w \in V(G_1)} \prod_{k \in V(G_2)} ((d_{G_1}(w) + p_2)^2 (d_{G_2}(k) + 1)^2) \)

\( = (r_1 + p_2)^{4q_1} \times (r_2 + 1)^{4p_1q_2} \times ((r_1 + p_2)^2 (r_2 + 1)^2)^{p_1p_2} \)

(8)

From (7) and (8) the bound is tight.

**7. Conclusion**

In this paper, we have defined the multiplicative second hyper Zagreb index and derived the sharp upper bound for this index of various graph operations like join, composition, cartesian and corona product of graphs are derived. And we have proved that the sharp upper bound is tight.

**Conflict of Interests**

The author(s) declare that there is no conflict of interests.

**References**


