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WEAKLY COMMUTING MAPPING IN LINEAR 2-NORMED SPACES

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Abstract. The main idea of this paper is establishing a common fixed point theorem for four self-mappings of a complete linear 2-normed space using the weak commutating condition and A-contraction type condition and give some inclusion relations between these concepts.

Keywords: linear 2-normed space; weakly commuting mappings; fixed point.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

In 1963, S.Gahler ([5],[6]) introduced The concept of linear 2-normed spaces and 2- metric spaces. They are very important in mathematics, A. White,Y J Cho,R W Freese, S C Gupta, A H Siddique and others established and proved many theorems in linear 2-normed spaces and 2-metric spaces ([7],[14],[15],[4],[2],[9],[11][12],[13]). They have many applications in Metric Geometry, Functional Analysis and Topology as a new branch. Recently many researchers presented results in 2-normed spaces, analogous with that in classical normed spaces and Banach

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spaces. By a (K)-space, we mean a linear 2-normed space such that the 2-metric induced by the 2-norm satisfies the (K) property ([3]).

Here we prove a common fixed point theorem for two pairs of weakly commuting mappings using the idea of A-contraction and then extend the theorem for a family of self-mappings in a linear 2-normed space. Before proving our main theorem we need to state some preliminary ideas and definitions of weakly commuting mappings in a linear 2-normed space.

2. Preliminary Assertions

In 1963, S. Gahler ([5],[8]) introduced the concept of linear 2-normed space. Since then a number of mathematician have been investigating the different aspects of fixed point theory in the setting of linear 2-normed space.

2.1. Linear 2-Normed Space.

Definition 2.1. [12] Let X be a linear space over $\mathbb{R}$ with dimension greater then or equal to 2. If the function $\|.,.\| : X^2 \to \mathbb{R}^+ \cup \{0\}$ satisfies the following axioms then $(X, \|.,.\|)$ is called a linear 2- normed space:. Then

1. $\|x,y\| \geq 0$ for all $x,y \in X$, $\|x,y\| = 0$ if and only if x and y are linearly dependent,
2. $\|x,y\| = \|y,x\|$ for all $x,y \in X$,
3. $\|\alpha x,y\| = |\alpha|\|x,y\|$ for all $x,y \in X$ and $\alpha \in \mathbb{R}$,
4. $\|x,y+z\| \leq \|x,y\| + \|y,z\|$ for all $x,y,z \in X$.

If $\|,.,\|$ is called a 2-norm and the pair $(X, \|,.,\|)$ is called a linear 2- normed space. So a 2-norm $\|.,.\|$ always satisfies[24] $\|x,y+\alpha x\| = \|x,y\|$ for all $x,y \in X$ and scalar α

If we fix $\{u_i\}_{i=1}^{d}$ to be a basis for X, we can give the following lemma.

Lemma 2.2. [12] *Let $(X, \|.,.\|)$ be a 2-normed space. Then a sequence $\{x_n\}$ converges to x in X if and only if* $\lim_{n\to\infty} \max \|x_n - x, u_i\| = 0$.

Definition 2.3. [12] A 2-normed space $(X, \|.,.\|)$ is a 2-Banach space if any Cauchy sequence in X is convergent to an x in X.

Definition 2.4. Let S and T be two mappings from a linear 2-normed space $(X, \|.,.\|)$ into itself. Then a pair of mappings (S,T) is said to be weakly commuting on x, if $\|STx - TSx, u\| \leq \|Tx - Sx, u\|$ for all $u \in X$

Note that a commuting pair (S,T) on a linear 2-normed space $(X, \|.,.\|)$ is weakly commuting, but the converse is not true (see [10]). On the otherhand Cho, Khan, Singh ([1]) have proved some common fixed point theorems for weakly commuting self mappings in a linear 2-normed space. Here we shall prove some common fixed point theorems in linear 2-normed space in a more generalised conditions.

Let a nonempty set A consisting of all functions $\alpha : R_+^3 \to R_+$ satisfying
(i) α is continuous on the set R_+^3 of all triplets of nonnegative reals(with respect to the Euclidean metric on R^3).
(ii) $a \leq kb$ for some $k \in [0,1)$ whenever $a \leq \alpha(a,b,b)$ or $a \leq \alpha(b,a,b)$ or $a \leq \alpha(b,b,a)$, for all a,b.

Definition 2.5. A self map T on a metric space X is said to be A-contraction if it satisfies the condition:

$$\|Tx, Ty\| \leq \alpha(\|x,y\|, \|x,Tx\|, \|y,Ty\|). \tag{2.1}$$

for all $x,y \in X$ and some αA.

3. Main Result

Theorem 3.1. *Let I,J,S and T be four self mappings of a complete linear 2-normed space $(X, \|.,.\|)$ satisfying*

$$I(X) \subseteq T(X) \ \text{ and } J(X) \subseteq S(X). \tag{3.1}$$

For $\alpha \in A$ and for all $x,y,u \in X$

$$\|Ix - Jy, u\| \leq \alpha(\|Sx - Ty, u\|, \|Sx - Ix, u\|, \|Ty - Jy, u\|). \tag{3.2}$$

If one of I,J,S and T is continuous and if I and J weakly commute with S and T respectively, then I,J,S and T have a unique common fixed point z in X.

Proof. Let x_0 be an arbitrary element of X. We define $I_{x2n+1} = y_{2n+2}, Tx_{2n} = y_{2n}$ and $Jx_{2n} = y_{2n+1}, Sx_{2n+1} = y_{2n+1}; n = 1,2,...$ Taking $x = x_{2n+1}$ and $y = x_{2n}$ in (3.2) we have

$$\|Ix_{2n+1} - Jx2n, u\| \leq$$
$$\alpha(\|Sx_{2n+1} - Tx_{2n}, u\|, \|Sx_{2n+1} - Ix_{2n+1}, u\|, \|Tx_{2n} - Jx_{2n}, u\|)$$

or

$$\|y_{2n+2} - y_{2n+1}, u\| \leq \alpha(\|y_{2n+1} - y_{2n}, u\|, \|y_{2n+1} - y_{2n+2}, u\|, \|y_{2n} - y_{2n+1}, u\|).$$

So by axiom (1) of function α,

$$\|y_{2n+1} - y_{2n+2}, u\| \leq k.\|y_{2n} - y_{2n+1}, u\| \text{ where } k \in [0,1) \tag{3.3}$$

Similarly by putting $x = x_{2n-1}$ and $y = x_{2n}$ in (3.2) we get

$$\|Ix_{2n-1} - Jx2n, u\| \leq$$
$$\alpha(\|Sx_{2n-1} - Tx_{2n}, u\|, \|Sx_{2n-1} - Ix_{2n-1}, u\|, \|Tx_{2n} - Jx_{2n}, u\|)$$

or

$$\|y_{2n} - y_{2n+1}, u\| \leq \alpha(\|y_{2n-1} - y_{2n}, u\|, \|y_{2n-1} - y_{2n}, u\|, \|y_{2n} - y_{2n+1}, u\|).$$

So by axiom (2) of function α,

$$\|y_{2n} - y_{2n+1}, u\| \leq k.\|y_{2n-1} - y_{2n}, u\| \text{ where } k \in [0,1) \tag{3.4}$$

So by (3.3) and (3.4) we get

$$\|y_{2n+1} - y_{2n+2}, u\| \leq k.\|y_{2n} - y_{2n+1}, u\| \leq k^2.\|y_{2n-1} - y_{2n}, u\|.$$

Proceeding in this way

$$\|y_{2n+1} - y_{2n+2}, u\| \leq k^{2n+1}.\|y_0 - y_1, u\|$$

and

$$\|y_{2n} - y_{2n+1}, u\| \leq k^{2n}.\|y_0 - y_1, u\|$$

So in general

$$\|y_n - y_{n+1}, u\| \leq k^n.\|y_0 - y_1, u\| \tag{3.5}$$

Then using property (4) of linear 2-normed space we get

$$\|y_n - y_{n+2}, u\| \leq \|y_n - y_{n+2}, y_{n+1}\| + \|y_n - y_{n+1}, u\| + \|y_{n+1} - y_{n+2}, u\| \tag{3.6}$$

$$\leq \|y_n - y_{n+2}, y_{n+1}\| + \sum_{r=0}^{1} \|y_{n+r} - y_{n+r+1}, u\|. \tag{3.7}$$

Here we consider two possible cases to show that $\|y_n, y_{n+2}, y_{n+1}\| = 0$.

Case I

$n =$ even $= 2m$ (say), therefore

$$\begin{aligned}
\|y_n - y_{n+2}, y_{n+1}\| &= \|y_{2m} - y_{2m+2}, y_{2m+1}\| \\
&= \|y_{2m+2} - y_{2m+1}, y_{2m}\| \\
&\leq \|Ix_{2m+1} - Jx_{2m}, y_{2m}\| \\
&\leq \alpha(\|Sx_{2m+1} - Tx_{2m}, y_{2m}\|, \|Sx_{2m+1} - Ix_{2m}, y_{2m}\|, \|Tx_{2m} - Jx_{2m}, y_{2m}\|) \\
&= \alpha(\|y_{2m+1} - y_{2m}, y_{2m}\|, \|y_{2m+1} - y_{2m}, y_{2m}\|, \|y_{2m} - y_{2m+1}, y_{2m}\|) \\
&= \alpha(0, \|y_{2m+1} - y_{2m}, y_{2m}\|, 0).
\end{aligned}$$

So by axiom (1) of function α,

$$\|y_n - y_{n+2}, y_{n+1}\| = \|y_{2m} - y_{2m+2}, y_{2m+1}\| \leq k.0 \text{ where} k \in [0, 1)$$

which implies $\|y_n - y_{n+2}, y_{n+1}\| = 0$.

Case II

$n =$ odd $= 2m+1$ (say),therefore

$$\begin{aligned}\|y_n - y_{n+2}, y_{n+1}\| =& \|y_{2m+1} - y_{2m+3}, y_{2m+2}\| \\ =& \|y_{2m+3} - y_{2m+2}, y_{2m+1}\| \\ \leq& \|Jx_{2m+2} - Ix_{2m+1}, y_{2m+1}\| \\ \leq& \alpha(\|Sx_{2m+1} - Tx_{2m+2}, y_{2m+1}\|, \|Sx_{2m+1} - Ix_{2m+1}, y_{2m+1}\|, \\ & \|Tx_{2m+2} - Jx_{2m+2}, y_{2m+1}\|) \\ =& \alpha(\|y_{2m+1} - y_{2m+2}, y_{2m+1}\|, \|y_{2m+1} - y_{2m+2}, y_{2m+1}\|, \\ & \|y_{2m+2} - y_{2m+3}, y_{2m+1}\|) \\ =& \alpha(0, 0, \|y_{2m+2} - y_{2m+3}, y_{2m+1}\|).\end{aligned}$$

So by axiom (1) of function α,

$$\|y_n - y_{n+2}, y_{n+1}\| = \|y_{2m+1} - y_{2m+3}, y_{2m+2}\| \leq k.0 \text{ where} k \in [0,1)$$

So in either cases $\|y_n - y_{n+2}, y_{n+1}\| = 0$. Therefore from (3.6) we have

$$\|y_n - y_{n+2}, u\| \leq \sum_{r=0}^{1} \|y_{n+r} - y_{n+r+1}, u\|.$$

Proceeding in the same fashion we have for any $p > 0$,

$$\|y_n - y_{n+p}, u\| \leq \sum_{r=0}^{p-1} \|y_{n+r} - y_{n+r+1}, u\|.$$

Then by (3.5) we get

$$\|y_n - y_{n+p}, u\| \leq \frac{k^n}{1-k} \|y_0 - y_1, u\| \to 0 \text{ as} n \to \infty, p > 0 \text{ and} k \in [0,1).$$

Hence $\{y_n\}$ is a Cauchy sequence. Then by completeness of $X, \{y_n\}$ converges to a point $z \in X$i.e. $y_n \to z \in X$ as $n \to \infty$.

Since $\{y_n\}$ is a Cauchy sequence and taking limit as $n \to \infty$, we get $Ix_{2n} = Tx_{2n+1} \to z, Jx_{2n-1} = Sx_{2n} \to z$ and also $Jx_{2n+1} \to z$. Next suppose that S is continuous. Then $\{SIx_{2n}\}$ converges to

Sz. Then by property (4) of linear 2-normed space, we have

$$\begin{aligned}\|ISx_{2n}-Sz,u\| &\leq \|ISx_{2n}-Sz,SIx_{2n}\|+\|ISx_{2n}-SIx_{2n},u\|+\|SIx_{2n}-Sz,u\| \\ &\leq \|ISx_{2n}-Sz,SIx_{2n}\|+\|Sx_{2n}-Ix_{2n},u\|+\|SIx_{2n}-Sz,u\|\end{aligned}$$

since I and S weakly commute.

Letting $n\to\infty$, it follows that $\{ISx_{2n}\}$ converges to Sz. Again by using (3.2) we have

$$\begin{aligned}\|ISx_{2n}-Jx_{2n+1},u\| &\leq \|ISx_{2n}-Sz,SIx_{2n}\|+\|ISx_{2n}-SIx_{2n},u\|+\|SIx_{2n}-Sz,u\| \\ &\leq \alpha(\|S^2x_{2n}-Tx_{2n+1},u\|+\|S^2x_{2n}-ISx_{2n},u\|+\|Tx_{2n+1}-Jx_{2n+1},u\|).\end{aligned}$$

Since α is continuous, taking limit as $n\to\infty$ we get

$$\|Sz-z,u)\leq \alpha(\|Sz-z,u\|,\|Sz-Sz,u\|,\|z-z,u\|)$$

implies

$$\|Sz-z,u\|\leq \alpha(\|Sz-z,u\|,0,0)$$

So by axiom (1) of function α,

$$\|Sz-z,u\|\leq k.0=0 \text{ which gives } Sz=z. \tag{3.8}$$

Again using the inequality (3.2) we have

$$\|Iz-Jx_{2n+1},u\|\leq \alpha(\|Sz-Tx_{2n+1},u\|,\|Sz-Iz,u\|,\|Tx_{2n+1}-Jx_{2n+1},u\|).$$

Passing limit as $n\to\infty$ we get

$$\|Iz-z,u\|\leq \alpha(\|Sz-z,u\|,\|z-Iz,u\|,\|z-z,u\|)$$

implies

$$\|Iz-z,u\|\leq \alpha(0,\|z-Iz,u\|,0).$$

Then by axiom (1) of function α,

$$\|Iz-z,u\|\leq k.0=0 \text{ which gives } Iz=z. \tag{3.9}$$

Since $I(X) \subseteq T(X)$, there exists a point $\in X$ such that $= z = Iz$, so by (3.2) we have

$$\begin{aligned}\|z - Jz, u\| &= \|z - Jz, u\| \\ &\leq \alpha(\|Sz - Tz, u\|, \|Sz - Iz, u\|, \|Tz - Jz, u\|) \\ &= \alpha(\|z - z, u\|, \|z - z, u\|, \|z - Jz, u\|) \\ &= \alpha(0, 0, \|z - Jz, u\|)\end{aligned}$$

Then by axiom (1) of function α,

$$\|z - Jz, u)\| \leq k.0 = 0 \text{ which implies } Jz = z.$$

As J and T weakly commute

$$\|JTz - TJz, u\| \leq \|Tz - Jz, u\|$$

which gives $JTz = TJz$ implies

(3.10) $$Jz = JTz = TJz = Tz$$

$$\begin{aligned}\|z - Tz, u\| &\|Iz - Jz, u\| \\ &\leq \alpha(\|Sz - Tz, u\|, \|Sz - Iz, u\|, \|Tz - Jz, u\|) \\ &= \alpha(\|z - Tz, u\|, 0, 0).\end{aligned}$$

Then by axiom (1) of function α,

(3.11) $$\|z - Tz, u\| \leq k.0 = 0 \text{ which implies } Tz = z.$$

So by (3.8),(3.9),(3.10) and (3.11) we conclude that z is a common fixed point of I, J, S and T .
For uniqueness, Let w be another common fixed point in X such that

$$Iz = Jz = Sz = Tz = z \; and \; Iw = Jw = Sw = Tw = w.$$

Then by (3.2) we have

$$\begin{aligned}\|w-z,u\| &= \|Iw-Jz,u\| \\ &\leq \alpha(\|Sw-Tz,u\|,\|Sw-Iw,u\|,\|Tz-Jz,u\|) \\ &= \alpha(\|w-z,u\|,0,0)\end{aligned}$$

Then by axiom (1) of function α,

$$\|w-z,u\| \leq k.0 = 0 \text{ which implies} w = z.$$

So uniqueness of z is proved. The same result holds if any one of I,J and T is continuous. □

Corollary 3.2. *Let S,T,I and J be four self mappings of a complete linear 2-normed space $(X,\|.,.\|)$ satisfying*

$$I(X) \subseteq T(X) \text{ and} J(X) \subseteq S(X) \tag{3.12}$$

$$\|Ix-Jy,u\| \leq c.\max\{\|Sx-Ty,u\|,\|Sx-Ix,u\|,\|Ty-Jy,u\|\}. \tag{3.13}$$

for all x,y,u in X, where $0 \leq c < 1$. If one of S,T,I and J is continuous and if I and J weakly commute with S and T respectively, then I,J,S and T have a unique common fixed point z in X. This result is a Linear 2-normed space analogue of the theorem of [14].

For any $f:(X,\|.,.\|) \to (X,\|.,.\|)$ we denote $F_f = \{x \in X : x = f(x)\}$.

Lemma 3.3. *Let I,J,S and T be four self mappings of a complete Linear2-normed space $(X,\|.,.,\|)$. If the inequality (3.2) holds for $\alpha \in A$ and for all $x,y,u \in X$.*

Then $(F_S \bigcap F_T)\bigcap F_I = (F_S \bigcap F_T)\bigcap F_J$.

Proof. Let $x \in (F_S \bigcap F_T)\bigcap F_I$. Then by(3.2)

$$\begin{aligned}\|x-Jx,u\| &= \|Ix-Jx,u\| \\ &\leq \alpha(\|Sx-Tx,u\|,\|Sx-Ix,u\|,\|Tx-Jx,u\|) \\ &= \alpha(0,0,\|x-Jx,u\|)\end{aligned}$$

Then by axiom (1) of function α,

$$\|x-Jx,u\| \leq k.0 = 0 \text{ implies} x = Jx$$

thus

$$(F_S \bigcap F_T) \bigcap F_I \subseteq (F_S \bigcap F_T) \bigcap F_J.$$

Similarly we have

$$(F_S \bigcap F_T) \bigcap F_J \subseteq (F_S \bigcap F_T) \bigcap F_I.$$

and so $(F_S \bigcap F_T) \bigcap F_I \subseteq (F_S \bigcap F_T) \bigcap F_J$. □

Theorem 3.4. *Let S,T and $\{I_n\}_{n\in N}$ be mappings from a complete Linear 2-normed space $(X, \|.,.\|)$ into itself satisfying*

$$I_1(X) \subseteq T(X) \text{ and } \bigcap I_2(X) \subseteq S(X) \tag{3.14}$$

For $\alpha \in A$ and for all $x,y,u \in X$,

$$\|I_n x - I_{n+1} y, u\| \leq \alpha(\|Sx-Ty,u\|, \|Sx-I_n x,u\|, \|Ty-I_{n+1}y,u\|). \tag{3.15}$$

holds for all $n \in N$. If one of S,T,I_1 and I_2 is continuous and if I_1 and I_2 weakly commute with S and T respectively, then S,T and $\{I_n\}_n \in N$ have a unique common fixed point z in X.

Proof. By Theorem (3.1), S,T,I_1 and I_2 have a unique common fixed point z in X. Now z is a unique common fixed point of S,T,I_1 and also by Lemma (3.3), $(F_S \bigcap F_T) \bigcap F_{I_1} = (F_S \bigcap F_T) \bigcap F_{I_2}, z$ is a common fixed point of S,T,I_2. Also z is unique common fixed point of S,T,I_2. If not, let w be another common fixed point of S,T,I_2. Then by (3.15)

$$\begin{aligned}
\|z-w,u\| &= \|I_1 z - I_2 w, u\| \\
&\leq \alpha(\|Sz-Tw,u\|, \|Sz-I_1 z,u\|, \|Tw-I_2 w,u\|) \\
&= \alpha(\|z-w,u\|, \|z-z,u\|, \|w-w,u\|) \\
&= \alpha(\|z-w,u\|, 0, 0)
\end{aligned}$$

Then by axiom (1) of function α,

$$\|z-w,u\| \leq k.0 = 0 \text{ implies} z = w$$

In the similar manner we can show that z is a unique common fixed point of S, T, I_1 and I_2. Continuing in this way, we arrive at desired result □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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