WEAKLY COMMUTING MAPPING IN LINEAR 2-NORMED SPACES

DOAA RIZK$^{1,*}$, D. DHAMODHARAN$^{2}$, A. MOHAMED ALI$^{3}$

$^1$Department of Mathematics, College of Science and Arts, Qassim University, Al-Asyah, Saudi Arabia
$^2$Department of Mathematics, Jamal Mohamed College (Autonomous), Tiruchirappalli-620020, India
$^3$Department of Mathematics, G.T.N. Arts College (Autonomous), Dindigul-624005, India

Abstract. The main idea of this paper is establishing a common fixed point theorem for four self-mappings of a complete linear 2-normed space using the weak commutating condition and $A$-contraction type condition and give some inclusion relations between these concepts.

Keywords: linear 2-normed space; weakly commuting mappings; fixed point.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

In 1963, S. Gahler ([5],[6]) introduced the concept of linear 2-normed spaces and 2-metric spaces. They are very important in mathematics, A. White, Y J Cho, R W Freese, S C Gupta, A H Siddique and others established and proved many theorems in linear 2-normed spaces and 2-metric spaces ([7],[14],[15],[4],[2],[9],[11],[12],[13]). They have many applications in Metric Geometry, Functional Analysis and Topology as a new branch. Recently many researchers presented results in 2-normed spaces, analogous with that in classical normed spaces and Banach...
spaces. By a \((K)\)-space, we mean a linear 2-normed space such that the 2-metric induced by the 2-norm satisfies the \((K)\) property ([3]).

Here we prove a common fixed point theorem for two pairs of weakly commuting mappings using the idea of \(A\)-contraction and then extend the theorem for a family of self-mappings in a linear 2-normed space. Before proving our main theorem we need to state some preliminary ideas and definitions of weakly commuting mappings in a linear 2-normed space.

2. Preliminary Assertions

In 1963, S. Gahler ([5],[8]) introduced the concept of linear 2-normed space. Since then a number of mathematician have been investigating the different aspects of fixed point theory in the setting of linear 2-normed space.

2.1. Linear 2-Normed Space.

**Definition 2.1.** [12] Let \(X\) be a linear space over \(\mathbb{R}\) with dimension greater then or equal to 2. If the function \(\|.,.\|: X^2 \to \mathbb{R}^+ \cup \{0\}\) satisfies the following axioms then \((X,\|.,.\|)\) is called a linear 2-normed space. Then

1. \(\|x,y\| \geq 0\) for all \(x,y \in X\), \(\|x,y\| = 0\) if and only if \(x\) and \(y\) are linearly dependent,
2. \(\|x,y\| = \|y,x\|\) for all \(x,y \in X\),
3. \(\|\alpha x,y\| = |\alpha|\|x,y\|\) for all \(x,y \in X\) and \(\alpha \in \mathbb{R}\),
4. \(\|x,y+z\| \leq \|x,y\| + \|y,z\|\) for all \(x,y,z \in X\).

If \(\|.,.\|\) is called a 2-norm and the pair \((X,\|.,.\|)\) is called a linear 2-normed space. So a 2-norm \(\|.,.\|\) always satisfies[24] \(\|x,y + \alpha x\| = \|x,y\|\) for all \(x,y \in X\) and scalar \(\alpha\).

If we fix \(\{u_i\}_{i=1}^d\) to be a basis for \(X\), we can give the following lemma.

**Lemma 2.2.** [12] Let \((X,\|.,.\|)\) be a 2-normed space. Then a sequence \(\{x_n\}\) converges to \(x\) in \(X\) if and only if \(\lim_{n \to \infty} \max_{i=1}^d \|x_n - x, u_i\| = 0\).

**Definition 2.3.** [12] A 2-normed space \((X,\|.,.\|)\) is a 2-Banach space if any Cauchy sequence in \(X\) is convergent to an \(x\) in \(X\).
Definition 2.4. Let $S$ and $T$ be two mappings from a linear 2-normed space $(X, \|.,.\|)$ into itself. Then a pair of mappings $(S, T)$ is said to be weakly commuting on $x$, if $\|STx - TSx, u\| \leq \|Tx - Sx, u\|$ for all $u \in X$.

Note that a commuting pair $(S, T)$ on a linear 2-normed space $(X, \|.,.\|)$ is weakly commuting, but the converse is not true (see [10]). On the otherhand Cho, Khan, Singh ([1]) have proved some common fixed point theorems for weakly commuting self mappings in a linear 2-normed space. Here we shall prove some common fixed point theorems in linear 2-normed space in a more generalised conditions.

Let a nonempty set $A$ consisting of all functions $\alpha : R_3^+ \to R_+$ satisfying

(i) $\alpha$ is continuous on the set $R_3^+$ of all triplets of nonnegative reals(with respect to the Euclidean metric on $R^3$).

(ii) $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$, for all $a, b$.

Definition 2.5. A self map $T$ on a metric space $X$ is said to be $A$-contraction if it satisfies the condition:

\begin{equation}
\|Tx, Ty\| \leq \alpha(\|x, y\|, \|x, Tx\|, \|y, Ty\|).
\end{equation}

for all $x, y \in X$ and some $\alpha A$.

3. Main Result

Theorem 3.1. Let $I, J, S$ and $T$ be four self mappings of a complete linear 2-normed space $(X, \|.,.\|)$ satisfying

\begin{equation}
I(X) \subseteq T(X) \text{ and } J(X) \subseteq S(X).
\end{equation}

For $\alpha \in A$ and for all $x, y, u \in X$

\begin{equation}
\|Ix - Jy, u\| \leq \alpha(\|Sx - Ty, u\|, \|Sx - Ix, u\|, \|Ty - Jy, u\|).
\end{equation}
If one of $I,J,S$ and $T$ is continuous and if $I$ and $J$ weakly commute with $S$ and $T$ respectively, then $I,J,S$ and $T$ have a unique common fixed point $z$ in $X$. 

**Proof.** Let $x_0$ be an arbitrary element of $X$. We define $I_{x_{2n+1}} = y_{2n+2}, Tx_{2n} = y_{2n}$ and $Jx_{2n} = y_{2n+1}, Sx_{2n+1} = y_{2n+1}; n = 1,2,\ldots$ Taking $x = x_{2n+1}$ and $y = x_{2n}$ in (3.2) we have

$$
\|Ix_{2n+1} - Jx_{2n},u\| \leq \\
\alpha(\|Sx_{2n+1} - Tx_{2n}, u\|,\|Sx_{2n+1} - Ix_{2n+1}, u\|,\|Tx_{2n} - Jx_{2n}, u\|)
$$

or

$$
\|y_{2n+2} - y_{2n+1}, u\| \leq \alpha(\|y_{2n+1} - y_{2n}, u\|,\|y_{2n+1} - y_{2n+2}, u\|,\|y_{2n} - y_{2n+1}, u\|).
$$

So by axiom (1) of function $\alpha$,

(3.3) $$
\|y_{2n+1} - y_{2n+2}, u\| \leq k\|y_{2n} - y_{2n+1}, u\| \text{ where } k \in [0,1)
$$

Similarly by putting $x = x_{2n-1}$ and $y = x_{2n}$ in (3.2) we get

$$
\|Ix_{2n-1} - Jx_{2n}, u\| \leq \\
\alpha(\|Sx_{2n-1} - Tx_{2n}, u\|,\|Sx_{2n-1} - Ix_{2n-1}, u\|,\|Tx_{2n} - Jx_{2n}, u\|)
$$

or

$$
\|y_{2n} - y_{2n+1}, u\| \leq \alpha(\|y_{2n-1} - y_{2n}, u\|,\|y_{2n-1} - y_{2n}, u\|,\|y_{2n} - y_{2n+1}, u\|).
$$

So by axiom (2) of function $\alpha$,

(3.4) $$
\|y_{2n} - y_{2n+1}, u\| \leq k\|y_{2n-1} - y_{2n}, u\| \text{ where } k \in [0,1)
$$

So by (3.3) and (3.4) we get

$$
\|y_{2n+1} - y_{2n+2}, u\| \leq k\|y_{2n} - y_{2n+1}, u\| \leq k^2\|y_{2n-1} - y_{2n}, u\|.
$$

Proceeding in this way

$$
\|y_{2n+1} - y_{2n+2}, u\| \leq k^{2n+1}\|y_{0} - y_{1}, u\|
$$

and

$$
\|y_{2n} - y_{2n+1}, u\| \leq k^{2n}\|y_{0} - y_{1}, u\|
$$
So in general

\[(3.5) \quad \|y_n - y_{n+1}, u\| \leq k^n. \|y_0 - y_1, u\|\]

Then using property (4) of linear 2-normed space we get

\[(3.6) \quad \|y_n - y_{n+2}, u\| \leq \|y_n - y_{n+2}, y_{n+1}\| + \|y_n - y_{n+1}, u\| + \|y_{n+1} - y_{n+2}, u\|\]

\[(3.7) \quad \leq \|y_n - y_{n+2}, y_{n+1}\| + \sum_{r=0}^{1} \|y_{n+r} - y_{n+r+1}, u\|\]

Here we consider two possible cases to show that \([y_n, y_{n+2}, y_{n+1}] = 0\).

Case I

\[n = \text{ even } = 2m \text{ (say), therefore}\]

\[\|y_n - y_{n+2}, y_{n+1}\| = \|y_{2m} - y_{2m+2}, y_{2m+1}\|\]

\[= \|y_{2m+2} - y_{2m+1}, y_{2m}\|\]

\[\leq \|I_{2m+1} - J_{2m}, y_{2m}\|\]

\[\leq \alpha(\|S_{2m+1} - T_{2m}, y_{2m}\|, \|S_{2m+1} - I_{2m}, y_{2m}\|, \|T_{2m} - J_{2m}, y_{2m}\|)\]

\[= \alpha(\|y_{2m+1} - y_{2m}, y_{2m}\|, \|y_{2m+1} - y_{2m}, y_{2m}\|, \|y_{2m} - y_{2m+1}, y_{2m}\|)\]

\[= \alpha(0, \|y_{2m+1} - y_{2m}, y_{2m}\|, 0)\].

So by axiom (1) of function \(\alpha\),

\[\|y_n - y_{n+2}, y_{n+1}\| = \|y_{2m} - y_{2m+2}, y_{2m+1}\| \leq k. 0 \quad \text{where} \ k \in [0, 1)\]

which implies \([y_n - y_{n+2}, y_{n+1}] = 0\).

Case II
\[ n = \text{odd} = 2m + 1 \text{ (say), therefore} \]

\[
\|y_n - y_{n+2}, y_{n+1}\| = \|y_{2m+1} - y_{2m+3}, y_{2m+2}\| = \|y_{2m+3} - y_{2m+2}, y_{2m+1}\| \\
\leq \|Jx_{2m+2} - Ix_{2m+1}, y_{2m+1}\| \\
\leq \alpha(\|Sx_{2m+1} - Tx_{2m+2}, y_{2m+1}\|, \|Sx_{2m+1} - Ix_{2m+1}, y_{2m+1}\|, \|Tx_{2m+2} - Jx_{2m+2}, y_{2m+1}\|) \\
= \alpha(\|y_{2m+1} - y_{2m+2}, y_{2m+1}\|, \|y_{2m+1} - y_{2m+2}, y_{2m+1}\|, \|y_{2m+2} - y_{2m+3}, y_{2m+1}\|) \\
= \alpha(0, 0, \|y_{2m+2} - y_{2m+3}, y_{2m+1}\|). \]

So by axiom (1) of function \(\alpha\),

\[
\|y_n - y_{n+2}, y_{n+1}\| = \|y_{2m+1} - y_{2m+3}, y_{2m+2}\| \leq k.0 \text{ where } k \in [0, 1) \]

So in either cases \(\|y_n - y_{n+2}, y_{n+1}\| = 0\). Therefore from (3.6) we have

\[
\|y_n - y_{n+2}, u\| \leq \sum_{r=0}^{1} \|y_{n+r} - y_{n+r+1}, u\|. \]

Proceeding in the same fashion we have for any \(p > 0\),

\[
\|y_n - y_{n+p}, u\| \leq \sum_{r=0}^{p-1} \|y_{n+r} - y_{n+r+1}, u\|. \]

Then by (3.5) we get

\[
\|y_n - y_{n+p}, u\| \leq \frac{k^n}{1-k} \|y_0 - y_1, u\| \to 0 \text{ as } n \to \infty, p > 0 \text{ and } k \in [0, 1). \]

Hence \(\{y_n\}\) is a Cauchy sequence. Then by completeness of \(X\), \(\{y_n\}\) converges to a point \(z \in X\) i.e. \(y_n \to z \in X\) as \(n \to \infty\).

Since \(\{y_n\}\) is a Cauchy sequence and taking limit as \(n \to \infty\), we get \(Ix_{2n} = Tx_{2n+1} \to z, Jx_{2n-1} = Sx_{2n} \to z\) and also \(Jx_{2n+1} \to z\). Next suppose that \(S\) is continuous. Then \(\{SIx_{2n}\}\) converges to
$Sz$. Then by property (4) of linear 2-normed space, we have

$$\|ISx_{2n} - Sz, u\| \leq \|ISx_{2n} - Sz, Sx_{2n}\| + \|ISx_{2n} - Sx_{2n}, u\| + \|Sx_{2n} - Sx_{2n}, u\|$$

$$\leq \|ISx_{2n} - Sx_{2n}\| + \|Sx_{2n} - Sx_{2n}, u\| + \|Sx_{2n} - Sx_{2n}, u\|$$

since $I$ and $S$ weakly commute.

Letting $n \to \infty$, it follows that $\{ISx_{2n}\}$ converges to $Sz$. Again by using (3.2) we have

$$\|ISx_{2n} - Jx_{2n+1}, u\| \leq \|ISx_{2n} - Sx_{2n}, Sx_{2n}\| + \|ISx_{2n} - Sx_{2n}, u\| + \|Sx_{2n} - Sx_{2n}, u\|$$

$$\leq \alpha(\|S^2x_{2n} - Tx_{2n+1}, u\| + \|S^2x_{2n} - ISx_{2n}, u\| + \|Tx_{2n+1} - Jx_{2n+1}, u\|).$$

Since $\alpha$ is continuous, taking limit as $n \to \infty$ we get

$$\|Sz - z, u\| \leq \alpha(\|Sz - z, u\|, \|Sz - Sz, u\|, \|z - z, u\|)$$

implies

$$\|Sz - z, u\| \leq \alpha(\|Sz - z, u\|, 0, 0)$$

So by axiom (1) of function $\alpha$,

(3.8) $\|Sz - z, u\| \leq k.0 = 0$ which gives $Sz = z$.

Again using the inequality (3.2) we have

$$\|Iz - Jx_{2n+1}, u\| \leq \alpha(\|Sz - Tx_{2n+1}, u\|, \|Sz - Iz, u\|, \|Tx_{2n+1} - Jx_{2n+1}, u\|).$$

Passing limit as $n \to \infty$ we get

$$\|Iz - z, u\| \leq \alpha(\|Sz - z, u\|, \|z - Iz, u\|, \|z - z, u\|)$$

implies

$$\|Iz - z, u\| \leq \alpha(0, \|z - Iz, u\|, 0).$$

Then by axiom (1) of function $\alpha$,

(3.9) $\|Iz - z, u\| \leq k.0 = 0$ which gives $Iz = z$. 
Since $I(X) \subseteq T(X)$, there exists a point $z \in X$ such that $z = I_z$, so by (3.2) we have

\[
\|z - J_z, u\| = \|z - J_z, u\| \\
\leq \alpha(\|S - T_z, u\|, \|S_z - I_z, u\|, \|T - J_z, u\|) \\
= \alpha(\|z - z, u\|, \|z - z, u\|, \|z - J_z, u\|) \\
= \alpha(0, 0, \|z - J_z, u\|)
\]

Then by axiom (1) of function $\alpha$,

\[
\|z - J_z, u\| \leq k.0 = 0 \text{ which implies } J_z = z.
\]

As $J$ and $T$ weakly commute

\[
\|j T z - T J z, u\| \leq \|T z - J_z, u\|
\]

which gives $JTz = TJz$ implies

\[
(3.10) \quad J_z = J T z = T J z = T z
\]

\[
\|z - T z, u\| \|I z - J_z, u\| \\
\leq \alpha(\|S - T_z, u\|, \|S_z - I_z, u\|, \|T - J_z, u\|) \\
= \alpha(\|z - T_z, u\|, 0, 0).
\]

Then by axiom (1) of function $\alpha$,

\[
(3.11) \quad \|z - T z, u\| \leq k.0 = 0 \text{ which implies } T z = z.
\]

So by (3.8),(3.9),(3.10) and (3.11) we conclude that $z$ is a common fixed point of $I, J, S$ and $T$.

For uniqueness, Let $w$ be another common fixed point in $X$ such that

\[
I z = J z = S z = T z = z \text{ and } I w = J w = S w = T w = w.
\]
Then by (3.2) we have
\[ \|w - z, u\| = \|Iw - Jz, u\| \]
\[ \leq \alpha(\|Sw - Tz, u\|, \|Sw - Iw, u\|, \|Tz - Jz, u\|) \]
\[ = \alpha(\|w - z, u\|, 0, 0) \]

Then by axiom (1) of function \( \alpha \),
\[ \|w - z, u\| \leq k.0 = 0 \] which implies \( w = z \).

So uniqueness of \( z \) is proved. The same result holds if any one of \( I, J \) and \( T \) is continuous. \( \square \)

**Corollary 3.2.** Let \( S, T, I \) and \( J \) be four self mappings of a complete linear 2-normed space \((X, \|\cdot,\|)\) satisfying
\[ I(X) \subseteq T(X) \quad \text{and} \quad J(X) \subseteq S(X) \quad (3.12) \]
\[ \|Ix - Jy, u\| \leq c \cdot \max \{ \|Sx - Ty, u\|, \|Sx - Ix, u\|, \|Ty - Jy, u\| \}. \quad (3.13) \]

for all \( x, y, u \) in \( X \), where \( 0 \leq c < 1 \). If one of \( S, T, I \) and \( J \) is continuous and if \( I \) and \( J \) weakly commute with \( S \) and \( T \) respectively, then \( I, J, S \) and \( T \) have a unique common fixed point \( z \) in \( X \).

This result is a Linear 2-normed space analogue of the theorem of [14].

For any \( f : (X, \|\cdot,\|) \to (X, \|\cdot,\|) \) we denote \( F_f = \{ x \in X : x = f(x) \} \).

**Lemma 3.3.** Let \( I, J, S \) and \( T \) be four self mappings of a complete Linear2-normed space \((X, \|\cdot,\|)\). If the inequality (3.2) holds for \( \alpha \in A \) and for all \( x, y, u \in X \).
Then \((F_S \cap F_T) \cap F_I = (F_S \cap F_T) \cap F_J \).

**Proof.** Let \( x \in (F_S \cap F_T) \cap F_I \). Then by (3.2)
\[ \|x - Jx, u\| = \|Ix - Jx, u\| \]
\[ \leq \alpha(\|Sx - Tx, u\|, \|Sx - Ix, u\|, \|Tx - Jx, u\|) \]
\[ = \alpha(0, 0, \|x - Jx, u\|) \]
Then by axiom (1) of function $\alpha$,
\[
\|x - Jx, u\| \leq k.0 = 0 \text{ implies } x = Jx
\]
thus
\[
(F_S \cap F_T) \cap F_I \subseteq (F_S \cap F_T) \cap F_J.
\]
Similarly we have
\[
(F_S \cap F_T) \cap F_J \subseteq (F_S \cap F_T) \cap F_I.
\]
and so $(F_S \cap F_T) \cap F_I \subseteq (F_S \cap F_T) \cap F_J$. □

**Theorem 3.4.** Let $S, T$ and $\{I_n\}_{n \in \mathbb{N}}$ be mappings from a complete Linear 2-normed space $(X, \| \cdot, \|)$ into itself satisfying

(3.14) \[I_1(X) \subseteq T(X) \text{ and } \bigcap I_2(X) \subseteq S(X)\]

For $\alpha \in A$ and for all $x, y, u \in X$,

(3.15) \[\|I_n x - I_{n+1} y, u\| \leq \alpha(\|Sx - Ty, u\|, \|Sx - I_n x, u\|, \|Ty - I_{n+1} y, u\|)\]

holds for all $n \in \mathbb{N}$. If one of $S, T, I_1$ and $I_2$ is continuous and if $I_1$ and $I_2$ weakly commute with $S$ and $T$ respectively, then $S, T$ and $\{I_n\}_{n \in \mathbb{N}}$ have a unique common fixed point $z$ in $X$.

**Proof.** By Theorem (3.1), $S, T, I_1$ and $I_2$ have a unique common fixed point $z$ in $X$. Now $z$ is a unique common fixed point of $S, T, I_1$ and also by Lemma (3.3), $(F_S \cap F_T) \cap F_I = (F_S \cap F_T) \cap F_J$, $z$ is a common fixed point of $S, T, I_2$. Also $z$ is unique common fixed point of $S, T, I_2$. If not, let $w$ be another common fixed point of $S, T, I_2$. Then by (3.15)

\[
\|z - w, u\| = \|I_1 z - I_2 w, u\|
\]
\[
\leq \alpha(\|Sz - Tw, u\|, \|Sz - I_1 z, u\|, \|Tw - I_2 w, u\|)
\]
\[
= \alpha(\|z - w, u\|, \|z - z, u\|, \|w - w, u\|)
\]
\[
= \alpha(\|z - w, u\|, 0, 0)
\]

Then by axiom (1) of function $\alpha$,

\[
\|z - w, u\| \leq k.0 = 0 \text{ implies } z = w
\]
In the similar manner we can show that $z$ is a unique common fixed point of $S$, $T$, $I_1$ and $I_2$. Continuing in this way, we arrive at desired result $\square$

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

**REFERENCES**


